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The structure of the 2-Sylow-subgroup of $K_2(\mathfrak{o})$, I

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Introduction

Let E be a totally real number field with ring of integers $\mathfrak{o}$. The object of this paper is to show that part of the structure of the 2-Sylow-subgroup of the tame kernel $K_2(\mathfrak{o})$ is determined by the arithmetic of the relative quadratic extension F/E, where $F = E(\sqrt{-1})$, and that this connection has some implications on the validity of the 2-primary part of the Birch–Tate-conjecture. Our main result is a formula for the 2^n-rank of $K_2(\mathfrak{o})$ for all $n \geqq 2$, such that F contains a primitive 2^n-th root of unity: Let $A(F/E)$ denote the 2-Sylow-subgroup of the relative S-class-group of F over E, where S consists of all infinite and all dyadic primes of E, and let m be the number of dyadic primes of E, which decompose in F. Then the following formula holds (Theorem 3.1):

$$rk_{2^n}(K_2(\mathfrak{o})) = m + rk_{2^{n-1}}(A(F/E)/\mathrm{im}\,({}_2A(E)),$$

where ${}_2A(E)$ consists of the elements of order $\leqq 2$ in the S-class-group of E.

These considerations were motivated by results of K. S. Brown [4] on the p-fractional part (p any prime) of the value of the zeta-function ζ_E at -1. Combining his results with the rank-formula we get a verification of the 2-primary part of the Birch–Tate-conjecture in some new cases (Theorem 3.4) including all fields E, for which the 2-Sylow-subgroup of $K_2(\mathfrak{o})$ is elementary abelian. This generalizes results of Browkin–Schinzel [3], Hurrelbrink [7], Hurrelbrink–Kolster [8], Kolster [9], Urbanowicz [15] and G. Gras [5]. It seems worth mentioning that the field E is not assumed to be abelian over $\mathbb{Q}$.

1. Elements of 2-power order in $K_2(\mathfrak{o})$

Throughout the paper we use the following notations: E is a totally real algebraic number field with ring of integers $\mathfrak{o}$, $F = E(\sqrt{-1})$, S (resp. T) is the set of all infinite and all dyadic primes of E (resp. F), U_S (resp. U_T) is the group of S-units of E (resp. T-units of F) and $A(E)$ (resp. $A(F)$) is the 2-Sylow-subgroup of the S-class-group of E (resp. T-class-group of F).

If H is any finite abelian group, we denote by ${}_2H$ the subgroup of elements of order $\leqq 2$ and for $t \geqq 1$ by $\mathrm{rk}_{2^t}(H)$ the 2^t-rank of H, i.e. the number of cyclic components of H, whose order is divisible by 2^t.

Let

$$H = \langle y_1 \rangle \times \langle y_2 \rangle \times \cdots \times \langle y_n \rangle$$

be a decomposition of H into a product of cyclic groups $\langle y_i \rangle$. We shall call $y_1, \ldots, y_n$ a *basis* for H.

The norm map from $A(F)$ to $A(E)$ is surjective (cf. Washington [16], Theorem 10.1), and we denote the kernel by $A(F/E)$. Then $A(F/E)$ contains the image of ${}_2A(E)$, and we abbreviate the quotient by $A'(F/E)$.

Let n_0 be the maximal natural number n, such that F contains a primitive 2^n-th root of unity ζ_n. Thus $n_0 \geqq 2$. Our first aim is to describe elements of order 2^n in $K_2(E)$ for $n \leqq n_0$. If $w \in K_2(E)$ has order 2 we have $w = \{-1, x\}$ for some $x \in E^*$ (cf. Tate [14], Theorem 6.1). If $w \in K_2(E)$ has order 4 we have $w^2 = \{-1, x\}$ and by a result of Bass and Tate (cf. Milnor [11], Theorem 15.12) x must be a norm from F. Let $x = N_{F/E}(z)$, $z \in F^*$, and let $\mathrm{Tr}_{F/E}: K_2(F) \to K_2(E)$ denote the transfer map. Then we get

$$w^2 = \{-1, N_{F/E}(z)\} = \mathrm{Tr}_{F/E}(\{-1, z\}) = \mathrm{Tr}_{F/E}(\{i, z\})^2,$$

$i = \sqrt{-1}$, hence

$$w = \mathrm{Tr}_{F/E}(\{i, z\}) \cdot \{-1, y\}$$

for some $y \in E^*$.

Assume now that $w \in K_2(E)$ has order 2^n, $2 \leqq n \leqq n_0$. Proceeding as above we get

$$w^{2^{n-1}} = \{-1, N_{F/E}(z)\} = \mathrm{Tr}_{F/E}(\{\zeta_n, z\})^{2^{n-1}},$$

hence w and $\mathrm{Tr}_{F/E}(\{\zeta_n, z\})$ differ by an element of order $\leqq 2^{n-1}$. Using induction we get

LEMMA 1.1. *Every element $w \in K_2(E)$ of order 2^n, $1 \leqq n \leqq n_0$, has the form*

$$w = \mathrm{Tr}_{F/E}(\{\zeta_n, z\}) \cdot \{-1, y\}$$

for some $z \in F^*$, $y \in E^*$.

