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On the homology of the special linear group over a number field

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Introduction

Let F be a number field and $SL(F)$ the infinite special linear group over F. The homology groups $H_i(SL(F); \mathbb{Z})$ are in general not finitely generated. The purpose of the present paper is to study the structure of these homology groups. Our main result is:

THEOREM. *For all $i \geq 0$ $H_i(SL(F); \mathbb{Z})$ is the direct sum of a torsion group and a free abelian group of finite rank.*

The first two sections establish the proof of this theorem. In Section 1 we show that the Postnikov k-invariants of a connected infinite loop space are cohomology classes of finite order. In Section 2 we describe the space $BSL(F)^+$ which is an infinite loop space with the same homology as the group $SL(F)$ and apply the results of Section 1 in order to obtain the main theorem.

We then consider in Section 3 representations $\rho: G \to GL(F)$ of discrete groups G over a number field $F \subset \mathbb{C}$ and are interested in the order of their Chern classes $c_i(\rho) \in H^{2i}(G; \mathbb{Z})$, $i \geq 1$. We deduce from the above theorem and [5] that these Chern classes are torsion classes and we get an upper bound for their order. More precisely there exist integers $\bar{E}_F(i)$ depending on F and i (cf. [4]) such that:

THEOREM. *For any discrete group G and any representation $\rho: G \to GL(F)$ one has: $\bar{E}_F(i)c_i(\rho) = 0$ for $i \geq 1$.*

In Section 4 we finally apply this statement to the Chern classes of rational representations of discrete groups ($F = \mathbb{Q}$).

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1. Postnikov invariants of infinite loop spaces

In order to get our main result (§2) we need some information on the Postnikov k-invariants of infinite loop spaces. All spaces we consider in this section are CW-complexes, not necessarily of finite type, and we use the following notation: let X be a connected CW-complex; $X \xrightarrow{\alpha_n} X[n]$ denotes an $(n+1)$-anticonnected extension of X (Postnikov approximation of X): i.e., $X[n]$ is a CW-complex, obtained from X by adjoining cells of dimension $\geqslant n+2$, with $\Pi_i X[n]=0$ for all $i>n$ and $(\alpha_n)_*: \Pi_i X \xrightarrow{\cong} \Pi_i X[n]$ for all $i \leqslant n$. Let $X(n)$ denote the fibre of α_n; $X(n)$ is an n-connected CW-complex and $X(n) \to X$ induces an isomorphism $\Pi_i X(n) \xrightarrow{\cong} \Pi_i X$ for all $i>n$. The spaces $X[n]$ and $X(n)$ are uniquely determined by X up to homotopy (cf. [9], p. 418). We use the same notation for CW-spectra.

The k-invariants $k^{n+1}(X)$ are maps $X[n-1] \to K(\Pi_n X, n+1)$ and therefore cohomology classes in $H^{n+1}(X[n-1]; \Pi_n X)$, $n \geqslant 1$.

Our first objective is the following theorem: the k-invariants of a connected infinite loop space are cohomology classes of finite order (cf. Theorem 3).

For the proof of this assertion we need some results on the ordinary integral homology of spectra. Let $\underline{K}(G, n)$ denote the spectrum whose homotopy groups vanish in all dimensions except in dimension n where its homotopy group is G.

LEMMA 1. *Let G be any abelian group. Then for each $i>0$ the homology group $H_i(\underline{K}(G,0); \mathbb{Z})$ is a torsion group of finite exponent. More precisely, if M_i denotes the product of all prime numbers $p \leqslant i/2+1$, then $M_i H_i(\underline{K}(G,0); \mathbb{Z})=0$ for $i>0$.*

Proof. This is a direct consequence of [3], Thm. 2.

