On a fundamental variational lemma for extremal quasiconformal mappings.

Comment. Math. Helvetici 61 (1986) 565–580 0010-2571/86/040565-16$01.50 + 0.20/0

On a fundamental variational lemma for extremal quasiconformal mappings

RICHARD FEHLMANN

1. Introduction

In [R2] E. Reich considers the following extremal problem in qc (quasiconformal) mappings. Given are a closed set σ on the boundary ∂D of the unit disk $D = \{w \mid |w| < 1\}$ which contains at least four points and a measurable set E in D where $D \backslash E$ has positive area-measure and where, if σ is an infinite set, $\bar{E}$ is assumed to be compact in $\bar{D}\backslash\sigma$. Furthermore a quasisymmetric boundary mapping $h: \partial D \to \partial D$ is given and a measurable non-negative function $b(w)$ on E with $\operatorname{ess\,sup}_{w \in E} b(w) < 1$ which is called the "dilatation bound function": $Q(h, \sigma, E, b)$ then denotes the class of all qc mappings $F: D \to D$ which satisfy the side-condition

$$F|_\sigma = h|_\sigma \quad \text{and} \quad |\kappa_F(w)| \le b(w) \quad \text{a.e. in } E,$$

where $\kappa_F = F_{\bar{w}}/F_w$ is the complex dilatation of F. In this class a mapping F is called *extremal* if it minimizes the value

$$\operatorname*{ess\,sup}_{w \in D\backslash E} |\kappa_F(w)|$$

and is called *uniquely extremal* if it is the only such mapping.

In the case when E is the empty set a necessary and sufficient condition for extremality is the Hamilton-condition as has been shown in [H] and [RS]. In [R2] E. Reich has given a generalization of this condition which is necessary and sufficient for extremality in $Q(h, \sigma, E, b)$ and by which extremal mappings can be characterized. But in his work an additional requirement had to be posed on $b(w)$, namely that it is bounded away from zero. Later F. Gardiner succeeded in proving the analogous condition in the case when σ is finite and $b(w) \equiv 0$ in E [G2]. He used a result from Teichmüller theory which he had proved in [G1].

In this note we use Gardiner's result to generalize a fundamental variational lemma which is needed in Reich's treatment. In its generalized form it turns out to be adequate for the general case. In section 3 we apply it to handle the case where σ is infinite and $b(w)\equiv 0$ in E. The proof then follows exactly the same pattern as the one in Reich's paper. In a forthcoming paper of K. Sakan [Sa] it then will be applied to arbitrary dilatation bound functions $b(w)$.

In section 4 we give, based on Reich's treatment, alternative proofs of Gardiner's result in two special cases. Namely, if the area-measure of the boundary ∂E of the set E is zero, then this result follows immediately by approximation and if E is supposed to be a closed set, it can be proved similarly.

Finally, I want to add that the idea of setting variable dilatation bounds as a side-condition for extremal problems goes back to O. Teichmüller ([T], p. 15), and to my knowledge R. Kühnau has been the first one who attacked such problems successfully. In [K1] he solved a problem of this sort (Satz 1) which enables him in [K2] to give a complete solution of our extremal problem above in the case where σ consists of four points by an essentially different method. No requirements as $b(w)\geq\varepsilon>0$ had to be made except for some regularity assumptions on E and $b(w)$.

2. Notations and the variational lemma

For a qc mapping F we denote its complex dilatation by κ_F, the dilatation of F at the point w by $D_F(w)=(1+|\kappa_F(w)|)/(1-|\kappa_F(w)|)$ and its maximal dilatation by $K[F]$. We put $\sigma'=h(\sigma)$, $E_0=\{w\in E\mid b(w)=0\}$ and for a fixed element $F\in Q(h,\sigma,E,b)$ we introduce

$$f=F^{-1},\qquad \kappa=\kappa_f,\qquad k_F=\operatorname*{ess\,sup}_{w\in D\setminus E}|\kappa_F(w)|$$

and

$$\hat\kappa(z)=\begin{cases}\kappa(z) & z\in D\setminus F(E)\\ k_F\dfrac{\kappa(z)}{b(f(z))} & z\in F(E\setminus E_0)\\ 0 & z\in F(E_0)\end{cases}\tag{2.1}$$

We note that $\|\hat\kappa\|_\infty:=\operatorname{ess\,sup}_{z\in D}|\hat\kappa(z)|=k_F$. Then the Banach-space $B_{\sigma'}=\{\varphi\mid\varphi$ holomorphic in D, $\|\varphi\|<\infty$, $\varphi\,dz^2$ real along $\partial D\setminus\sigma'\}$ over the field $\mathbb{R}$ will be used, where

$$\|\varphi\|=\iint_D|\varphi(z)|\,dx\,dy,\quad (z=x+iy)$$

as well as the unit sphere in $B_{\sigma'}$

$$B_{\sigma'1} = \{\varphi \in B_{\sigma'} \mid \|\varphi\| = 1\}.$$

For measurable sets A in D we will put

$$\|\varphi\|_A = \iint_A |\varphi(z)|\, dx\, dy.$$

If $Q(h, \sigma, E, b)$ is not empty, then there exist extremal mappings in this class as follows by normality and the following result of Strebel [St]: If a sequence of qc mappings F_n converge locally uniformly in D to a qc mapping, F, then

$$|\kappa_F(w)| \le \overline{\lim_{n\to\infty}}\, |\kappa_{F_n}(w)| \quad \text{a.e. in } D.$$