Remark 1.2. It is easy to calculate $\mathrm{Tr}_{F/E}(\{\zeta_n, z\})$: Let $z = a(1 + b\zeta_n)$, a, $b \in E^*$. Then we have

$$\{\zeta_n, z\} = \{\zeta_n, a\} \cdot \{\zeta_n, 1 + b\zeta_n\} = \{\zeta_n, a\} \cdot \{1 + b\zeta_n, -b\},$$

hence

$$\mathrm{Tr}_{F/E}(\{\zeta_n, z\}) = \mathrm{Tr}_{F/E}(\{1 + b\zeta_n, -b\}) = \{1 + b(\zeta_n + \bar{\zeta}_n) + b^2, -b\}.$$

Let $\mathfrak{p}$ be a finite prime of E with corresponding valuation $v_{\mathfrak{p}}$. The tame symbol $\tau_{\mathfrak{p}}: K_2(E) \to (\mathfrak{o}/\mathfrak{p})^*$ is defined by

$$\tau_{\mathfrak{p}}(\{u, v\}) = (-1)^{v_{\mathfrak{p}}(u)v_{\mathfrak{p}}(v)} \frac{u^{v_{\mathfrak{p}}(v)}}{v^{v_{\mathfrak{p}}(u)}} \bmod \mathfrak{p}$$

and $w \in K_2(E)$ is contained in $K_2(\mathfrak{o})$ if and only if $\tau_{\mathfrak{p}}(w) = 1$ for all finite primes $\mathfrak{p}$. Now, if w has 2-power order, the tame symbols vanish at the dyadic primes, hence in this case it is enough to show $\tau_{\mathfrak{p}}(w) = 1$ at all $\mathfrak{p} \notin S$. Assume $w = \{-1, x\}$ has order 2. Then $w \in K_2(\mathfrak{o})$ if and only if $v_{\mathfrak{p}}(x) \equiv 0(2)$ at all $\mathfrak{p} \notin S$, hence if we define

$$\Delta_S(E) = \{x \in E^* \mid v_{\mathfrak{p}}(x) \equiv 0(2) \text{ for all } \mathfrak{p} \notin S\}$$

we get a surjective homomorphism

$$\Delta_S(E)/E^{*2} \to {}_2K_2(\mathfrak{o}),$$

whose kernel has order 2 (cf. Tate [14], Theorem 6.3).

LEMMA 1.3. *$\alpha_E := 2 + \zeta_{n_0} + \bar{\zeta}_{n_0}$ generates the kernel of $\Delta_S(E)/E^{*2} \to {}_2K_2(\mathfrak{o})$.*

Proof. We have $\alpha_E = N_{F/E}(1 + \zeta_{n_0})$, hence α_E is an S-unit, in fact the norm of a T-unit in F. Let ζ_{n_0+1} be a primitive 2^{n_0+1}-th root of unity such that $\zeta_{n_0+1}^2 = \zeta_{n_0}$. By definition of n_0 the field $E(\zeta_{n_0+1} + \bar{\zeta}_{n_0+1})$ has degree 2 over E. Since $\alpha_E = (\zeta_{n_0+1} + \bar{\zeta}_{n_0+1})^2$ this field is equal to $E(\sqrt{\alpha_E})$. Hence α_E is not a square in E. In order to show that $\{-1, \alpha_E\}$ is trivial, it is enough to show that $\{-1, 1 + \zeta_{n_0}\}$ is trivial in $K_2(F)$. But this is obvious, since $-1 = \zeta_{n_0}^{2^{n_0-1}}$ and $n_0 \geqq 2$.
The order of the group $\Delta_S(E)/E^{*2}$ is easily determined from the exact sequence

$$0 \to U_S/U_S^2 \to \Delta_S(E)/E^{*2} \xrightarrow{\varphi} A(E)_2 \to 0,$$

where $\varphi([x]) = [\prod_{\not\!\mathfrak{p} \notin S} \mathfrak{p}^{(1/2)v_{\mathfrak{p}}(x)}]$. We get

$$\mathrm{rk}_2\,(\Delta_S(E)/E^{*2}) = |S| + \mathrm{rk}_2\,(A(E)),$$

hence the well-known formula for the 2-rank of $K_2(\mathfrak{o})$:

LEMMA 1.4. $\mathrm{rk}_2\,(K_2(\mathfrak{o})) = |S| - 1 + \mathrm{rk}_2\,(A(E))$.

To detect elements of higher 2-power order in $K_2(\mathfrak{o})$ we have to look at certain subgroups of $\Delta_S(E)/E^{*2}$: Let $\Delta_0(E) := \Delta_S(E) \cap N_{F/E}(F^*)$. An element $x \in \Delta_0(E)$ is called *n-admissible* $(2 \leqq n \leqq n_0)$, if there is $z \in F^*$ with $N_{F/E}(z) = x$ and $v_{\mathscr{P}}(z \cdot \bar{z}^{-1}) \equiv 0(2^{n-1})$ for all $\mathscr{P} \notin T$. Since $v_{\mathscr{P}}(z \cdot \bar{z}^{-1}) = v_{\mathfrak{p}}(N_{F/E}(z)) - 2v_{\mathscr{P}}(\bar{z}) \equiv 0(2)$, all elements from Δ_0(E) are 2-admissible. In general, for fixed n, the n-admissible elements form a subgroup of $\Delta_0(E)$. Moreover the property of being n-admissible depends only on the class mod E^{*2}, so that the n-admissible classes form a subgroup of $\Delta_0(E)/E^{*2}$. Assume now that $[x]$ is an n-admissible class and let $z \in F^*$ satisfy $N_{F/E}(z) \in [x]$ and $v_{\mathscr{P}}(z \cdot \bar{z}^{-1}) \equiv 0(2^{n-1})$ at all $\mathscr{P} \notin T$. To $[x]$ and z we attach the following ideal class mod squares:

$$\psi_n([x]; z) = \left[\prod_{\mathfrak{p} \notin S} \mathfrak{p}^{(1/2^{n-1})v_{\mathscr{P}}(z \cdot \bar{z}^{-1})}\right] \bmod A(E)^2,$$

where $\mathscr{P} \notin T$ is any prime dividing $\mathfrak{p}$. We define

$$\Delta_n(E) := \{[x] \in \Delta_0(E)/E^{*2} \mid \psi_n([x]; z) = 0 \text{ for some } z\}.$$

