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The influence of the boundary behaviour on hypersurfaces with constant mean curvature in H^{n+1}

M. P. do Carmo, J. de M. Gomes and G. Thorbergsson[1]

1. Introduction

This paper deals with complete, properly embedded hypersurfaces M^n with constant mean curvature H of the hyperbolic space H^{n+1}, and addresses itself to the following general question. How is the behaviour of such hypersurfaces influenced by their behaviour at infinity?

H^{n+1} has a natural compactification $\bar{H}^{n+1}$ and we will call $\partial_\infty M = \bar{M} \cap \partial \bar{H}^{n+1}$ the *asymptotic boundary* of M where $\bar{M}$ is the closure of M in $\bar{H}^{n+1}$. Some recent work ([dCL], [GRR], [Hs], [LR]) has shown the strong influence of $\partial_\infty M$ on M. To describe our contribution, we first observe that $\partial \bar{H}^{n+1}$ has a natural conformal structure where the conformal transformations are induced by the isometries of H^{n+1}. Thus it makes sense to talk about k-dimensional spheres S^k in $\partial \bar{H}^{n+1}$, $0 \leq k \leq n-1$.

In Section 2 we define a conformally invariant distance between two compact sets in $\partial \bar{H}^{n+1}$ and show (Theorem 1) that for $H \in [0, 1)$ there exists a real number d_H that is an upper bound for the distance between any connected component A of $\partial_\infty M$ and its complement $\partial_\infty M - A$ (supposed nonempty). Furthermore if the bound d_H is attained for some component A, M is a *rotation hypersurface of spherical type,* i.e., M is invariant by a group of isometries that leave a geodesic pointwise fixed. Since the distance is defined in such a way that the distance from a point to a compact set not containing it is unbounded, it follows that $\partial_\infty M$ contains no isolated points for $H \in [0, 1)$ (Corollary 1). The result in Corollary 1 is sharp, since for any $H \geq 1$ there exists examples of embedded hypersurfaces with constant mean curvature H whose asymptotic boundary consists of two points (see [Go] or [GRR]).

It has been noticed that some condition at infinity is necessary for some of the theorems in the quoted literature (see, e.g., the final remark in [dCL]). In Section

[1] The third author was supported by the exchange program of GMD (Germany) and CNPq (Brazil).

3 we present a definition of regularity at infinity that is a slight modification of the one given in [LR] for the minimal case and turns out to be very strong. With such a condition, we show that the fact that H belongs to one of the intervals $[0, 1)$, $(1, \infty)$, or $H = 1$ can be completely characterized in terms of the boundary behaviour of M (Theorem 2). In particular we show that if M is a properly embedded hypersurface regular at infinity with constant mean curvature $H > 1$, it must be compact.

In Section 4 we prove (Theorem 3) that if $\partial_\infty M$ consists of two disjoints $(n-1)$-spheres, M is regular at infinity and $H \neq 1$, then M is a rotation hypersurface of spherical type. This extends a result in [LR], where the theorem is proved for $H = 0$ (as usual in cases where M is minimal, no embeddness is assumed here), and is related to [GRR] where similar results were obtained for isometries of H^{n+1} that leave fixed one point in $\partial \bar{H}^{n+1}$ (parabolic isometries) or two points in $\partial \bar{H}^{n+1}$ (hyperbolic isometries). The idea of the proof of Theorem 3 can be used to give simpler proofs of some results in [dCL] and [LR]. We do not know whether Theorem 3 holds true for $H = 1$.

The method used in proving the above results is essentially Alexandrov maximum principle, that we will call *the tangency principle,* in the form given in Proposition 1.5 of [dCL]. In section 2 we make essential use of some facts from the classification of rotation hypersurfaces of spherical type. These facts were proved in Gomes' thesis at IMPA; we describe them and refer to [Go] for the proofs.

2. Non-existence of isolated points in the asymptotic boundary

$\bar{H}^{n+1}$ will denote the natural compactification of hyperbolic $(n+1)$-dimensional space, and $S^n(\infty)$ is the boundary $\partial \bar{H}^{n+1}$ of H^{n+1}. The *asymptotic boundary* $\partial_\infty A$ of a set $A \subset H^{n+1}$ is $\partial_\infty A = \bar{A} \cap S^n(\infty)$, where $\bar{A}$ is the closure of A in $\bar{H}^{n+1}$.

Let S_1 and S_2 be two disjoint codimension-one spheres in $S^n(\infty)$; we will denote by D_1 and D_2 the components of $S^n(\infty) - (S_1 \cup S_2)$ that are homeomorphic to disks. Given two (not necessarily connected) subsets A_1 and A_2 in $S^n(\infty)$ we say that S_1 and S_2 *separate* A_1 and A_2 if $A_1 \subset D_1$ and $A_2 \subset D_2$. The distance $d(S_1, S_2)$ will mean the hyperbolic distance of the two totally geodesic submanifolds H_1, H_2, where $\partial_\infty H_i = S_i$, $i = 1,2$.