The result of Gardiner then is the

THEOREM 2.1 [G2]. *If σ is finite and $b(w) \equiv 0$ in E, then $F \in Q(h, \sigma, E, b)$ is extremal iff*

$$\sup_{\substack{\varphi \in B_{\sigma'} \\ \|\varphi\|_{D\setminus F(E_0)} = 1}} \operatorname{Re} \iint_D \hat{\kappa}(z)\varphi(z)\, dx\, dy = k_F.$$

Since σ is finite, the space $B_{\sigma'}$ is finite dimensional and it is easy to see that the sup must be attained. Namely, if φ_n is a sequence in $B_{\sigma'}$ with $\|\varphi_n\|_{D\setminus F(E_0)} = 1$, then the norms $\|\varphi_n\|$ stay bounded. Otherwise, by normality of $B_{\sigma'1}$, $\psi_n := (\varphi_n / \|\varphi_n\|)$ would contain a subsequence which converges to zero locally uniformly in $D\setminus F(E_0)$, an impossibility because of the finite dimension of $B_{\sigma'}$. Hence φ_n is a normal sequence and if it is a maximizing sequence for the functional above, then the limit of a convergent subsequence maximizes the functional. Therefore this theorem implies the

COROLLARY 2.1 [G2]. *If σ is finite, $b(w) \equiv 0$ in E and F extremal in $Q(h, \sigma, E, b)$, then there is a $\varphi \in B_{\sigma'}\setminus\{0\}$ with*

$$\kappa(z) = \begin{cases} k_F \dfrac{\bar{\varphi}(z)}{|\varphi(z)|} & z \in D\setminus F(E) \\ 0 & z \in F(E). \end{cases}$$

Our main tool will be the Main Inequality of Reich and Strebel [RS], p. 380 (see also [R1], p. 110), or more precisely, two statements following from it. First

(M1) If $\varphi \in B_{\sigma' 1}$ and f and g are two qc mappings from D onto itself which agree on σ', then

$$1 \leq \iint_D |\varphi(z)| \frac{\left|1 - \kappa_f(z) \frac{\varphi(z)}{|\varphi(z)|}\right|^2}{1 - |\kappa_f(z)|^2} D_{g^{-1}}(f(z))\, dx\, dy.$$

Then, as is shown in [R1], p. 119, the Main Inequality applied to extremal n-gon Teichmüller mappings, yields

(M2) If σ_n' consists of n points on ∂D and f_n is a Teichmüller mapping with complex dilatation $(K_n - 1)/(K_n + 1)(\bar{\varphi}_n / |\varphi_n|)$, where $\varphi_n \in B_{\sigma_n' 1}$, then for every qc selfmapping g of D which agrees with f_n on σ_n', we have

$$K_n \leq \iint_D |\varphi_n(z)| \frac{\left|1 + \kappa_g(z) \frac{\varphi_n(z)}{|\varphi_n(z)|}\right|^2}{1 - |\kappa_g(z)|^2}\, dx\, dy.$$

Before coming to the variational lemma we will derive the

LEMMA 2.1. *Let h, σ, σ' and E be as above and $K \geq 1$ be a fixed number. Then there is a $q < 1$ such that*

$$\|\varphi\|_{G(E)} \leq q$$

for every $\varphi \in B_{\sigma' 1}$ and every $G \in Q_K(h, \sigma) := \{G \mid G : D \to D,\ K - qc,\ G|_\sigma = h|_\sigma\}$.

Proof. If this lemma were false, there would be a sequence φ_n in $B_{\sigma' 1}$ and G_n in $Q_K(h, \sigma)$ with

$$\|\varphi_n\|_{G_n(E)} \to 1, \qquad n \to \infty.$$

The set $B_{\sigma' 1}$ is normal and, since σ contains at least three points, the set $Q_K(h, \sigma)$ is normal and closed. So by passing to subsequences we may assume that there is a $\varphi_\infty \in B_{\sigma'}$ and a $G_\infty \in Q_K(h, \sigma)$ where

$$\varphi_n \xrightarrow{n \to \infty} \varphi_\infty \text{ locally uniformly in } \bar{D} \backslash \sigma'$$

$$G_n \xrightarrow{n \to \infty} G_\infty \text{ uniformly in } \bar{D}.$$

Taking into account that for infinite σ the set $G_\infty(E)$ is supposed to be relatively compact in $\bar{D}\backslash\sigma'$ we infer that

$$\|\varphi_\infty\|_{G_\infty(E)} = \lim_{n\to\infty} \|\varphi_n\|_{G_n(E)} = 1. \tag{2.2}$$

(This equality seems to be obvious, but in lack of a precise reference we add a proof of it in the appendix). Clearly $\|\varphi_\infty\| \le 1$ and hence

$$\iint_{D\backslash G_\infty(E)} |\varphi_\infty(z)|\, dx\, dy = 0$$

which is a contradiction because $D\backslash G_\infty(E)$ has positive measure and $\varphi_\infty \neq 0$ is holomorphic.