Clearly for all n, $2 \leqq n \leqq n_0$, this is a subgroup of $\Delta_0(E)/E^{*2}$. Moreover, since α_E is the norm of a T-unit, we have $[\alpha_E] \in \Delta_n(E)$ for all n.

Our main result in this section is the following:

PROPOSITION 1.5. *Let* $[x] \in \Delta_0(E)/E^{*2}$ *be a non-trivial class different from* $[\alpha_E]$. *Then* $\{-1, x\}$ *is a* 2^{n-1}*-th power in* $K_2(\mathfrak{o})$, $2 \leqq n \leqq n_0$, *if and only if* $[x] \in \Delta_n(E)$.

We immediately get the following

COROLLARY 1.6. *Let* $2 \leqq n \leqq n_0$. *Then we have*

$$\mathrm{rk}_{2^n}\,(K_2(\mathfrak{o})) = \mathrm{rk}_2\,(\Delta_n(E)) - 1.$$

Proof of Prop. 1.5. Let $w \in K_2(\mathfrak{o})$ satisfy $w^{2^{n-1}} = \{-1, x\}$. By Lemma 1.1

$$w = \mathrm{Tr}_{F/E}(\{\zeta_n, z\}) \cdot \{-1, y\}$$

with $N_{F/E}(z) \in [x]$, $y \in E^*$. To go further we have to evaluate the tame symbols $\tau_{\mathfrak{p}}$ at $\mathrm{Tr}_{F/E}(\{\zeta_n, z\})$ for $\mathfrak{p} \notin S$. This can be done using Remark 1.2, but it is simpler to use the following approach: Let $\mu(E_{\mathfrak{p}})$ (resp. $\mu(F_{\mathfrak{P}})$) denote the group of roots of unity of the local field $E_{\mathfrak{p}}$ (resp. $F_{\mathfrak{P}}$) and let $\lambda_{\mathfrak{p}}: K_2(E) \to \mu(E_{\mathfrak{p}})$ and $\lambda_{\mathfrak{P}}: K_2(F) \to \mu(F_{\mathfrak{P}})$ denote the norm residue symbols. According to Bak–Rehmann [1], Prop. 2 for each $\mathfrak{p} \notin S$ there is a commutative square

$$\begin{tikzcd} K_2(F) \arrow[r, "\prod\limits_{\mathfrak{P}|\mathfrak{p}} \lambda_{\mathfrak{P}}"] \arrow[d, "\mathrm{Tr}_{F/E}"'] & \prod\limits_{\mathfrak{P}|\mathfrak{p}} \mu(F_{\mathfrak{P}}) \arrow[d, "\prod\limits_{\mathfrak{P}|\mathfrak{p}} \iota_{\mathfrak{P}}"] \\ K_2(E) \arrow[r, "\lambda_{\mathfrak{p}}"] & \mu(E_{\mathfrak{p}}) \end{tikzcd}$$

where $\iota_{\mathfrak{P}}: \mu(F_{\mathfrak{P}}) \to \mu(E_{\mathfrak{p}})$ is raising to the $|\mu(F_{\mathfrak{P}})|/|\mu(E_{\mathfrak{p}})|$-th power.

Since $\mathfrak{p} \notin S$, the extension F/E is unramified at $\mathfrak{p}$, hence up to odd torsion the order of the norm residue symbol coincides with the order of the tame symbol. Thus if we replace the norm residue symbols by the tame symbols we get a corresponding square which commutes up to odd torsion.

Now, if $\mathfrak{p} \notin S$ is inert in F and $\mathfrak{P} | \mathfrak{p}$ we have $\tau_{\mathfrak{P}}(\{\zeta_n, z\}) = \zeta_n^{v_{\mathfrak{P}}(z)} \bmod \mathfrak{P}$ and $\tau_{\mathfrak{p}}(\mathrm{Tr}_{F/E}(\{\zeta_n, z\}))$ is a certain power of this. But this power must be divisible by 2^n, since if we take any element $a \in E$ with $v_{\mathfrak{p}}(a) = 1$, we have $\tau_{\mathfrak{P}}(\{\zeta_n, a\}) = \zeta_n \bmod \mathfrak{P}$, whereas $\mathrm{Tr}_{F/E}(\{\zeta_n, a\}) = 1$. Hence we see that $\mathrm{Tr}_{F/E}(\{\zeta_n, z\})$ has trivial tame symbols at inert $\mathfrak{p}$. Assume now that $\mathfrak{p} = \mathfrak{P}\bar{\mathfrak{P}}$ decomposes in F. Then we have to take the product of $\zeta_n^{v_{\mathfrak{P}}(z)} \bmod \mathfrak{P}$ and $\zeta_n^{v_{\bar{\mathfrak{P}}}(z)} \bmod \bar{\mathfrak{P}}$ to get the order of $\tau_{\mathfrak{p}}(\mathrm{Tr}_{F/E}(\{\zeta_n, z\})$, hence the product of $\zeta_n^{v_{\mathfrak{P}}(z)} \bmod \mathfrak{P}$ and $\bar{\zeta}_n^{v_{\mathfrak{P}}(\bar{z})} \bmod \mathfrak{P}$, which is equal to $\zeta_n^{v_{\mathfrak{P}}(z \cdot \bar{z}^{-1})} \bmod \mathfrak{p}$.