Given two compact sets $A_1, A_2 \subset S^n(\infty)$, we define the *distance* $d(A_1, A_2)$ from A_1 to A_2 by

$$d(A_1, A_2) = \begin{cases} 0 \text{ if there does not exist spheres } S_1 \text{ and} \\ \quad S_2 \text{ that separate } A_1 \text{ and } A_2; \\ \sup \{d(S_1, S_2); S_1 \text{ and } S_2 \text{ separate } A_1 \text{ and } A_2\}. \end{cases}$$

Since conformal transformations of $S^n(\infty)$ are induced by hyperbolic isometries of H^{n+1}, $d(A_1, A_2)$ is conformally invariant. Notice for $n \geq 2$ that the distance of a compact set to a point away from this set is infinite; also if $d(A_1, A_2) < +\infty$, by compactness there exists S_{A_1}, and S_{A_2} such that $d(A_1, A_2) = d(H_{A_1}, H_{A_2})$, where $\partial_\infty H_{A_1} = S_{A_1}$, and $\partial_\infty H_{A_2} = S_{A_2}$. Although we called d a distance, we observe that the triangle inequality does not hold in general.

THEOREM 1. *Let $M^n \subset H^{n+1}$ be a complete connected, properly embedded hypersurface with constant mean curvature $H \in [0, 1)$. Assume that the asymptotic boundary $\partial_\infty M$ has at least two components and let A be any such component. Then there exists a constant d_H (depending only on H, and computable) such that*

$$d(A, \partial_\infty M - A) \leq d_H,$$

and equality holds if and only if M is a rotation hypersurface of spherical type.

Before proving the theorem, we will mention the following

COROLLARY 1. *Let $M^n \subset H^{n+1}$ be a complete connected, properly embedded hypersurface of H^{n+1} with constant mean curvature $H \in [0, 1)$. Then the asymptotic boundary of M has no isolated points.*

Proof of the corollary. If the asymptotic boundary reduces to one point, the result follows from the characterization of horospheres by do Carmo–Lawson [dCL]. Otherwise, there are at least two connected components in $\partial_\infty M$, and we can apply the theorem. Since the distance from a compact set to a point is infinite, the corollary follows.

Before starting the proof of the theorem, we need some facts from the classification of rotation hypersurfaces of spherical type in hyperbolic space, with constant mean curvature. These questions were treated by Wu-Yi Hsiang [Hs], and do Carmo–Dajczer [dCD], but the facts that we need here were proved in Gomes' thesis [Go].

We will use the half-space model of the hyperbolic space:

$$H^{n+1} = \{(x_1, \ldots, x_{n+1}) \in R^{n+1}; \qquad x_{n+1} > 0\},$$

with the metric $g_{ij} = \delta_{ij}/x_{n+1}^2$. In this model, certain rotation hypersurfaces M of spherical type with constant mean curvature H can be described as follows. Let γ be the axis of rotation, that we take to be perpendicular to the hyperplane $x_{n+1} = 0$ (see Fig. 1).

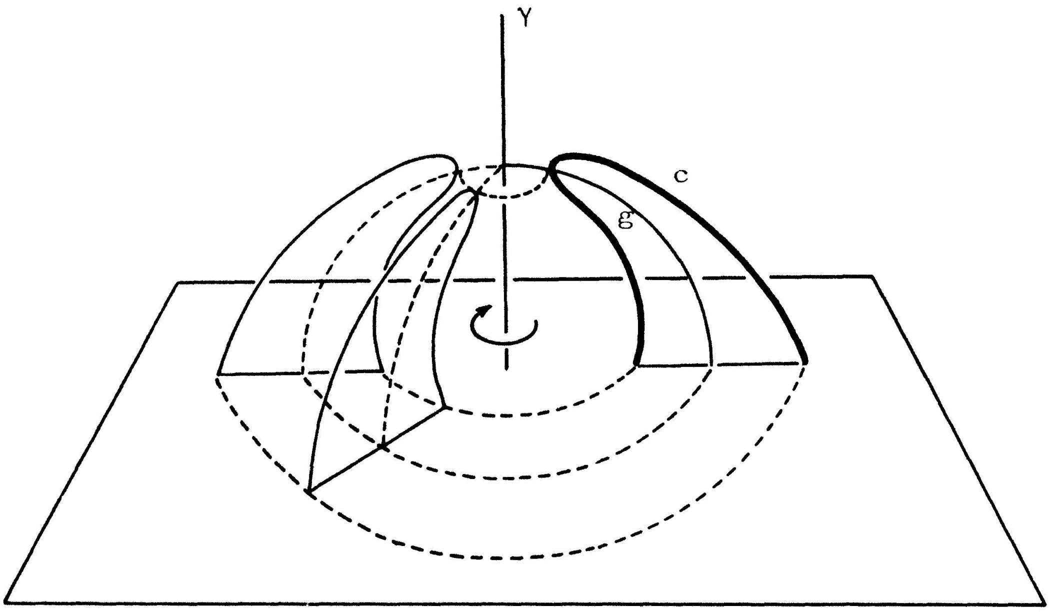

Figure 1

Let g be any geodesic perpendicular to γ and consider a totally geodesic plane σ containing γ and g. There exists, in the plane σ, a curve c that is symmetric relative to g (Fig. 1) and is such that M is obtained by rotating c about γ. The properties of M that have a bearing in our proof are as follows:

i) For each $H \in [0, 1)$, and each g perpendicular to γ, there exists a one-parameter family M_λ, $\lambda \in (0, \infty)$, of rotation hypersurfaces of spherical type with mean curvature H. The points where M_λ intersects the totally geodesic hypersurface generated by the rotation of g have a constant (hyperbolic) distance to γ, and this distance is the value of the parameter λ. Furthermore, the asymptotic boundary $\partial_\infty M_\lambda$ of each hypersurface in the family M_λ consists of two disjoint codimension-one spheres.

ii) Consider a hypersurface M_λ of the family defined in (i), let S_1 and S_2 be the two components of $\partial_\infty M_\lambda$ and set $d(\lambda) = d(S_1, S_2)$. The function of $d = d(\lambda)$ satisfies $d(0) = 0$, increases initially, reaches a maximum d_H, and decreases asymptotically to zero as $\lambda \to \infty$. The maximum value d_H *depends only on* H, and it is given in terms of an integral; thus d_H can be explicitly computed to any degree of accuracy. If $H > 0$, then $d_H > d_0$, and the mean curvature vector of M_λ points to the connected component of $H^{n+1} - M_\lambda$ that contains the axis of rotation γ.

Proof of the theorem. We may suppose that $\partial_\infty M$ is contained in the hyperplane $x_{n+1} = 0$. Consider a totally geodesic submanifold H_A, such that A is contained in the disk bounded by $\partial_\infty H_A = S_A$. Let γ be the geodesic in H^{n+1} represented in R^{n+1} as a half-line emanating from the center of the sphere S_A (see

Fig. 2). We will assume that

$$d(A, \partial_\infty M - A) > d_H,$$

and we will derive a contradiction.

Set $B = \partial_\infty M - A$, and let H_B be a totally geodesic submanifold orthogonal to γ and such that B is contained in the disk bounded by $\partial_\infty H_B = S_B$; H_B exists since $d(A, B) > d_H > 0$. Let p_A and p_B be the intersections of γ with H_A and H_B respectively, and let p be the (hyperbolic) middle point of the segment $\overline{p_A p_B}$ along γ. Let g be a geodesic orthogonal to γ at p, and consider the family M_λ described in (i). Then

$$d(p_A, p_B) > d_H \geq d_0,$$

where the last inequality comes from (ii).

It follows, for each $H \in [0, 1)$, that the family M_λ has the property that $\partial_\infty M_\lambda$ does not intersect either A or B. Furthermore, given $\varepsilon > 0$, there exists $\bar{\lambda} = \lambda(\varepsilon)$ with $\bar{\lambda} < \varepsilon$. Since λ is the distance from M_λ to the axis of rotation γ, and, for λ sufficiently large, $M_\lambda \cap M = \emptyset$, there exists λ_0 such that M_{λ_0} touches M for the first time, say at a point $q \in M$.

Since M is embedded, $H^{n+1} - M$ has two connected components. Let us denote by O the component that contains M_λ for large λ and by I the other one. We orient M in such a way that the mean curvature $H \geq 0$. We claim that either M is minimal, or the mean curvature vector of M points towards I. Otherwise,