FUNDAMENTAL VARIATIONAL LEMMA. *Let E' be a measurable subset of D where $D\backslash E'$ has positive measure and σ' be a closed set on ∂D which contains at least four points and where, if σ' is an infinite set, $\overline{E'}$ is compact in $\bar{D}\backslash\sigma'$. If g is a qc mapping from D onto itself where its complex dilatation κ_g satisfies*

$$\kappa_g(z) \equiv 0 \text{ in } E' \quad \text{and} \quad \operatorname{Re} \iint_D \kappa_g \varphi\, dx\, dy = 0 \quad \forall \varphi \in B_{\sigma'},$$

then there is a qc mapping $g^*: D \to D$ with $g^* \circ g = id$ on σ' and with a complex dilatation κ_{g^*} that satisfies

$$\kappa_{g^*}(z) \equiv 0 \text{ in } g(E') \quad \text{and} \quad \|\kappa_{g^*}\|_\infty = O(\|\kappa_g\|_\infty^2) \quad \text{as} \quad \|\kappa_g\|_\infty \to 0.$$

Proof. The best choice for g^* is to be an extremal element in $Q(g^{-1}|_{\partial D}, \sigma'', E'', 0)$ where $\sigma'' = g(\sigma')$, $E'' = g(E')$. If σ'' is not finite, we choose σ''_n to consist of n points on σ'' which become to be denser and denser as n tends to infinity. For every n there is an extremal mapping G_n in $Q(g^{-1}|_{\partial D}, \sigma''_n, E'', 0)$ and by Corollary 2.1 there is a $\varphi_n \in B_{\sigma'_n 1}$ ($\sigma'_n = g^{-1}(\sigma''_n)$) such that the complex dilatation κ_n of $g_n := G_n^{-1}$ satisfies

$$\kappa_n(z) = \begin{cases} k_n \dfrac{\bar{\varphi}_n(z)}{|\varphi_n(z)|} & z \in G_n(D\backslash E'') \\ 0 & z \in G_n(E'') \end{cases} \tag{2.3}$$

Evidently $k_n \leq \|\kappa_{g^*}\|_\infty$ and k_n is an increasing sequence. By normality we may assume that the qc mappings G_n converge locally uniformly to a qc mapping $G_\infty : D \to D$ where obviously $G_\infty|_{\sigma''} = g^{-1}|_{\sigma''}$ and by Strebel's result

$$|\kappa_{G_\infty}(z)| \leq \overline{\lim_{n\to\infty}}\, |\kappa_{G_n}(z)| \quad \text{a.e. in } D.$$

We hence conclude that $\kappa_{G_\infty}(z) = 0$ in E'' and $\operatorname{ess\,sup}_{z\in D\setminus E''} |\kappa_{G_\infty}(z)| \leq \lim_{n\to\infty} k_n \leq \|\kappa_{g^*}\|_\infty$. So $G_\infty \in Q(g^{-1}|_{\partial D}, \sigma'', E'', 0)$ and G_∞ is therefore extremal in this class, i.e., $\|\kappa_{g^*}\|_\infty = \lim_{n\to\infty} k_n$ and we can take G_∞ for g^*.

For the purpose of estimating the numbers k_n we introduce the extremal Teichmüller n-gon mappings $f_n : D \to D$ which agree on σ'_n with g. Their complex dilatations $\tilde{\kappa}_n$ are equal to

$$\tilde{k}_n \frac{\bar{\tilde{\varphi}}_n}{|\tilde{\varphi}_n|} \quad \text{a.e. in } D$$

where $\tilde{\varphi}_n \in B_{\sigma'_n 1}$.

We use the statement (M1) where we put the quadruple $(\sigma'_n, \varphi_n, g_n, f_n)$ for (σ', φ, f, g)

$$1 \leq \iint_D |\varphi_n(z)| \frac{\left|1 - \kappa_n(z) \dfrac{\varphi_n(z)}{|\varphi_n(z)|}\right|^2}{1 - |\kappa_n(z)|^2} \tilde{K}_n \, dx\, dy \quad \left(\tilde{K}_n = \frac{1+\tilde{k}_n}{1-\tilde{k}_n}\right).$$

Using (2.3), splitting up the integral and putting $K_n = (1+k_n)/(1-k_n)$ we get

$$1 \leq \frac{\tilde{K}_n}{K_n}(1 - \|\varphi_n\|_{G_n(E'')}) + \tilde{K}_n \|\varphi_n\|_{G_n(E'')}.$$

We multiply with K_n and subtract $K_n \|\varphi_n\|_{G_n(E'')}$, so

$$K_n(1 - \|\varphi_n\|_{G_n(E'')}) \leq \tilde{K}_n(1 - \|\varphi_n\|_{G_n(E'')}) + (\tilde{K}_n - 1)K_n \|\varphi_n\|_{G_n(E'')}$$

and finally since $\|\varphi_n\|_{G_n(E'')} < 1$, $K_n \leq K[g^*]$

$$K_n \leq \tilde{K}_n + (\tilde{K}_n - 1)\frac{K[g^*]}{1 - \|\varphi_n\|_{G_n(E'')}}. \tag{2.4}$$