Since $\tau_{\mathfrak{p}}(w) = 1$ at all $\mathfrak{p} \notin S$ and $\tau_{\mathfrak{p}}(\{-1, y\}) = (-1)^{v_{\mathfrak{p}}(y)} \bmod \mathfrak{p}$, we see that $v_{\mathfrak{P}}(z \cdot \bar{z}^{-1}) \equiv 0(2^{n-1})$ at all $\mathfrak{P} \notin T$ and $(1/2^{n-1})v_{\mathfrak{P}}(z \cdot \bar{z}^{-1}) + v_{\mathfrak{p}}(y) \equiv 0(2)$, hence $[x]$ is n-admissible and $\psi_n([x]; z) = 0$.

Conversely, the vanishing of $\psi_n([x]; z)$ for some z implies that there is $y \in E^*$ such that $(1/2^{n-1})v_{\mathfrak{P}}(z \cdot \bar{z}^{-1}) + v_{\mathfrak{p}}(y)$ is even at all $\mathfrak{p} \notin S$, hence the computations above show that $\mathrm{Tr}_{F/E}(\{\zeta_n, z\}) \cdot \{-1, y\}$ lies in $K_2(\mathfrak{o})$.

2. Auxiliary computations

Let $\Delta_T(F) := \{z \in F^* \mid v_{\mathcal{P}}(z) \equiv 0(2) \text{ for all } \mathcal{P} \notin T\}$. We extract some useful information contained in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_T/U_T^2 & \longrightarrow & \Delta_T(F)/F^{*2} & \longrightarrow & {}_2A(F) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U_S/U_S^2 & \longrightarrow & \Delta_S(E)/E^{*2} & \longrightarrow & {}_2A(E) & \longrightarrow & 0 \end{array} \qquad (*)$$

where the vertical arrows are the various norm maps.

LEMMA 2.1. *The kernel of the norm map from $\Delta_T(F)/F^{*2}$ to $\Delta_S(E)/E^{*2}$ is isomorphic to $\Delta_S(E)/E^{*2} \cup -E^{*2}$.*

Proof. An application of Hilbert 90 shows that the kernel is contained in the image of $\Delta_S(E)/E^{*2}$. Moreover if $x \in \Delta_S(E)$ becomes a square in F, then either x or $-x$ is a square in E.

The kernel of the norm map from ${}_2A(F)$ to ${}_2A(E)$ is ${}_2A(F/E)$ and contains the image of ${}_2A(E)$ in ${}_2A(F)$. Thus if we let $c = \mathrm{rk}_2(A(E))$, $c' = \mathrm{rk}_2(\ker(A(E) \to A(F))$, then this image has 2-rank $c - c'$ and we define $g \geqq 0$ by $\mathrm{rk}_2(A(F/E)) = c - c' + g$. Since by Lemma 2.1 the kernel of the norm map $N_{F/E}: \Delta_T(F)/F^{*2} \to \Delta_S(E)/E^{*2}$ has 2-rank $|S| + c - 1$ and maps onto the image of ${}_2A(E)$ in ${}_2A(F)$, which has 2-rank $c - c'$ we get

LEMMA 2.2. *The 2-rank of the kernel of $N_{F/E}: U_T/U_T^2 \to U_S/U_S^2$ is equal to $|S| + c' - 1$.*

Let m be the number of dyadic primes of E, which decompose in F. Thus $m = |T| - |S|$. We get

COROLLARY 2.3. *The image of U_T/U_T^2 in U_S/U_S^2 has 2-rank $m - c' + 1$.*

Now this image contains the S-unit α_E, hence

COROLLARY 2.4. *$c' \leqq m$. In particular: $A(E)$ injects into $A(F)$, if all dyadic primes of E are undecomposed in F.*

Finally, since there are g components in ${}_2A(F/E)$ which survive in the cokernel of $U_T/U_T^2 \to U_S/U_S^2$, we get from Corollary 2.3:

COROLLARY 2.5. *The group* $U_S \cap N_{F/E}(\Delta_T(F))/U_S^2$ *has 2-rank* $m - c' + 1 + g$.

3. Rank formulas for $K_2(\mathfrak{o})$

The main result of this paper is the following:

THEOREM 3.1. *Let* $2 \leqq n \leqq n_0$. *Then the following formula holds:*

$$\mathrm{rk}_{2^n}(K_2(\mathfrak{o})) = m + \mathrm{rk}_{2^{n-1}}(A'(F/E)).$$

Before we give the proof we derive some consequences. If we take $n = 2$ we see that $\mathrm{rk}_4(K_2(\mathfrak{o})) = 0$ if and only if $m = 0$ and $A(F/E) = \mathrm{im}\,{}_2A(E)$. Now by Corollary 2.4 $m = 0$ implies that $A(E)$ injects into $A(F)$. Thus we get

COROLLARY 3.2. *The 2-Sylow-subgroup of* $K_2(\mathfrak{o})$ *is elementary abelian if and only if all dyadic primes of E are undecomposed in F and* $A(F/E) \cong {}_2A(E)$.