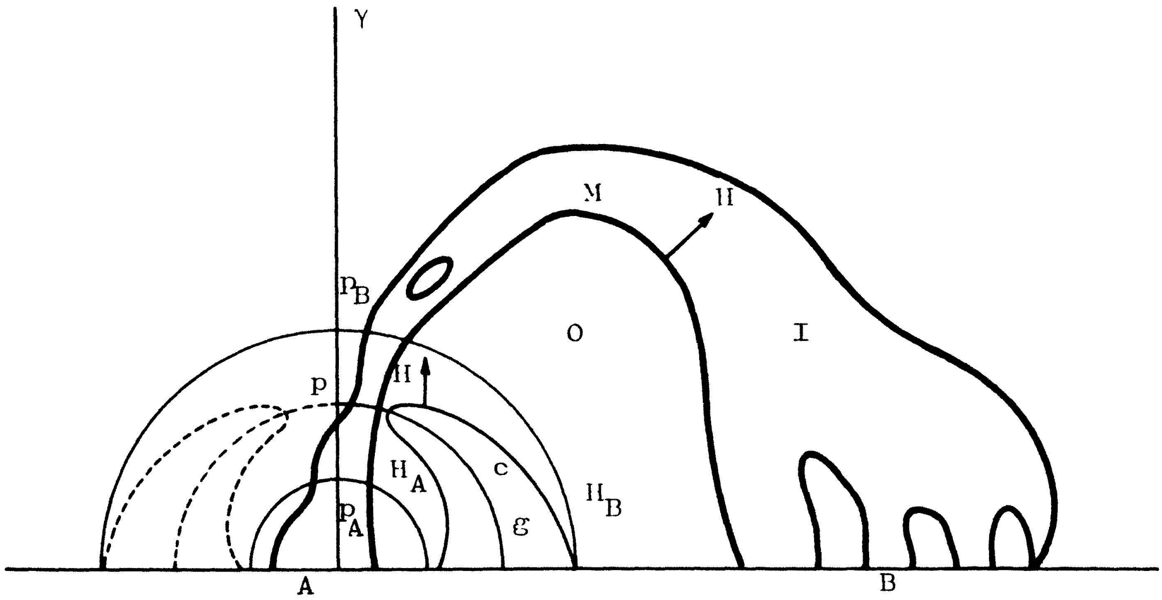

Figure 2

consider the minimal spherical hypersurface M_{λ_0} that is tangent to M. By looking at the normal sections at the tangency point, it is easy to see that the mean curvature of M_{λ_0} is greater than or equal to the mean curvature of M, and this proves the claim.

If M is minimal, an application of the tangency principle shows that M agrees with M_{λ_0}, a contradiction to the fact that $d(A, B) > d_H > d_0$. If M has constant mean curvature $H > 0$, the mean curvature vector points towards I, and a application of the tangency principle gives again a contradiction.

This shows that $d(A, B) \leq d_H$, and proves the first part of the theorem.

Now assume that $d(A, B) = d_H$, and choose S_A and S_B as in the above proof, so that $d(H_A, H_B) = d(A, B)$. Proceeding as in the proof, we obtain that $M = M_{\lambda_0}$. This proves the second part, and completes the proof of the theorem.

3. Boundary regularity

Until further notice, we will use the unit ball model for the hyperbolic space H^{n+1}. $\bar{H}^{n+1}$ will denote the closed unit ball and $S^n(\infty)$ will denote the unit sphere. All topological notions used here will refer to the topology of the closed unit ball.

Given an embedded hypersurface $M \subset H^{n+1}$, we will say that M is *C^k-regular at infinity*, $k \geq 1$, (or simply C^k-regular) if $\bar{M} \subset \bar{H}^{n+1}$ is a C^k-submanifold with boundary of $\bar{H}^{n+1}$, and $\partial_\infty M$ is a C^k-submanifold of $S^n(\infty)$; in particular, if M is C^1-regular, it has a well defined tangent space at each point in $\bar{M} \cap S^n(\infty)$ which is the limit of tangent spaces of M (this definition is a slight modification of the one given in [LR]).

It is sometimes convenient to localize the definition of regularity at infinity and say that M is *C^k-regular at a point $p \in \bar{M} \cap S^n(\infty)$* if there exists an open neighbourhood U of p in $\bar{H}^{n+1}$ such that $\bar{M} \cap U$ is a C^k-submanifold with boundary of $\bar{H}^{n+1}$, and $\partial_\infty M \cap U$ is a C^k-submanifold of $S^n(\infty)$.

A general discussion about regularity at infinity, with many examples is given in Chapter IV of [Go].

When M is C^k-regular, it will be convenient to consider the boundary $\partial\bar{M}$ of the submanifold $\bar{M}$. It is easily checked that $\partial\bar{M} \subset \partial_\infty M$, but equality may fail to occur. For instance, if M is a horosphere, $\partial\bar{M} = \emptyset$ and $\partial_\infty M = \{\text{one point}\}$. Also note that proper embeddedness together with the additional hypothesis that $\partial_\infty M = \emptyset$ means that M is compact.

If M is an embedded hypersurface in H^{n+1} with constant mean curvature $H \neq 0$, we will orient M in such a way that $H > 0$.

THEOREM 2. *Let $M^n \subset H^{n+1}$ be a connected, complete, properly embedded* *hypersurface with constant mean curvature H. Assume that M is C^2-regular at infinity. Then*

i) $H > 1 \Leftrightarrow \partial_\infty M = \emptyset$;
if, in addition, $\partial_\infty M \neq \emptyset$,
ii) $H < 1 \Leftrightarrow \partial_\infty M = \partial \bar{M}$. *In this case, M is nowhere tangent to* $S^n(\infty)$.
iii) $H = 1 \Leftrightarrow \partial \bar{M} = \emptyset$.

Proof. M divides H^{n+1} into two components denoted by I and O. Assume that the normal vector points towards I. We will first prove some assertions that will imply the theorem.

ASSERTION 1. Let $\partial_\infty M \neq \emptyset$ and $\partial \bar{M} \neq \emptyset$. Then $H < 1$.