Next we estimate $\tilde{K}_n$. We use the statement (M2) and put the quadrupel $(\sigma'_n, \tilde{\varphi}_n, f_n, g)$ for $(\sigma'_n, \varphi_n, f_n, g)$

$$\tilde{K}_n \leq \iint_D |\tilde{\varphi}_n(z)| \frac{\left|1 + \kappa_g(z)\dfrac{\tilde{\varphi}_n(z)}{|\tilde{\varphi}_n(z)|}\right|^2}{1 - |\kappa_g(z)|^2} \, dx \, dy.$$

Following Reich's calculation in [R2] the integral becomes

$$\iint_D |\tilde{\varphi}_n| \frac{1 + |\kappa_g|^2}{1 - |\kappa_g|^2} \, dx \, dy + 2 \operatorname{Re} \iint_D \frac{\kappa_g \tilde{\varphi}_n}{1 - |\kappa_g|^2} \, dx \, dy$$
$$\leq \frac{1 + \|\kappa_g\|_\infty^2}{1 - \|\kappa_g\|_\infty^2} + 2 \operatorname{Re} \iint_D \left(\frac{\kappa_g \tilde{\varphi}_n}{1 - |\kappa_g|^2} - \kappa_g \tilde{\varphi}_n \right) dx \, dy$$

because of the hypothesis

$$\operatorname{Re} \iint_D \kappa_g \tilde{\varphi}_n \, dx \, dy = 0, \qquad \tilde{\varphi}_n \in B_{\sigma'_n 1} \subset B_{\sigma'}.$$

The second term can be estimated by

$$2 \left| \iint_D \frac{|\kappa_g|^2 \kappa_g \tilde{\varphi}_n}{1 - |\kappa_g|^2} \, dx \, dy \right| \leq \frac{2 \|\kappa_g\|_\infty^3}{1 - \|\kappa_g\|_\infty^2}.$$

Hence

$$\tilde{K}_n \leq 1 + 2 \frac{\|\kappa_g\|_\infty^2 + \|\kappa_g\|_\infty^3}{1 - \|\kappa_g\|_\infty^2} = 1 + 2 \frac{\|\kappa_g\|_\infty^2}{1 - \|\kappa_g\|_\infty}$$

With (2.4) finally

$$K_n \leq 1 + 2 \frac{\|\kappa_g\|_\infty^2}{1 - \|\kappa_g\|_\infty} + 2 \frac{\|\kappa_g\|_\infty^2}{1 - \|\kappa_g\|_\infty} \frac{K[g^*]}{1 - \|\varphi_n\|_{G_n(E'')}}.$$

We let n tend to infinity, i.e. $G_n \xrightarrow{n \to \infty} G_\infty$ locally uniformly in D and by normality of $B_{\sigma' 1}$ we may pass to a further subsequence such that

$$\varphi_n \xrightarrow{n \to \infty} \varphi_\infty \text{ locally uniformly in } \bar{D} \setminus \sigma'$$

where $\varphi_\infty \in B_{\sigma'}$, $\|\varphi_\infty\| \leq 1$.

This approximation takes place only if σ' is infinite, so E'' is relatively compact in $\bar{D}\backslash\sigma''$ and hence

$$\|\varphi_n\|_{G_n(E'')} \xrightarrow{n\to\infty} \|\varphi_\infty\|_{G_\infty(E'')}.$$

Hence

$$K[g^*] = \lim_{n\to\infty} K_n \le 1 + \frac{2\,\|\kappa_g\|_\infty^2}{1-\|\kappa_g\|_\infty}\left(1 + \frac{K[g^*]}{1-\|\varphi_\infty\|_{G_\infty(E'')}}\right).$$

Now we apply Lemma 2.1 with the quadrupel $(id, \sigma', \sigma', E')$ instead of (h, σ, σ', E) for a fixed number $K > K[g]^2$. Namely, $G_\infty(E'') = G_\infty \circ g(E')$ and $G_\infty \circ g|_{\sigma'} = id|_{\sigma'}$, so $G_\infty \circ g \in Q_K(id, \sigma')$ and by $\|\varphi_\infty\| \le 1$ and $(\varphi_\infty/\|\varphi_\infty\|) \in B_{\sigma'1}$, there is a $q < 1$ which does not depend on g (only on K), such that

$$\|\varphi_\infty\|_{G_\infty\circ g(E')} \le \left\|\frac{\varphi_\infty}{\|\varphi_\infty\|}\right\|_{G_\infty\circ g(E')} \le q$$

and hence

$$K[g^*] \le 1 + \left(1 + \frac{K}{1-q}\right)\frac{2\,\|\kappa_g\|_\infty^2}{1-\|\kappa_g\|_\infty}$$

which shows that

$$\|\kappa_{g^*}\|_\infty = O(\|\kappa_g\|_\infty^2)$$

for $\|\kappa_g\|_\infty \to 0$, since K can stay fixed as $K[g] \to 1$.