Let $m = 0$ and let h_S^- denote the order of $A(F/E)$. Then the order of $A'(F/E)$ is equal to $h_S^-/2^c$. If the exponent of the 2-Sylow-subgroup of $K_2(\mathfrak{o})$ is less than 2^{n_0}, we can compare the orders:

COROLLARY 3.3. *Assume that* $m = 0$ *and that the 2-Sylow-subgroup of* $K_2(\mathfrak{o})$ *has exponent less than* 2^{n_0}. *Then its order is equal to* $2^{|S|-1} \cdot h_S^-$.

Let us now describe the consequences of this result towards the 2-primary part of the Birch–Tate-conjecture: Let ζ_E denote the ζ-function of E and for each prime number p let n_p be the maximal natural number n, such that the cyclotomic field $E(\zeta_{p^n})$ is quadratic over E. Thus in particular n_2 coincides with n_0. If we put

$$w_2(E) := 2^{n_2+1} \cdot \prod_{p \text{ odd}} p^{n_p},$$

the Birch–Tate-conjecture predicts that

$$|K_2(\mathfrak{o})| = w_2(E) \cdot |\zeta_E(-1)|.$$

(cf. Birch [2], Tate [13]). The proof of the Main Conjecture in Iwasawa-theory given recently by Mazur–Wiles, implies that the conjecture holds for all abelian number fields up to 2-torsion (cf. Mazur–Wiles [10], Theorem 5). In contrast with this there are still only a few results on the 2-primary part: It is known to hold for certain real quadratic number fields by the work of Browkin–Schinzel [3], Hurrelbrink [7] and Urbanowicz [15] and for certain maximal real subfields of cyclotomic fields (cf. Kolster [9] and Hurrelbrink–Kolster [8]). In all these examples the 2-Sylow-subgroup of $K_2(\mathfrak{o})$ is elementary abelian. A more general result in this direction was obtained recently by G. Gras assuming that the 2-Sylow-subgroup of $K_2(\mathfrak{o})$ is elementary abelian and of 2-rank $[E:\mathbb{Q}]$, which of course implies that $A(E)$ is trivial and E has only one dyadic prime.

Combining the rank-formula with results of K. S. Brown we obtain a result which generalizes those from above:

THEOREM 3.4. *If $m=0$ and $h_S^- < 2^{c+n_0-1}$ the 2-primary part of the Birch–Tate-conjecture holds. In particular it holds if $K_2(\mathfrak{o})$ has an elementary abelian 2-Sylow-subgroup.*

Proof. In the case $m=0$ K. S. Brown (cf. [4], Prop. 9(ii)) has shown that the 2-fractional part of $\zeta_E(-1)/2^{|S|+c-3}$ is equal to $h_S^-/2^{c+n_0-1}$. Since by assumption $h_S^- < 2^{c+n_0-1}$ we get

$$w_2(E)\cdot|\zeta_E(-1)| = h_S^-\cdot 2^{|S|-1} \text{ up to odd torsion.}$$

Since $A'(F/E)$ has order $h_S^-/2^c < 2^{n_0-1}$, the exponent of the 2-Sylow-subgroup of $K_2(\mathfrak{o})$ is less than 2^{n_0}, hence the result follows from Corollary 3.3.

Proof of Theorem 3.1.

We start with the following simple group-theoretic lemma:

LEMMA 3.5. *Let H be a finite abelian 2-group of rank n and let N be an elementary abelian subgroup of rank k. There exists a basis $y_1,\dots,y_n$ of H, $|y_i| = 2^{n_i}$, such that $y_1^{2^{n_1-1}},\dots,y_k^{2^{n_k-1}}$ is a basis of N.*

Proof. Let $z_1,\dots,z_n$ be an arbitrary basis of H and $x_1,\dots,x_k$ an arbitrary basis of N. Furthermore let $|z_i| = 2^{m_i}$. We may assume that

$$x_1 = \prod_{j=1}^{s} z_j^{2^{m_j-1}}$$

and that $m_1 \leqq m_j$ for $j = 1, \ldots, s$. Let

$$y_1 = \prod_{j=1}^{s} z_j^{2^{m_j - m_1}}.$$

Then we get $y_1^{2^{m_1-1}} = x_1$ and $y_1, z_2, \ldots, z_n$ is again a basis of H. Changing – if necessary – x_i to $x_i x_1$, we may assume that $\langle x_2 \rangle \times \cdots \times \langle x_k \rangle$ is contained in $\langle z_2 \rangle \times \cdots \times \langle z_n \rangle$, hence the claim follows by induction.

If – in the situation of Lemma 3.5 – we have $k = n$, we call $y_1, \ldots, y_n$ a *joint basis* for H and N.

Let us apply Lemma 3.5 to the group $A(F/E)$ and the elementary abelian subgroup $\mathrm{im}\,({}_2A(E))$. We see in particular that $A(F/E)$ has some decomposition into a product $A(F/E)_{\mathrm{I}} \times A(F/E)_{\mathrm{II}}$, where $A(F/E)_{\mathrm{I}}$ and $\mathrm{im}\,({}_2A(E))$ have a joint basis. Thus

$$\mathrm{rk}_{2^{n-1}}(A'(F/E)) = \mathrm{rk}_{2^n}(A(F/E)_{\mathrm{I}}) + \mathrm{rk}_{2^{n-1}}(A(F/E)_{\mathrm{II}}).$$

In view of Corollary 1.6 we have to show that

$$\begin{aligned}\mathrm{rk}_2(\Delta_n(E)) &= m + 1 + \mathrm{rk}_{2^{n-1}}(A'(F/E)) \\ &= m + 1 + \mathrm{rk}_{2^n}(A(F/E)_{\mathrm{I}}) + \mathrm{rk}_{2^{n-1}}(A(F/E)_{\mathrm{II}}).\end{aligned}$$

We prove this in two steps using the map $\varphi : \Delta_S(E)/E^{*2} \to {}_2A(E)$.