LEMMA 2. *Let $\underline{X}$ be a connective CW-spectrum (i.e., $\Pi_i \underline{X}=0$ for $i<0$) and let m be an integer $\geqslant 0$ such that $\Pi_i \underline{X}=0$ for $i>m$. Then for each $i>m$ the group $H_i(\underline{X}; \mathbb{Z})$ has finite exponent.*

Proof. Let i be an integer $>m$. We first look at the spectrum $\underline{X}(m-1)=\underline{K}(\Pi_m \underline{X}, m)$: by Lemma 1, $H_i(\underline{X}(m-1); \mathbb{Z})=H_{i-m}(\underline{K}(\Pi_m \underline{X}, 0); \mathbb{Z})$ has finite exponent. Now we use the following induction argument: for $k<m$ the long exact homology sequence associated to the cofibration $\underline{X}(k) \to \underline{X}(k-1) \to \underline{K}(\Pi_k \underline{X}, k)$ implies that, if $H_i(\underline{X}(k); \mathbb{Z})$ has finite exponent, then $H_i(\underline{X}(k-1); \mathbb{Z})$ has also finite exponent; the induction starts with $k=m-1$ and stops with $k=0$ $(\underline{X}(-1)=\underline{X})$.

Remark. Let $\underset{\sim}{X}$ be as in Lemma 2. It follows from our induction argument that, for $i > m$, the positive integer $N_{i,m} := \prod_{k=0}^{m} M_{i-k}$ is an upper bound for the exponent of $H_i(\underset{\sim}{X}; \mathbb{Z}) : N_{i,m} H_i(\underset{\sim}{X}; \mathbb{Z}) = 0$.

We are now able to prove the main result of this section.

DEFINITION. $S_{n+1} := \prod_{i=2}^{n+1} M_i$ (for $n \geq 1$), where M_i is defined in Lemma 1. Note that a prime number p divides the positive integer S_{n+1} if and only if $p \leq (n+3)/2$.

THEOREM 3. *For any connected infinite loop space X and for each $n \geq 1$ the k-invariant $k^{n+1}(X)$ satisfies:*

$$S_{n+1} k^{n+1}(X) = 0.$$

Remarks. 1) The integers S_{n+1} do not depend on X.

2) The statement of this theorem is valid without any finiteness condition on X.

Proof. Let X be a connected infinite loop space and let $\underset{\sim}{X}$ denote the associated (connective) spectrum. For $n \geq 1$ the k-invariant $k^{n+1}(X) \in H^{n+1}(X[n-1]; \Pi_n X)$ is the image of an element of $H^{n+1}(\underset{\sim}{X}[n-1]; \Pi_n \underset{\sim}{X})$ under the cohomology suspension (cf. [9], p. 438). Since $\underset{\sim}{X}[n-1]$ is a spectrum such that $\Pi_i \underset{\sim}{X}[n-1] = 0$ for $i > n-1$ it follows from Lemma 2 and the universal coefficient theorem that the group $H^{n+1}(\underset{\sim}{X}[n-1]; \Pi_n \underset{\sim}{X})$ has finite exponent which is bounded by $lcm(N_{n+1,n-1}, N_{n,n-1}) = lcm(M_2 M_3 \cdots M_{n+1}, M_1 M_2 \cdots M_n) = S_{n+1}$. Therefore $k^{n+1}(X)$ is a cohomology class of finite order: $S_{n+1} k^{n+1}(X) = 0$.

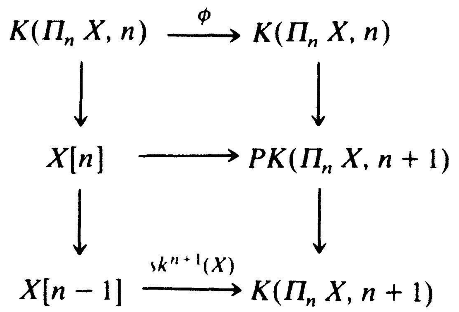
LEMMA 4. *Let X be a connected, simple CW-complex and let the k-invariant $k^{n+1}(X)$ be a cohomology class of finite order s in $H^{n+1}(X[n-1]; \Pi_n X)$, then there exists a map $f: X \to K(\Pi_n X, n)$ such that the induced homomorphism $f_*: \Pi_n X \to \Pi_n X$ is multiplication by s.*