3. Application: The case $b(w) \equiv 0$

An elaboration of a technique employed by Krushkal [Kr] which is done in Reich's paper [R2] now yields the necessity part of the

THEOREM 3.1. *A qc mapping F is extremal in $Q(h, \sigma, E, 0)$ iff*

$$\sup_{\substack{\varphi\in B_{\sigma'}\\ \|\varphi\|_{D\backslash F(E_0)}=1}} \operatorname{Re} \iint_D \hat{\kappa}\varphi \, dx\, dy = k_F$$

Proof. Let F be extremal. Since $k_F = \|\hat{\kappa}\|_\infty$ we now assume that

$$a := \sup_{\substack{\varphi \in B_{\sigma'} \\ \|\varphi\|_{D\setminus F(E)}=1}} \operatorname{Re} \iint_D \hat{\kappa}\varphi \, dx \, dy < k_F$$

and hence $k_F > 0$.

Then a is the norm of the linear operator

$$\varphi \mapsto \operatorname{Re} \iint_{D\setminus F(E)} \hat{\kappa}\varphi \, dx \, dy$$

($\hat{\kappa} = 0$ on $F(E)$!) defined on the Banach-space $\{\varphi|_{D\setminus F(E)} \mid \varphi \in B_{\sigma'}\}$ which is a subspace of $L_1(D\setminus F(E))$ over the field $\mathbb{R}$. By the Hahn–Banach Theorem there is an extension of this operator on $L_1(D\setminus F(E))$ with norm a and by the Riesz-representation theorem there is a complex-valued function β on $D\setminus F(E)$ with $\|\beta\|_\infty = a$ which realizes this extension, i.e.,

$$\operatorname{Re} \iint_{D\setminus F(E)} \hat{\kappa}\varphi \, dx \, dy = \operatorname{Re} \iint_{D\setminus F(E)} \beta\varphi \, dx \, dy \quad \forall \varphi \in B_{\sigma'}.$$

We put

$$v(z) = \begin{cases} \hat{\kappa}(z) - \beta(z) & z \in D\setminus F(E) \\ 0 & z \in F(E) \end{cases}$$

and have $\|v\|_\infty > 0$ ($\hat{\kappa} \neq \beta$!) and

$$\operatorname{Re} \iint_D v\varphi \, dx \, dy = 0 \quad \text{for} \quad \varphi \in B_{\sigma'}.$$

For t, $0 \le t < (1/\|v\|_\infty)$ we put $g: D \to D$ to be a qc mapping with $g(1) = 1$, $g(i) = i$, $g(-1) = -1$ and

$$\kappa_g = tv.$$

Here we apply the Fundamental Variational Lemma on g, $\sigma' = h(\sigma)$ and $E' = F(E)$. Hence there is a qc mapping $g^*: D \to D$ where $g^* \circ g = id$ on σ', $\kappa_{g^*} = 0$ in $g(E')$ and

$$\|\kappa_{g^*}\|_\infty = O(t^2) \quad \text{as} \quad t \to 0.$$

We have $g^* \circ g \circ F \in Q(h, \sigma, E, 0)$ and show that

$$\operatorname*{ess\,sup}_{w \in D \setminus E} |\kappa_{g^* \circ g \circ F}(w)| < k_F$$

for $t > 0$, sufficiently small. This contradicts then the extremality of F. One computes for $z \in D \setminus E'$

$$|\kappa_{f \circ g^{-1}}(g(z))| = \left| \frac{\kappa(1-t) + t\beta}{1 - t\bar{\nu}\kappa} \right|$$

and the computation in [R2], p. 109 and 110, assures the existence of numbers $\delta > 0$, $t_0 > 0$ with

$$|\kappa_{f \circ g^{-1}}(g(z))| \leq k_F - \delta_1 t \quad \text{for} \quad 0 \leq t \leq t_0 \quad \text{and} \quad z \in D \setminus E'.$$

By $\|\kappa_{g^*}\|_\infty = O(t^2)$ the values $|\kappa_{f \circ g^{-1} \circ g^{*-1}}(g^*(g(z)))|$ can be estimated in the same manner in $D \setminus E'$ and this yields the result.

The sufficiency part is immediate. We do not need any restriction on $b(w)$ for it. Let $F \in Q(h, \sigma, E, b)$ and

$$\sup_{\substack{\varphi \in B_{\sigma'} \\ \|\varphi\|_{D \setminus F(E_0)} = 1}} \operatorname{Re} \iint_D \hat{\kappa} \varphi \, dx \, dy = k_F.$$

If this sup is attained, then there is a $\varphi \in B_{\sigma'}$, $\|\varphi\|_{D \setminus F(E_0)} = 1$, with

$$\hat{\kappa}(z) = k_F \frac{\bar{\varphi}(z)}{|\varphi(z)|} \quad \text{a.e. in } D \setminus F(E_0).$$

If $k_F = 0$ we have extremality. If $k_F > 0$ we conclude from (2.1)

$$\kappa(z) = \begin{cases} k_F \dfrac{\bar{\varphi}(z)}{|\varphi(z)|} & z \in D \setminus F(E) \\[2ex] b(f(z)) \dfrac{\bar{\varphi}(z)}{|\varphi(z)|} & z \in F(E) \end{cases}$$

Then by [R2], Theorem 5, F is even uniquely extremal in $Q(h, \sigma, E, b)$.