Claim A. $\mathrm{rk}_2(\Delta_n(E) \cap \ker \varphi) = m - c' + 1 + \mathrm{rk}_{2^{n-1}}(A(F/E)_{\mathrm{II}})$.

Claim B. $\mathrm{rk}_2(\varphi(\Delta_n(E)) = c' + \mathrm{rk}_{2^n}(A(F/E)_{\mathrm{I}})$.

Proof of Claim A:

We first prove a Lemma:

LEMMA 3.6. $[x] \in \Delta_S(E)/E^{*2}$ *is contained in* $\Delta_n(E) \cap \ker \varphi$ *if and only if there exists* $u \in F^*$ *with* $N_{F/E}(u) \in [x]$, *such that* $v_{\mathcal{P}}(u) \equiv 0(2^{n-1})$ *for all* $\mathcal{P} \notin T$ *and* $[\prod_{\mathcal{P} \notin T} \mathcal{P}^{(1/2^{n-1}) v_{\mathcal{P}}(u)}] \in A(F/E)$.

Proof. Assume that $[x] \in \Delta_n(E) \cap \ker \varphi$ and let $z \in F^*$ satisfy $N_{F/E}(z) \in [x]$, $v_{\mathcal{P}}(z \cdot \bar{z}^{-1}) \equiv 0(2^{n-1})$ at all $\mathcal{P} \notin T$ and $\psi_n([x]; z)$ vanishes $\mathrm{mod}\, A(E)^2$. Since $[x] \in \ker \varphi$, we may assume that $N_{F/E}(z)$ is an S-unit. Then

$$\left[\prod_{\not{p} \notin S} \not{p}^{(1/2^{n-1}) v_{\not{p}}(z \cdot \bar{z}^{-1})}\right] = \left[\prod_{\not{p} \notin S} \not{p}^{-(1/2^{n-2}) v_{\not{p}}(\bar{z})}\right],$$

hence there is $y \in E^*$, such that

$$-\frac{1}{2^{n-2}} v_{\mathscr{P}}(\bar{z}) + v_{\not p}(y) \equiv 0(2) \quad \text{for all} \quad \mathscr{P} \notin T.$$

Let $u = y^{2^{n-2}}/\bar{z}$. Then u satisfies the conditions of the Lemma. Conversely, let $u \in F^*$ satisfy the conditions of the Lemma. Then it is easy to see that $[x] \in \ker \varphi$ and that $\psi_n([x]; u)$ vanishes in $A(E)/A(E)^2$.

Lemma 3.6 implies in particular that we have the following inclusions

$$N_{F/E}(U_T)/U_S^2 \subset \Delta_n(E) \cap \ker \varphi \subset U_S \cap N_{F/E}(\Delta_T)/U_S^2. \qquad (**)$$

By Corollary 2.3 the group $N_{F/E}(U_T)/U_S^2$ has rank $m - c' + 1$, hence we have to show that the rank of the quotient $B := \Delta_n(E) \cap \ker \varphi / N_{F/E}(U_T)$ is equal to $\mathrm{rk}_{2^{n-1}}(A(F/E)_{\mathrm{II}})$. Now by $(**)$ B embeds into the quotient of $U_S \cap N_{F/E}(\Delta_T)/U_S^2$ by $N_{F/E}(U_T)/U_S^2$, which by Corollary 2.4 and the definition of $A(F/E)_{\mathrm{II}}$ is isomorphic with the group of elements of order $\leqq 2$ in $A(F/E)_{\mathrm{II}}$, where the isomorphism is given by

$$\left[\prod_{\mathscr{P} \notin T} \mathscr{P}^{(1/2) v_{\mathscr{P}}(w)}\right] \to [N_{F/E}(w)] \bmod N(U_T).$$

Let $\mathrm{rk}_2(B) = k$. By Lemma 3.5 we find a basis $[\prod_{\mathscr{P} \notin T} \mathscr{P}^{(1/2^{n_i}) v_{\mathscr{P}}(w_i)}]$, $1 \leqq i \leqq g$, of $A(F/E)_{\mathrm{II}}$, such that $[w_i] \bmod N_{F/E}(U_T)$, $1 \leqq i \leqq k$, is a basis of B. Lemma 3.6 implies that we have $n_i < n - 1$ for all $i > k$, and that for $i \leqq k$ we find $u_i \in F^*$, such that $[N_{F/E}(u_i)] = [N_{F/E}(w_i)]$, $v_{\mathscr{P}}(u_i) \equiv 0(2^{n-1})$ at $\mathscr{P} \notin T$ and $[\prod_{\mathscr{P} \notin T} \mathscr{P}^{(1/2) v_{\mathscr{P}}(u_i)}]$ is a 2^{n-2}-th power in $A(F/E)$. Thus $n_i \geqq n - 1$ for $i \leqq k$, which proves Claim A.