Since $\partial \bar{M} \neq \emptyset$, both $\partial_\infty I$ and $\partial_\infty O$ have interior points. Let $p \in \operatorname{Int}(\partial_\infty I)$ and let H_t be the family of horospheres with p as asymptotic boundary; the parameter t is chosen in such a way that a geodesic $\gamma(t)$ with $\gamma(\infty) = p$ satisfies $\gamma(t) \in H_t$. Since $p \in \operatorname{Int}(\partial_\infty I)$, the intersection $H_t \cap M = \emptyset$ for large t. Therefore, there exists t_0 such that H_{t_0} is tangent to M and $H_{t_0} \subset \bar{I}$. Since the normal vector of M points towards I, we see that $H \leq 1$. If $H = 1$, by the tangency principle, $H_{t_0} = M$, and this contradicts the fact that $\partial \bar{M} \neq \emptyset$. Thus $H < 1$, and this proves Assertion 1.

ASSERTION 2. Let $\partial_\infty M \neq \emptyset$ and $H < 1$. Then $\bar{M}$ is nowhere tangent to $S^n(\infty)$.

Assume the contrary, i.e., there exists $p \in \partial_\infty M$ where $\bar{M}$ is tangent to $S^n(\infty)$. By C^2-regularity, there exists a codimension-one sphere $\Sigma_0 = \partial B$ in $S^n(\infty)$ that is tangent to $\partial_\infty M$ at p, and is such that $\operatorname{Int} B \cap \bar{M} = \emptyset$. Foliate B by codimension-one spheres Σ_t, $0 \leq t \leq 1$, and consider one of the two continuous families of hyperspheres h_t that satisfy $\partial_\infty h_t = \Sigma_t$ and have mean curvature H. Clearly, for some of the two possible choices of Σ_0 and some $0 < t_0 \leq 1$, the hypersphere h_{t_0} is tangent to M. We can assume that the normal vector of h_{t_0} and M agree at the tangency point; if this is not the case, we just have to choose the other family of hyperspheres with the same mean curvature and same asymptotic boundaries. By the tangency principle, $h_{t_0} = M$ and this contradicts the fact that $\partial_\infty h_{t_0} \cap \partial_\infty M = \emptyset$. This proves Assertion 2.

ASSERTION 3. Let $\partial_\infty M \neq \emptyset$ and $\partial \bar{M} = \emptyset$. Then $H \leq 1$.

Choose $p \in \partial_\infty M$. Since $\bar{M}$ is a submanifold with boundary, and $\partial \bar{M} = \emptyset$, $\bar{M}$ is tangent to $S^n(\infty)$ at p. By C^2-regularity, there exists a codimension-one *euclidean* sphere Σ_p in $\bar{H}^{n+1}$ such that Σ_p is tangent to M and $\Sigma_p \subset \bar{I}$. By decreasing the euclidean radius of Σ_p, if necessary, we can find a continuous family of euclidean spheres Σ_q, $q \in \bar{M} \cap V$, where V is a sufficiently small neighbourhood of p. From the view point of hyperbolic geometry, Σ_p is a horosphere and if $p \neq q$, Σ_q is a hyperbolic sphere tangent to M, with the same normal vector as M. Let $\varphi(q) = H_q - H$, where H_q is the mean curvature of the hyperbolic sphere Σ_q. By looking at the normal sections we see that at q, $\varphi(g) \geq 0$. Since Σ_1 is a horosphere, $\varphi(p) = 1 - H$, and by continuity $1 - H \geq 0$. This proves Assertion 3.

Now we come to the proof of the Theorem itself.

Assertions (1) and (3) imply that if $\partial_\infty M \neq \emptyset$ then $H \leq 1$. This proves (i)$\Rightarrow$. The converse comes from the tangency principle, and this completes the proof of (i).

We now prove (ii). Assume that $H < 1$ and that there exists $p \in \partial_\infty M$ with $p \notin \partial\bar{M}$. Then $\bar{M}$ is tangent to $S^n(\infty)$ at p and this contradicts Assertion 2. Thus $H < 1 \Rightarrow \partial_\infty M = \partial\bar{M}$. The converse follows from Assertion 1, and the last statement of (ii) follows from Assertion 2.

Finally we prove (iii). If $\partial_\infty M \neq \emptyset$ and $\partial\bar{M} \neq \emptyset$, then by Assertion 1, $H < 1$. Thus if $H \geq 1$ either $\partial_\infty M = \emptyset$ or $\partial\bar{M} = \emptyset$. But $\partial_\infty M = \emptyset$ is equivalent to $H > 1$ by (i). Thus $H = 1 \Rightarrow \partial_\infty\bar{M} = \emptyset$. Conversely, if $\partial\bar{M} = \emptyset$ and $\partial_\infty M \neq \emptyset$, then $\partial\bar{M} \neq \partial_\infty M$. Thus by (ii), $H \geq 1$ and, by (i), $H \leq 1$, hence $H = 1$. This completes the proof of the Theorem.