If the sup is not attained, and this can occur only if σ is infinite, then there is a sequence $\varphi_n \in B_{\sigma'}$, $\|\varphi_n\|_{D \setminus F(E_0)} = 1$ with $\operatorname{Re} \iint_D \hat{\kappa} \varphi_n \, dx \, dy \xrightarrow{n \to \infty} k_F$ and $\varphi_n \xrightarrow{n \to \infty} 0$ locally uniformly in $\bar{D} \setminus \sigma'$.

From the relative compactness of E in $\bar{D}\backslash\sigma'$ we conclude that $\|\varphi_n\|_{F(E_0)} \xrightarrow{n\to\infty} 0$ and hence $\|\varphi_n\| \xrightarrow{n\to\infty} 1$. So if we put

$$\hat{\varphi}_n := \frac{\varphi_n}{\|\varphi_n\|}$$

we get a degenerating Hamilton sequence $\hat{\varphi}_n$ for the complex dilatation $\hat{\kappa}$, this is a sequence $\hat{\varphi}_n \in B_{\sigma'1}$ where $\operatorname{Re} \iint_D \hat{\kappa}\hat{\varphi}_n \, dx \, dy \xrightarrow{n\to\infty} \|\hat{\kappa}\|_\infty = k_F$ and $\hat{\varphi}_n \xrightarrow{n\to\infty} 0$ locally uniformly in $\bar{D}\backslash\sigma'$. We denote by $\hat{f}$ a qc selfmapping from D with complex dilatation $\hat{\kappa}$. By the sufficiency of Hamilton's condition $\hat{f}$ is extremal for its own boundary values on σ', and by Satz 5.2 in [F] there exists a substantial boundary point on σ', i.e., a point with local dilatation equal to $K[\hat{f}] = (1 + k_F) \mid (1 - k_F)$ (for the boundary values $\hat{f}|_{\sigma'}$). Since $\hat{f} \circ F$ is conformal in $D\backslash E$ which contains a neighborhood of σ and since local dilatations of the boundary mapping are preserved under conformal mapping we conclude that there is a point on σ with local dilatation $K[\hat{f}]$ for the boundary values $h|_\sigma$. Hence every mapping in $Q(h, \sigma, E, b)$ needs to have its dilatation near that point at least as large as $K[\hat{f}] = (1 + k_F)/(1 - k_F)$, in particular, F is extremal.

4. Alternative proofs of Theorem 2.1 in two special cases

Let σ be finite and $b(w) \equiv 0$ in E. We put $b_n(w) \equiv 1/n$ in E and denote by F_n an extremal qc mapping in $Q(h, \sigma, E, 1/n)$. By Reich's result there is a $\varphi_n \in B_{\sigma'1}$ where the complex dilatation κ_n of $f_n := F_n^{-1}$ is

$$\kappa_n(z) = \begin{cases} k_n \dfrac{\bar{\varphi}_n(z)}{|\varphi_n(z)|} & z \in D\backslash F_n(E) \\[2ex] \dfrac{1}{n} \dfrac{\bar{\varphi}_n(z)}{|\varphi_n(z)|} & z \in F_n(E) \end{cases}$$

Furthermore, let F be extremal in $Q(h, \sigma, E, 0)$, hence $k_n \leq k_F$. Passing to subsequences we find a qc mapping F_∞ and a function φ_∞ where

$$F_n \xrightarrow{n\to\infty} F_\infty \quad \text{locally uniformly in } D$$

and

$$\varphi_n \xrightarrow{n\to\infty} \varphi_\infty \quad \text{locally uniformly in } \bar{D}\backslash\sigma'$$

As above we conclude that $F_\infty \in Q(h, \sigma, E, 0)$, and therefore $k_F \le k_{F_\infty} \le \lim_{n\to\infty} k_n \le k_F$, and since σ is finite we have $\varphi_\infty \in B_{\sigma'1}$. Furthermore

$$f_n \xrightarrow{n\to\infty} f_\infty := F_\infty^{-1} \text{ locally uniformly in } D.$$

F_∞ is hence extremal in $Q(h, \sigma, E, 0)$ and the complex dilatations κ_n converge pointwise a.e. in the interior of $F_\infty(E)$ or in the interior of $D\backslash F_\infty(E)$ to zero or $k_F(\overline{\varphi_\infty}/|\varphi_\infty|)$ respectively. Since qc mappings preserve sets of area-measure zero we infer from Theorem 5.2 in [LV], p. 187, that if the area-measure of ∂E is zero, then a.e.

$$\kappa_{f_\infty}(z) = \begin{cases} k_F \dfrac{\overline{\varphi_\infty}(z)}{|\varphi_\infty(z)|} & z \in D\backslash F_\infty(E) \\ 0 & z \in F_\infty(E). \end{cases} \tag{4.1}$$

By Theorem 5 in [R2] F_∞ is uniquely extremal in $Q(h, \sigma, E, 0)$ and hence $F = F_\infty$. Clearly

$$\sup_{\substack{\varphi\in B_{\sigma'} \\ \|\varphi\|_{D\backslash F(E)}=1}} \operatorname{Re} \iint_D \kappa_{f_\infty} \varphi \, dx\, dy = k_F$$

and we thus have proved the first part of

PROPOSITION 4.1. *Let F be extremal in $Q(h, \sigma, E, 0)$ where σ is finite. If the area-measure of ∂E is zero or if E is a closed set, then*

$$\sup_{\substack{\varphi\in B_{\sigma'} \\ \|\varphi\|_{D\backslash F(E)}=1}} \operatorname{Re} \iint_D \hat{\kappa} \varphi \, dx\, dy = k_F.$$