Proof of Claim B:

We show first that $\ker(A(E) \to A(F))$ is contained in $\varphi(\Delta_n(E))$. Thus let $[\mathscr{b}] \in \ker(A(E) \to A(F))$. Since the norm map from $A(F)$ to $A(E)$ is surjective, we find an ideal-class $[\mathscr{a}]$ in $A(F)$, say

$$[\mathscr{a}] = \left[\prod_{\mathscr{P} \notin T} \mathscr{P}^{(1/2^m) v_{\mathscr{P}}(w)}\right],$$

such that $[N_{F/E}(\mathscr{a})] = [\mathscr{b}]$, hence $[\prod_{\not p \notin S} \not p^{(1/2^m) v_{\not p}(N_{F/E}(w))}] = [\mathscr{b}]$. Since $[\mathscr{b}]$ vanishes in $A(F)$, there is $z \in F^*$, such that $(1/2^m) v_{\not p}(N_{F/E}(w)) = v_{\mathscr{P}}(z)$ at all $\mathscr{P} \notin T$ $(\mathscr{P} \mid \not p)$.

This implies that $v_{\mathcal{P}}(z \cdot \bar{z}^{-1}) = 0$ at all $\mathcal{P} \notin T$ and that $[\mathfrak{b}] = \varphi([N_{F/E}(z)])$ with $[N_{F/E}(z)] \in \Delta_n(E)$. Since $\ker(A(E) \to A(F))$ has rank c', we get a contribution of c' to the 2-rank of $\varphi(\Delta_n(E))$.

Assume now, that the image of $\varphi(\Delta_n(E))$ in $A(F/E)_{\mathrm{I}}$ has rank k. By Lemma 3.5 we can find a basis $[\mathfrak{a}_1], \ldots, [\mathfrak{a}_{c-c'}]$ of $A(F/E)_{\mathrm{I}}$, $\|[\mathfrak{a}_i]\| = 2^{n_i}$, such that for $1 \leqq i \leqq k$ the $2^{n_i - 1}$-th powers of the $[\mathfrak{a}_i]$ form a basis of the image of $\varphi(\Delta_n(E))$ in $A(F/E)_{\mathrm{I}}$. In order to prove Claim B we have to show that $n_i \geqq n$ for $1 \leqq i \leqq k$ and $n_i < n$ for $i > k$.

Let $[x] \in \Delta_n(E)$. Then for a suitable $z \in F^*$ with $N_{F/E}(z) \in [x]$ we have

$$\left[\prod_{\mathfrak{p} \notin S} \mathfrak{p}^{(1/2^{n-1}) v_{\mathcal{P}}(z \cdot \bar{z}^{-1})}\right] = [\mathfrak{b}]^2 \text{ in } A(E),$$

hence

$$\left[\prod_{\mathfrak{p} \notin S} \mathfrak{p}^{(1/2) v_{\mathcal{P}}(z \cdot \bar{z}^{-1})}\right] = [\mathfrak{b}]^{2^{n-1}} \text{ in } A(E).$$

The left-hand side equals $\varphi([x]) \cdot [\prod_{\mathfrak{p} \notin S} \mathfrak{p}^{-v_{\mathcal{P}}(\bar{z})}]$. Now in $A(F)$:

$$\begin{aligned}\left[\prod_{\mathfrak{p} \notin S} \mathfrak{p}^{v_{\mathcal{P}}(\bar{z})}\right] &= \left[\prod_{\substack{\mathfrak{p} \notin S \\ \mathfrak{p} = \mathcal{P}}} \mathcal{P}^{v_{\mathcal{P}}(\bar{z})}\right] \cdot \left[\prod_{\substack{\mathfrak{p} \notin S \\ \mathfrak{p} = \mathcal{P}\bar{\mathcal{P}}}} (\mathcal{P}\bar{\mathcal{P}})^{v_{\mathcal{P}}(\bar{z})}\right] \\ &= \left[\prod_{\substack{\mathfrak{p} \notin S \\ \mathfrak{p} = \mathcal{P}\bar{\mathcal{P}}}} \bar{\mathcal{P}}^{v_{\mathcal{P}}(\bar{z} \cdot z^{-1})}\right] = [\mathfrak{c}]^{2^{n-1}},\end{aligned}$$

where $\mathfrak{c} = \prod_{\substack{\mathfrak{p} \notin S \\ \mathfrak{p} = \mathcal{P}\bar{\mathcal{P}}}} \bar{\mathcal{P}}^{(1/2^{n-1}) v_{\mathcal{P}}(\bar{z} \cdot z^{-1})}$. Now the norm of $[\mathfrak{c}]$ is equal to $[\mathfrak{b}]^{-2}$, hence if we denote by $[\mathfrak{b}_F]$ the image of $[\mathfrak{b}]$ in $A(F)$, we have shown that $[\mathfrak{c}\mathfrak{b}_F] \in A(F/E)$ and that the image of $\varphi([x])$ is a 2^{n-1}-th power in $A(F/E)$. Thus in particular we get $n_i \geqq n$ for $1 \leqq i \leqq k$.