Remark 1. In [dCL] it is proved that if $M^n \subset H^{n+1}$ is a complete properly embedded hypersurface with constant mean curvature, $\partial_\infty M = S^{n-1}$, and M separates poles, then M is a hypersphere with S^{n-1} as asymptotic boundary. The condition that M separates poles in the proof of [dCL] is equivalent to the condition that both $\partial_\infty I$ and $\partial_\infty O$ have interior points, which by its turn is equivalent to both $\partial_\infty M \neq \emptyset$ and $\partial\bar{M} \neq \emptyset$. Thus in [dCL] we can, by Theorem 2, replace the condition "M separates poles" by the stronger condition that "M is C^2-regular at infinity".

Remark 2. It is easily checked that the implications $\Leftarrow$ of Theorem 2 hold if M is merely locally C^2-regular. Also if we localize the right hand sides of the implications in Theorem 2, the theorem holds true for local C^2-regularity.

Notice that we did not use the regularity hypothesis in the proof of Assertion 1. In fact that proof shows the following "dual" of Corollary 1.

COROLLARY 2. *Let M be a properly embedded hypersurface in H^{n+1} with constant mean curvature $H > 1$. Then the asymptotic boundary of M does not contain any component whose codimension in $S^n(\infty)$ is one.*

4. A characterization of embedded rotation hypersurface of spherical type

In this section we will prove the following

THEOREM 3. *Let $M^n \subset H^{n+1}$ be a connected complete properly embedded hypersurface in H^{n+1} with constant mean curvature $H \neq 1$. Assume that M is C^2-regular at infinity and that $\partial_\infty M$ is the union of two disjoint codimension-one spheres of $S^n(\infty)$. Then M is a rotation hypersurface of spherical type.*

Remark 3. For the case $H = 0$ and M not necessarily embedded, Theorem 3 was proved by Levitt and Rosenbert [LR]. Our proof is essentially the one in Alexandroff [Al]; see also Hopf [Ho].

We will need the following elementary lemma, the proof of which will be sketched for the sake of completeness.

LEMMA 1. *For any two codimension-one spheres S_1, and S_2 in the unit sphere $S^n \subset R^{n+1}$, such that $S_1 \cap S_2 = \varnothing$, there exists a conformal transformation of S^n that brings S_1 and S_2 into spheres of equal radii which lie in parallel hyperplanes of R^{n+1}.*

Proof. Let B be a ball bounded by S_1, that contains S_2. The ball B has a hyperbolic metric with the property that the isometries of such a metric are the conformal transformations of S^n that leave B and S_1 invariant. The family $S(t)$ of codimension-one spheres in $B \subset S^n \subset R^{n+1}$, parallel (i.e., in parallel hyperplanes) to S_1, is a family of hyperbolic spheres whose hyperbolic radii $R(t)$ varies in the interval $[0, \infty)$. Thus there exists $t_0 \in [0, \infty)$ such that the hyperbolic radius of $S(t_0)$ and S_2 are the same. Therefore there exists a conformal transformation that leaves S_1 invariant and brings S_2 into $S(t_0)$. It follows that we can assume that S_1 and S_2 are in parallel hyperplanes. By using the conformal transformations of S^n induced by the similarity transformations of R^n via a stereographic projection (say, from the center of S_1), the proof of the lemma is easily completed.

Proof of Theorem 3. We will use the ball model S^{n+1} for hyperbolic $(n+1)$-space. First we introduce the notation to be used in the proof. We can assume by Lemma 1 that the two spheres in $\partial_\infty M$ lie in parallel hyperplanes and have the same radius. They are therefore symmetric to each other relative to an equator of $S^n(\infty)$ which we denote by E (see Fig. 3 below).

Let $\gamma : (-\infty, +\infty) \to H^{n+1}$ be a geodesic of H^{n+1} whose image is a diameter of E, and such that $\gamma(0)$ is the center of E. Consider the family P_t, $-\infty < t < +\infty$, of