Proof. The identity map id: $K(\Pi_n X, n+1) \to K(\Pi_n X, n+1)$ is a cohomology class in $H^{n+1}(K(\Pi_n X, n+1); \Pi_n X)$; we can therefore consider the map $s \cdot id$: $K(\Pi_n X, n+1) \to K(\Pi_n X, n+1)$. The composition

$$X[n-1] \xrightarrow{k^{n+1}(X)} K(\Pi_n X, n+1) \xrightarrow{s \cdot id} K(\Pi_n X, n+1)$$

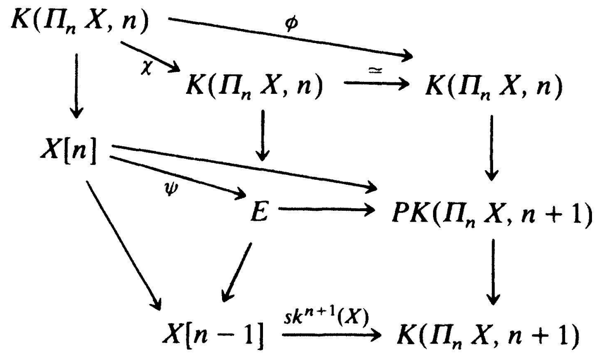
is actually $[k^{n+1}(X)]^*(s \cdot id) = s[k^{n+1}(X)]^*(id) = sk^{n+1}(X)$, where $[k^{n+1}(X)]^*$: $H^{n+1}(K(\Pi_n X, n+1); \Pi_n X) \to H^{n+1}(X[n-1]; \Pi_n X)$ is the homomorphism induced by $k^{n+1}(X)$.

We have the following map of fibrations:



where the second fibration is the path-fibration over $K(\Pi_n X, n+1)$. The map ϕ induces a multiplication by s on $\Pi_n X$.

Let now $K(\Pi_n X, n) \to E \to X[n-1]$ denote the fibration which is induced from the path-fibration by $sk^{n+1}(X)$. Because E is a pull-back we get a map $\psi: X[n] \to E$ and the following commutative diagram:



It follows that the induced homomorphism $\chi_*: \Pi_n X \to \Pi_n X$, and therefore also $\psi_*: \Pi_n X[n] \to \Pi_n E \cong \Pi_n X$, is multiplication by s.

Because $sk^{n+1}(X) = 0$ we obtain a homotopy equivalence $E \simeq X[n-1] \times K(\Pi_n X, n)$. We then can define a map

$$f: X \xrightarrow{\alpha_n} X[n] \xrightarrow{\psi} E \simeq X[n-1] \times K(\Pi_n X, n) \xrightarrow{\pi} K(\Pi_n X, n),$$

where π is the projection on the second factor, and $f: X \to K(\Pi_n X, n)$ induces a multiplication by s: $f_*: \Pi_n X \to \Pi_n X$.

COROLLARY 5. *Let X be a connected infinite loop space. Then there exists a* *map*

$$f: X \to \prod_{n=1}^{\infty} K\left(\Pi_n X, n\right)$$

such that, for each $n \geqslant 1$, *the induced homomorphism* $f_*: \Pi_n X \to \Pi_n X$ *is multiplication by some positive integer dividing* S_{n+1}.

Proof. This assertion follows from Theorem 3 and Lemma 4.