Let $[\mathfrak{a}]$ be any of the ideal-classes $[\mathfrak{a}_i]$ with $i > k$ and assume that the order of $[\mathfrak{a}]$ is equal to 2^m with $m \geqq n$. Thus for some $w \in F^*$ we have

$$[\mathfrak{a}] = [\textstyle\prod_{\mathcal{P} \notin T} \mathcal{P}^{(1/2^m) v_{\mathcal{P}}(w)}] \text{ and } [N_{F/E}(\mathfrak{a})] = [\textstyle\prod_{\mathfrak{p} \notin S} \mathfrak{p}^{(1/2^m) v_{\mathfrak{p}}(N_{F/E}(w))}]$$

vanishes in $A(E)$. Thus there exists $y \in E^*$, such that $v_{\mathfrak{p}}(N_{F/E}(w)) = v_{\mathfrak{p}}(y^{2^m})$. Since $[\mathfrak{a}^{2^{m-1}}]$ lies in the image of ${}_2A(E)$, there exists an ideal-class $[\mathfrak{c}] \in A(F)$, say $[\mathfrak{c}] = [\prod_{\mathcal{P} \notin T} \mathcal{P}^{(1/2^s) v_{\mathcal{P}}(u)}]$, such that $[\mathfrak{a}^{2^{m-1}}]$ is equal to the image of $[\prod_{\mathfrak{p} \notin S} \mathfrak{p}^{(1/2^s) v_{\mathfrak{p}}(N_{F/E}(u))}]$

in $A(F)$. This implies the existence of some $q \in F^*$, such that

$$\frac{1}{2^s} v_{\mathfrak{p}}(N_{F/E}(u)) = \tfrac{1}{2} v_{\mathfrak{P}}(w) + v_{\mathfrak{P}}(q) \quad \text{at all} \quad \mathfrak{P} \notin T(\mathfrak{P} \mid \mathfrak{p}).$$

Thus

$$\frac{1}{2^s} v_{\mathfrak{p}}(N_{F/E}(u)) = \frac{1}{2^{m-2}} v_{\mathfrak{p}}(y) + \tfrac{1}{2} v_{\mathfrak{p}}(N_{F/E}(q)) \quad \text{at all} \quad \mathfrak{p} \notin S.$$

Let $z = q \cdot y^{2^{m-2}}$. Then the calculations above imply that the image of $([N_{F/E}(z)])$ in $A(F/E)_{\mathrm{I}}$ is equal to $[\alpha]^{2^{m-1}}$. Furthermore we have

$$v_{\mathfrak{P}}(z \cdot \bar{z}^{-1}) = v_{\mathfrak{P}}(q \cdot \bar{q}^{-1}) = -\tfrac{1}{2} v_{\mathfrak{P}}(w \cdot \bar{w}^{-1}) \equiv 0(2^{n-1})$$

and

$$\Big[\prod_{\mathfrak{p} \notin S} \mathfrak{p}^{(1/2^{n-1}) v_{\mathfrak{P}}(z \cdot \bar{z}^{-1})}\Big] = \Big[\prod_{\mathfrak{p} \notin S} \mathfrak{p}^{-(1/2^n) v_{\mathfrak{P}}(w \cdot \bar{w}^{-1})}\Big] = \Big[\prod_{\mathfrak{p} \notin S} \mathfrak{p}^{(1/2^{n-1}) v_{\mathfrak{P}}(\bar{w})}\Big],$$

hence $\psi_n([x]; z)$ vanishes in $A(E)/A(E)^2$, since we assumed $m \geqq n$. Thus $[N_{F/E}(z)]$ lies in $\Delta_n(E)$, which gives the desired contradiction. Thus indeed $n_i < n$ for all $i > k$.

REFERENCES

[1] BAK, A. and REHMANN, U., *K_2-analogs of Hasse's norm theorems,* Comm. Math. Helv. *59* (1984), 1–11.

[2] BIRCH, B., *K_2 of global fields,* Proc. Symp. Pure Math. *20* (1969), 87–95.

[3] BROWKIN, J. and SCHINZEL, A., *On Sylow 2-subgroups of $K_2\mathfrak{o}_F$ for quadratic number fields F,* J. reine angew. Math. *331* (1982), 104–113.

[4] BROWN, K. S., *Euler characteristics of discrete groups and G-spaces,* Inv. math. *27* (1974), 229–264.

[5] GRAS, G., *Remarks on K_2 of number fields,* preprint.

[6] HASSE, H., *Zahlbericht, Teil I,* Physica-Verlag, Würzburg Wien 1965.

[7] HURRELBRINK, J., *On the size of certain K-groups,* Comm. in Algebra *10* (1982), 1873–1889.

[8] HURRELBRINK, J. and KOLSTER, M., *On the 2-primary part of the Birch–Tate conjecture for cyclotomic fields,* to appear in Contemporary Math., Proc. AMS Boulder Conf. 1983.

[9] KOLSTER, M., *On the Birch–Tate conjecture for maximal real subfields of cyclotomic fields; Alg. K-theory, Number Theory, Geometry and Analysis,* Proc. Bielefeld 1982, Lect. Notes in Math. 1046, Sprigner 1984, 229–234.

[10] MAZUR. B. and WILES, A., *Class fields of abelian extension of* $\mathbb{Q}$, Inv. math. *76* (1984), 179–330.

[11] MILNOR, J., *Introduction to Algebraic K-Theory,* Ann. of Math. Studies *72*, Princeton University Press, Princeton 1971.

[12] SERRE, J.-P., *Corps Locaux,* Hermann, Paris 1968.
[13] TATE, J., *Symbols in arithmetic,* Actes du Congrès Int. des Mathématiciens 1970, Tome 1, Gauthier Villars, Paris 1971, 201–211.
[14] TATE, J., *Relations between K_2 and Galois cohomology,* Inv. math. *36* (1976), 257–274.
[15] URBANOWICZ, J., *On the 2-primary part of a conjecture of Birch and Tate,* Acta Arithmetica 43.1. (1983), 69–81.
[16] WASHINGTON, L. C., *Introduction to Cyclotomic Fields,* Springer, New York–Heidelberg–Berlin 1982.

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