For the second part the reasoning in the proof above has to be slightly changed since we do not know if κ_n is convergent a.e. in D. For this purpose we change to the w-plane. First we observe that by $|\kappa_{F\infty}(w)| \le \overline{\lim}_{n\to\infty} |\kappa_{F_n}(w)|$ a.e. in D we conclude that $\kappa_{F_\infty}(w) = 0$ a.e. in E, hence $\kappa_{f_\infty}(z) = 0$ a.e. in $F_\infty(E)$. Next we use the fact that E is closed. Let $z_0 \in D\backslash F_\infty(E)$. There is a neighborhood U_{z_0} of z_0 with $U_{z_0} c D\backslash F_\infty(E)$ and by the local uniform convergence of F_n to F_∞ we find an open disk D_{z_0} with center z_0 in U_{z_0} such that for a number $n_0 \in \mathbb{N}$

$$D_{z_0} c D\backslash F_n(E) \quad \forall n \ge n_0.$$

We hence infer that

$$\kappa_n(z) = k_n \frac{\bar{\varphi}_n(z)}{|\varphi_n(z)|} \xrightarrow{n \to \infty} k_F \frac{\overline{\varphi_\infty}(z)}{|\varphi_\infty(z)|} \quad \text{a.e. in } D_{z_0}.$$

Again by Theorem 5.2 in [LV] we get

$$\kappa_{f_\infty}(z) = k_F \frac{\overline{\varphi_\infty}(z)}{|\varphi_\infty(z)|} \quad \text{a.e. in } D_{z_0}$$

and since z_0 was arbitrary in $D \backslash F_\infty(E)$, we again have (4.1) from which the result follows.

5. Appendix

As has been pointed out to me by K. Sakan, the equation (2.2) in Lemma 2.1 is not at all a triviality. So let me add a proof here. There are several ways to do it, e.g. one could infer this statement from results on the weak-convergence of Jacobians of qc mappings (see [L]). I prefer here to use a consequence of a result on area-distortion by Gehring and Reich [GR].

Let us denote the Jacobian of a qc mapping f by J_f. For a number $K \geq 1$ let F be the set of K-qc mappings of the unit disk D onto itself which fix the origin. From [GR] it then follows that the integrals $\iint_D J_f \, dx \, dy$ are uniformly absolutely continuous, i.e., for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\iint_E J_f \, dx \, dy < \varepsilon$$

for every $f \in F$ and every measurable set E in the disk with $|E| := \iint_E dx \, dy < \delta$.

This property will imply the

THEOREM 5.1. *Let f_n and f be K-qc selfmappings of the unit disk D where $f_n \xrightarrow{n \to \infty} f$ locally uniformly in D. Then we have for every measurable and bounded function φ in D*

$$\lim_{n \to \infty} \iint_D \varphi(z) J_{f_n}(z) \, dx \, dy = \iint_D \varphi(z) J_f(z) \, dx \, dy.$$

In case that φ_0 is continuous in $\bar{D}\backslash\sigma$ and E relatively compact in $\bar{D}\backslash\sigma$ we choose $\varphi = \varphi_0\chi_E$ (χ_E denotes the characteristic function of E), hence φ is bounded and from this theorem we derive that

$$\iint_E \varphi_0 J_{f_n}\,dx\,dy \xrightarrow{n\to\infty} \iint_E \varphi_0 J_f\,dx\,dy. \tag{5.1}$$

The equation (2.2) claims that

$$\iint_E |\varphi_n \circ G_n|\, J_{G_n}\,dx\,dy \xrightarrow{n\to\infty} \iint_E |\varphi_\infty \circ G_\infty|\, J_{G_\infty}\,dx\,dy$$

which clearly follows from (5.1) by putting $\varphi_0 = \varphi_\infty \circ G_\infty$ because $\varphi_n \circ G_n \to \varphi_\infty \circ G_\infty$ locally uniformly in $\bar{D}\backslash\sigma$.

Proof of Theorem 5.1. By [GR] we conclude that obviously also the integrals $\iint_D J_{f_n}\,dx\,dy$ are uniformly absolutely continuous. We first choose $\varphi = \chi_R$ where R is a rectangle whose closure is contained in D. Then the statement follows from Lebesgue's dominated convergence theorem since $\chi_{f_n(R)} \to \chi_{f(R)}$ a.e. in D. Hence for step-functions $s = \sum_{i=1}^N c_i\chi_{R_i}$ we have

$$\iint_D sJ_{f_n}\,dx\,dy \xrightarrow{n\to\infty} \iint_D sJ_f\,dx\,dy. \tag{5.2}$$