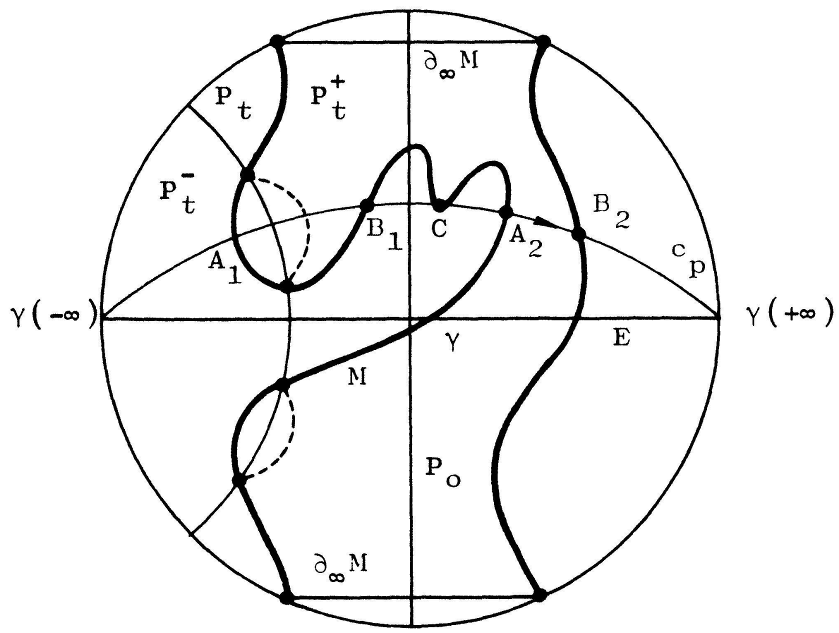

Figure 3

totally geodesic hypersurfaces orthogonal to γ at $\gamma(t)$. Notice that $\partial_\infty P_t$ is the family of parallel codimension-one spheres which approaches $\gamma(+\infty)$ or $\gamma(-\infty)$ as t tends to $+\infty$ and $-\infty$, respectively, and that $\partial_\infty \bar{P}_0$ is the great sphere in $S^n(\infty)$ orthogonal to E. $\bar{P}_t$ divides H^{n+1} into two components: the one that contains $\gamma(-\infty)$ is denoted by $\bar{P}_t^-$, and the other by $\bar{P}_t^+$. Set $\bar{M}_t = \bar{M} \cap \bar{P}_t^-$.

Consider the reflection $R_t: H^{n+1} \to H^{n+1}$ across P_t. We want to show that R_t extends continuously to a map $\bar{R}_t$ of $\bar{H}^{n+1}$ and we proceed as follows (cf. [Al] and also [LR]). For each $p \in H^{n+1}$, as t runs the interval $(-\infty, +\infty)$, $R_t p$ will describe the hypercycle passing through p and joining $\gamma(-\infty)$ to $\gamma(+\infty)$. Let c_p be such a hypercycle and parametrize it in such a way that $c_p(t) \in P_t$, $t \in (-\infty, +\infty)$ (see Fig. 3). As p approaches a point $q \in S^n(\infty)$, $q \neq \gamma(+\infty)$, $q \neq \gamma(-\infty)$, c_p approaches the (unique) arc of circle c_q of $S^n(\infty)$ that passes through q and joins $\gamma(-\infty)$ to $\gamma(+\infty)$. For convenience, we will say that c_q is the hypercycle passing through q, although this notion depends on the choice of γ (which will be kept fixed throughout the proof). We now define $\bar{R}_t$ in an obvious continuous way and call it the *reflection* across $\bar{P}_t$.

Let $\bar{M}_t'$ be the reflection of $\bar{M}_t$ accross $\bar{P}_t$, and denote by I the component of $\bar{H}^{n+1} - \bar{M}$ that contains the north and south poles relative to E. Notice that $S^n(\infty) - \partial_\infty M$ has three connected components; denote by I_N (resp. I_S) the component that contains the north (resp. south) pole relative to E. Since $H \neq 1$ and M is C^2-regular, it follows from (ii) of Theorem 2 that $\partial_\infty I = I_N \cup I_S$. Hence the reflection of $\partial_\infty M$ across $\bar{P}_t \cap S^n(\infty)$ is contained in I if $t \leq 0$, and it is not contained in I if $t > 0$.

We now start the proof. Set

$$t_1 = \inf \{t; \bar{M}'_t \text{ is not contained in } \bar{I}\}.$$

By the behaviour of the reflections of $\partial_\infty M$, it is clear that $t_1 \leq 0$. Furthermore, $\bar{M}'_{t_1} \subset \bar{I}$, since it is the limit of such sets. The crucial point of the proof is to show that $t_1 = 0$.

Assume that $t_1 < 0$. Choose an orientation for M, and define a function $\varphi : \bar{M} - \{\gamma(-\infty), \gamma(+\infty)\} \to R$ by $\varphi(p) = \langle N(p), c'_p \rangle$, where $N(p)$ is the normal vector of M at p (notice that M has normal vectors at infinity although it makes no sense to talk about their lengths). Set $A = \varphi^{-1}(-\infty, 0)$, $B = \varphi^{-1}(0, +\infty)$, and $C = \varphi^{-1}(0)$. Of course, c_p is tangent to M if $p \in C$, and by changing the orientation of M, if necessary, we can assume that the normal to M points towards I. We say that c_p *enters* M in the points of A, and *leaves* M in the points of the set B. In figure 3, the points A_1 and A_2 belong to A, the point C_1 belongs to C, and the points B_1 and B_2 belong to the set B.

We claim that $\bar{M}_{t_1} \subset A$. Since $\bar{M}_t = \emptyset$ for t near $-\infty$, if we set

$$t_0 = \sup \{x; \bar{M}_t = \emptyset\},$$

then $\bar{P}_{t_0} \cap \bar{M} \subset A$. If there exists $p \in \bar{M}_{t_1} \cap B$, by continuity of φ, we can find $t_2 \leq t_1$ and $q \in \bar{M}_{t_2} \cap C$. Thus $q \in \bar{P}_t$, $t_2 \leq t_1$ and $\bar{M}'_t \subset \bar{I}$. Since M is C^2-regular, and $H \neq 1$, it follows from Theorem 2 that M is nowhere tangent to $S^n(\infty)$, hence $q \in M$. Furthermore since $q \in C$, P_t intersects M orthogonally at q, which shows that $\bar{M}'_t$ is, around q, a manifold with boundary that is tangent to $\bar{M}_t$ at q. Since $\bar{M}'_t \subset \bar{I}$, we can apply the boundary tangency principle to conclude that $\partial_\infty \bar{M}' = \partial_\infty M$ and this contradicts the boundary behaviour of the reflection for $t_1 < 0$. It follows that there are no points of $\bar{M}_{t_1}$ either in C or in B, and our claim is proved.