COROLLARY 6. *Let* X *be a connected infinite loop space such that, for each* $i \geqslant 1$, $\Pi_i X$ *is the direct sum of a torsion group and a free abelian group of finite rank. Then, for each* $i \geqslant 0$, $H_i(X; \mathbb{Z})$ *is a group of the same form.*

Proof. Let C denote the Serre class of all torsion abelian groups. If we compose the natural map

$$g: \prod_{n=1}^{\infty} K(\Pi_n X, n) \to Y := \prod_{n=1}^{\infty} K(\Pi_n X/\text{torsion}, n)$$

with the map f of Corollary 5 we get a map $h = g \circ f: X \to Y$, which induces a C-isomorphism on homotopy groups and therefore also a C-isomorphism on homology groups: $h_*: H_i(X; \mathbb{Z}) \to H_i(Y; \mathbb{Z})$, $i \geqslant 0$. For each $i \geqslant 0$ the group $H_i(Y; \mathbb{Z})$ is finitely generated; consequently $H_i(X; \mathbb{Z})/\ker h_*$ is also finitely generated and $\ker h_* \in C$. It follows that $H_i(X; \mathbb{Z})$ modulo its torsion subgroup is a free abelian group of finite rank.

2. The main theorem

Let F be a number field, $SL(F) = \varinjlim SL_n(F)$ its infinite special linear group and $BSL(F)^+$ the space we obtain by performing the plus construction on its classifying space $BSL(F)$. The space $BSL(F)^+$ is a simply connected CW-complex and its singular homology groups are isomorphic to the homology groups of the group $SL(F): H_i(SL(F); \mathbb{Z}) \cong H_i(BSL(F); \mathbb{Z}) \cong H_i(BSL(F)^+; \mathbb{Z})$, $i \geqslant 0$.

Because $BSL(F)^+$ is the universal cover of $BGL(F)^+$ ($GL(F)$ denotes the infinite general linear group over F), we have for $i \geqslant 2$: $\Pi_i BSL(F)^+ \cong K_i F$. Let now A be the ring of algebraic integers in F. We consider the localization exact sequence (cf. [7], Thm. 8):

$$\cdots \to K_{i+1}A \to K_{i+1}F \to \bigoplus_m K_iA/m \to K_iA \to \cdots$$

where m runs over the set of maximal ideals of A. For every maximal ideal m, A/m is a finite field and therefore:

$$K_iA/m = \begin{cases} 0, & \text{if } i \text{ is even,} \\ \text{a finite cyclic group,} & \text{if } i \text{ is odd.} \end{cases}$$

We get the following exact sequences:

$$\cdots \to K_iA \to K_iF \to 0, \qquad \text{for } i \text{ odd,} \qquad i \geq 3,$$

$$0 \to K_iA \to K_iF \to \bigoplus_m K_{i-1}A/m, \qquad \text{for } i \text{ even,} \qquad i \geq 2.$$

D. Quillen proved in [6] that K_iA is finitely generated for all $i \geq 0$; using the fact that $K_iA \otimes \mathbb{Q} = 0$ for i even (cf. [2], Cor. of Thm. 2) we conclude that K_iA is finite for i even ($i \geq 2$). We obtain the following description of the homotopy groups of $BSL(F)^+$ for $i \geq 2$:

$$\Pi_i\, BSL(F)^+ \cong K_iF = \begin{cases} \text{a finitely generated abelian group for } i \text{ odd,} \\ \text{a torsion group for } i \text{ even.} \end{cases}$$

If i is odd the rank of the group K_iF is given by [2]. If i is even the group K_iF is not finitely generated; therefore there are homology groups of this space which are not finitely generated.

Since $BSL(F)^+$ is an infinite loop space (cf. [8]) we deduce from Corollary 6 our main result which describes the structure of the integral homology groups of $BSL(F)^+$.

THEOREM 7. *Let F be a number field. Then for all $i \geq 0$ $H_i(BSL(F)^+; \mathbb{Z})$ is the direct sum of a torsion group and a free abelian group of finite rank.*

Remarks. 1) The rank of this free abelian group is given by [2], Thm. 2.

2) Let A be the ring of algebraic integers in F, S a multiplicative subset of A, and let us consider the localization $S^{-1}A$; then the statement of Theorem 7 is also true for the space $BSL(S^{-1}A)^+$.