Finally let φ be measurable and bounded in D. Let s_m be a sequence of step-functions with $s_m(z) \xrightarrow{m\to\infty} \varphi(z)$ a.e. in D and M be a number with $|\varphi| \le M$ and $|s_m| \le M$ for all m. Let $\varepsilon > 0$ be given. By the uniform absolute continuity of the integrals $\iint_D J_{f_n}\,dx\,dy$, there is an $\eta > 0$ such that

$$\iint_E J_{f_n}\,dx\,dy < \varepsilon \quad \forall n \quad \text{and} \quad \iint_E J_f\,dx\,dy < \varepsilon$$

whenever E is a measurable set in D with $|E| < \eta$. By Egoroff's theorem there is a set $E_\eta \subset D$ with $|E_\eta| < \eta$ such that $s_m \xrightarrow{m\to\infty} \varphi$ uniformly on $D\backslash E_\eta$. We choose m_0 such that

$$|s_m(z) - \varphi(z)| < \varepsilon \quad \forall z \in D\backslash E_\eta, \quad \forall m \ge m_0.$$

For an $m \geq m_0$ we then have

$$\iint_D \varphi J_{f_n}\,dx\,dy - \iint_D \varphi J_f\,dx\,dy = \iint_{D\setminus E_\eta} (\varphi - s_m)J_{f_n} + \iint_{E_\eta} (\varphi - s_m)J_{f_n}$$
$$+ \iint_D s_m(J_{f_n} - J_f) + \iint_{D\setminus E_\eta} (s_m - \varphi)J_f + \iint_{E_\eta} (s_m - \varphi)J_f.$$

By

$$\iint_{D\setminus E_\eta} |\varphi - s_m|\,J_{f_n}\,dx\,dy \leq \varepsilon\pi \quad \text{and} \quad \iint_{E_\eta} |\varphi - s_m|\,J_{f_n} \leq 2M\varepsilon$$

(also with J_f instead of J_{f_n}) we conclude from (5.2)

$$\overline{\lim_{n\to\infty}} \left| \iint_D \varphi(J_{f_n} - J_f)\,dx\,dy \right| \leq 2\varepsilon\pi + 4M\varepsilon$$

and $\varepsilon \to 0$ proves the theorem.

REFERENCES

[F] R. FEHLMANN, *Ueber extremale quasikonforme Abbildungen.* Comment. Math. Helv. *56*, 558–580 (1981).

[G1] F. P. GARDINER, *The existence of Jenkins–Strebel differentials from Teichmüller theory.* Amer. J. Math. *99*, 1097–1104 (1975).

[G2] F. P. GARDINER, *On partially Teichmüller Beltrami differentials.* Michigan Math. J. *29*, 237–242 (1982).

[GR] F. W. GEHRING and E. REICH, *Area distortion under quasiconformal mappings.* Ann. Acad. Sci. Fenn. AI 388 (1966).

[H] R. S. HAMILTON, *Extremal quasiconformal mappings with prescribed boundary values.* Trans. Am. Math. Soc. *138*, 399–406 (1969).

[Kr] S. L. KRUSHKAL, *On the theory of extremal quasiconformal mappings.* Sibirsk. Mat. Zh. *10*, 573–583 (1969) (Russian); translated under the title *Extremal quasiconformal mappings,* Sib. Math. J. *10*, 411–418 (1969).

[Kü1] R. KÜHNAU, *Ueber gewisse Extremalprobleme der quasikonformen Abbildung.* Wiss. Z. d. Martin-Luther-Univ. Halle-Wittenberg, Math.-Nat. *13*, 35–40 (1964).

[Kü2] R. KÜHNAU, *Einige Extremalprobleme bei differentialgeometrischen und quasikonformen Abbildungen.* Math. Z. 94, 178–192 (1966).

[LV] O. LEHTO and K. I. VIRTANEN, *Quasiconformal mappings in the plane.* Berlin-Heidelberg-New York: Springer 1973.

[L] K. LESCHINGER, *Untersuchungen über Jacobi-Determinanten von zweidimensionalen quasikonformen Abbildungen.* Bonner Math. Schr. *72* (1974).

[R1] E. REICH, *Quasiconformal mappings of the disk with given boundary values.* In: *Proceedings of Seminars on Advances in Complex Function Theory* (Maryland 1973/74), 101–137. Lecture Notes in Math. 505. Berlin–Heidelberg–New York: Springer 1976.

[R2] E. REICH, *Quasiconformal mappings with prescribed boundary values and a dilatation bound.* Arch. Rational Mech. Anal. *68*, 99–112 (1978).

[RS] E. REICH and K. STREBEL, *Extremal quasiconformal mappings with given boundary values.* Contributions to Analysis, a collection of papers dedicated to Lipman Bers, 375–391. New York: Academic Press 1974.

[Sa] K. SAKAN, *Necessary and sufficient conditions for extremality in certain classes of quasiconformal mappings.* J. Math. Kyoto Univ. *26*, 31–37 (1986).

[St] K. STREBEL, *Ein Konvergenzsatz für Folgen quasikonformer Abbildungen.* Comment. Math. Helv. *44*, 469–475 (1969).

[T] O. TEICHMÜLLER, *Extremale quasikonforme Abbildungen und quadratische Differentiale.* Abh. Preuss. Akad. Wiss., math.-naturw. Kl. 22, 1–197 (1939).

University of Jyväskylä
Department of Mathematics
Seminaarinkatu 15
SF-40100 Jyväskylä
Finland

Received June 25, 1985