Since $\bar{M}_{t_1} \subset A$, $\bar{M}_{t_1}$ is compact and A is open, we conclude that $\bar{M}_{t_1+1/n} \subset A$ for $n > n_0$, n_0 sufficiently large. Since $\bar{M}'_{t_1} \subset \bar{I}$, a hypercycle c_p that enters $p \in \bar{M}_{t_1}$ at a time t is still in $\bar{I}$ at time $t + 2(t_1 - t)$. Since $M'_{t_1+1/n} \not\subset \bar{I}$, there exists a hypercycle c_p, $p \in \bar{M}_{t_1+1/n}$, and a real number δ, $0 < \delta < 1/n < 1/n_0$, such that $c_p(t + 2(t_1 + \delta - t)) \in B$. By letting n_0 approach infinity, we see that there exists c_q, $q \in \bar{M}_{t_1}$, such that $c_q(t + 2(t_1 - t)) = r \in B$.

Again, since $\bar{M}'_{t_1} \subset \bar{I}$, this implies that $\bar{M}'_{t_1}$ is tangent to $\bar{M}$ at r and on the same side of M. From the fact that $t_1 < 0$ and the boundary behaviour of the reflections, we see that $q \notin \partial_\infty M$. Thus $r \in M$ and we can apply the tangency principle to obtain the same contradiction as before. Therefore $t_1 = 0$, as we wished.

Let O be the closure of $\bar{H}^{n+1} - \bar{I}$. Since $t_1 = 0$, $\bar{M}'_0 \subset \bar{I}$. Thus the reflection of

$\bar{M} \cap \bar{P}_0^-$ across $\bar{P}_0$ is contained in O. On the other hand, by using the above construction for the geodesic $-\gamma$, we see that the reflection of $\bar{M} \cap \bar{P}_0^+$ across $\bar{P}_0$ is contained in I. It follows that M is symmetric about $\bar{P}_0$.

We now repeat the above argument for any geodesic whose image is a diameter of the equator E. Thus M is symmetric about all hyperplanes containing the geodesic g that joins the north and south poles relative to the equator E. Such symmetries generate the group $0(n)$ of isometries of H^{n+1} that leave g fixed. It follows that M is a rotation hypersurface of spherical type, and this completes the proof of the Theorem.

Remark 4. Once one knows that M is a rotation hypersurface of spherical type, it follows from the classification theorem of such hypersurfaces (see Gomes' thesis at IMPA [Go]) that both $M \cap P_0^-$, and $M \cap P_0^+$ are graphs over P_0. It also follows from Theorem 2 that the mean curvature H of M belongs to the interval $[0, 1)$.

Wu-Yi Hsiang has shown in [Hs] that if M is an embedded hypersurface with constant mean curvature and at a finite distance from a totally geodesic $(k+1)$-submanifold, then it is $0(n-k)$-invariant. It is clear that such a condition (finiteness of distance) implies that $\partial_\infty M$ is contained in a k-dimensional sphere (the converse is not true). The ideas of the proof of Theorem 2 above apply and yield a proof of the following result (Cf. [LR]).

COROLLARY 3, (of the proof). *Let M be a complete nonumbilic properly embedded hypersurface in H^{n+1} with constant mean curvature. Assume that $\partial_\infty M$ is contained in a k-dimensional sphere $S^k \subset S^n(\infty)$. Then M is an $0(n-k)$-rotational hypersurface whose "axis" is the totally geodesic $(k+1)$-submanifold with S^k as asymptotic boundary.*

Notice that there is no need of regularity at infinity in the above corollary. This reflects the fact that in the proof we only reach the asymptotic boundary of M at $t=0$.

We want to mention two special cases of Corollary 3 that were proved in [LR]. If $\partial_\infty M$ is contained in a codimension-one sphere, then M has a Z_2-symmetry. If $\partial_\infty M$ consists of two points, it is a rotational hypersurface of spherical type.

Corollary 3 can also be used to give a proof to the fact that if $\partial_\infty M$ reduces to one point p, then M is a horosphere with p as asymptotic boundary (cf. [dCL]). In fact, from the last result in the previous paragraph, M is symmetric relative to any geodesic γ in H^{n+1} with $\gamma(\infty)=p$. Thus if $q \in M \cap \gamma$, q is an umbilic point of M. Since γ is arbitrary, M is an umbilic hypersurface with $\partial_\infty M = \{p\}$, hence a horosphere.

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