3) Theorem 7 remains true if we replace the infinite special linear group by the infinite Steinberg group.

COROLLARY 8. *For all* $i \geqslant 0$ $H^i(BSL(F)^+; \mathbb{Z})$ *contains no infinitely divisible element except* 0.

Proof. Let $X := BSL(F)^+$. We know from the universal coefficient theorem that $H^i(X; \mathbb{Z}) \cong \mathrm{Hom}\,(H_iX, \mathbb{Z}) \oplus \mathrm{Ext}\,(H_{i-1}X, \mathbb{Z})$. Because the homomorphisms of $\mathrm{Hom}\,(H_iX, \mathbb{Z})$ take values in $\mathbb{Z}$, the unique infinitely divisible element of $\mathrm{Hom}\,(H_iX, \mathbb{Z})$ is 0. On the other side, by Theorem 7, $\mathrm{Ext}\,(H_{i-1}X, \mathbb{Z}) \cong \mathrm{Ext}\,(T, \mathbb{Z})$ where T is the torsion subgroup of $H_{i-1}X$ and the conclusion follows from Lemma 4.2. of [1].

COROLLARY 9. *For all* $i \geqslant 0$ $H^i(BGL(F)^+; \mathbb{Z})$ *contains no infinitely divisible element except* 0.

Proof. Because $BSL(F)^+$ is the universal cover of $BGL(F)^+$ we have the fibration $BSL(F)^+ \to BGL(F)^+ \to BF^\times$, where $F^\times$ denotes the multiplicative group of the units of F. The three spaces are H-spaces and there exists a section $BF^\times \to BGL(F)^+$ induced by the inclusion $F^\times = GL_1(F) \hookrightarrow GL(F)$; this implies a homotopy equivalence $BGL(F)^+ \simeq BSL(F)^+ \times BF^\times$. Since $F^\times$ is the direct sum of a finite cyclic group and a free abelian group, it follows from Theorem 7 that $H_i(BGL(F)^+; \mathbb{Z})$ is the direct sum of a torsion group and a free abelian group, $i \geqslant 0$. The assertion is then a consequence of the argument used to prove the previous corollary.

3. Chern classes of group representations over a number field

We consider in this section representations $\rho: G \to GL(F)$ of discrete groups G over a number field $F \subset \mathbb{C}$ and their Chern classes: our objective is to prove that these Chern classes are torsion classes.

Let G be a discrete group and $\rho: G \to GL(\mathbb{C})$ a complex representation of G. The Chern classes $c_i(\rho) \in H^{2i}(G; \mathbb{Z})$, $i \geqslant 1$, of the representation ρ are defined as the Chern classes of the associated flat complex vector bundle over the classifying space $BG = K(G, 1)$ of G (cf. [1], [4], or [5]). For a number field $F \subset \mathbb{C}$ let $c_i(GL(F))$ denote the i-th Chern class of the inclusion $GL(F) \hookrightarrow GL(\mathbb{C})$, $i \geqslant 1$, where we consider $GL(F)$ as a discrete group. If a representation ρ of the discrete group G is realizable over a number field $F \subset \mathbb{C}$, i.e., $G \xrightarrow{\rho} GL(F) \hookrightarrow GL(\mathbb{C})$, then the Chern classes of ρ and of $GL(F)$ are related by

$$c_i(\rho) = \rho^* c_i(GL(F))$$

where ρ^* is the homomorphism $H^{2i}(GL(F);\mathbb{Z})\to H^{2i}(G;\mathbb{Z})$ induced by ρ, $i\geq 1$. Therefore the order of $c_i(GL(F))$ in $H^{2i}(GL(F);\mathbb{Z})$ is the best upper bound for the order of $c_i(\rho)$ when ρ ranges over all F-representations of all discrete groups.

Let now $\bar{E}_F(i)$ be the integer described in [4] and [5]. Namely $\bar{E}_F(i):=\max\{n \text{ with } \exp(\mathrm{Gal}(F(\zeta_n)/F)) \text{ dividing } i\}$ where ζ_n denotes a primitive n-th root of unity and $\exp(\mathrm{Gal}(F(\zeta_n)/F))$ is the exponent of the Galois group of $F(\zeta_n)$ over F.

The main result of this section is the following

THEOREM 10. *Let $F\subset\mathbb{C}$ be a number field. For any discrete group G and any representation $\rho: G\to GL(F)$ one has*

$$\bar{E}_F(i)c_i(\rho)=0 \quad \textit{for} \quad i\geq 1.$$

Proof. It follows from [5] (Main Theorem) that $\bar{E}_F(i)c_i(GL(F))$ is an infinitely divisible element of $H^{2i}(GL(F);\mathbb{Z})$ and we conclude by Corollary 9 that $\bar{E}_F(i)c_i(GL(F))=0$. The theorem is then proved because the order of $c_i(\rho)$ divides the order of $c_i(GL(F))$.

Remark. This theorem gives for $i\geq 1$ an upper bound for the order of the i-th Chern class of all F-representations of all discrete groups but not necessarily the best upper bound. In fact we deduce from [4] that the best upper bound (= the order of $c_i(GL(F))$) is equal to $\bar{E}_F(i)$ unless i is even and F formally real (i.e., -1 is not a sum of squares in F), where it is $\bar{E}_F(i)$ or $\frac{1}{2}\bar{E}_F(i)$ (we do not know which is correct).

4. The case of the field $\mathbb{Q}$ of rational numbers

If $F=\mathbb{Q}$, the field of rational numbers, then we can give a more precise answer. In this case $\bar{E}_{\mathbb{Q}}(i)=2$ if i is odd and $\bar{E}_{\mathbb{Q}}(i)=$ denominator of $B_i/2i$ if i is even, where B_i is the i-th Bernoulli number ($B_2=\frac{1}{6}$, $B_4=\frac{1}{30}$, ...). The best upper bound for the order of Chern classes of rational representations of discrete groups is given by

THEOREM 11. *The order of $c_i(GL(\mathbb{Q}))$ in $H^{2i}(GL(\mathbb{Q});\mathbb{Z})$, $i\geq 1$, equals*

a) *2 for i odd,*
b) *$\bar{E}_{\mathbb{Q}}(i)$ for $i\equiv 2 \bmod 4$,*
c) *$\bar{E}_{\mathbb{Q}}(i)$ or $\frac{1}{2}\bar{E}_{\mathbb{Q}}(i)$ for $i\equiv 0 \bmod 4$.*

Proof. We write $c_i(GL(\mathbb{Z}))$ for the i-th Chern class of the inclusion $GL(\mathbb{Z}) \hookrightarrow GL(\mathbb{C})$; clearly its order divides the order of $c_i(GL(\mathbb{Q}))$. The proof follows from Theorem 10 and the study of the order of $c_i(GL(\mathbb{Z}))$ in [1].

Remark. The homomorphism $H_3(BSL(\mathbb{Z})^+; \mathbb{Z}) \to H_3(BSL(\mathbb{Q})^+; \mathbb{Z})$, induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$, is surjective, because the corresponding homomorphism $K_3\mathbb{Z} \to K_3\mathbb{Q}$ is surjective, and we know that $H_3(BSL(\mathbb{Z})^+; \mathbb{Z}) \cong \mathbb{Z}/24$ (cf. [1], Satz 1.5). Since the group $H^4(BSL(\mathbb{Q})^+; \mathbb{Z})$ ($\cong H_3(BSL(\mathbb{Q})^+; \mathbb{Z})$) contains the element $c_2(SL(\mathbb{Q}))$ of order $\bar{E}_{\mathbb{Q}}(2) = 24$ we conclude that $H_3(BSL(\mathbb{Z})^+; \mathbb{Z}) \to H_3(BSL(\mathbb{Q})^+; \mathbb{Z})$ is an isomorphism.

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