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Growth of the coefficient of quasiconformality and the boundary correspondence of automorphisms of a ball

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Abstract. A homeomorphism $f: B^n \to B^n$ of the unit ball in $R^n (n \geqslant 2)$ whose coefficient of quasiconformality in the ball of radius $r<1$ has asymptotic rate of growth $K(r) = \sup_{|x| \leqslant r} k(x, f) = O(\log (1/1-r))$ can be continued to a homeomorphism $\bar{f}: \bar{B}^n \to \bar{B}^n$ of the closed ball $\bar{B}^n$. For $n = 2$ this implies that the Caratheodory theory of prime ends for conformal mappings also holds for the class of homeomorphisms $f: B^2 \to D$ with $K(r) = O(\log (1/1-r))$.

The following theorem was recently given by Zorič [10]:
If $f: B^2 \to B^2$ is an automorphism of the unit disc B^2 such that

$$\int^1 \frac{dr}{(1-r)K(r)} = \infty, \qquad \int^1 K(r)\, dr < \infty,$$

where $K(r)$ is the coefficient of quasiconformality of f in the disc $B^2(r)$, then f can be extended to a *continuous* mapping $\bar{f}: \bar{B}^2 \to \bar{B}^2$ of the closed disc $\bar{B}^2$ into itself.

Zorič [10] also made the conjecture that the above theorem holds for $n \geqslant 3$ with $K^{n-1}(r)$ instead of $K(r)$.

In this paper we prove that every homeomorphism $f: B^n \to B^n$ of the unit ball $B^n (n \geqslant 2)$ such that $K(r) = O(\log (1/1-r))$, i.e. $K(r)$ increases as the logarithm, can be continued to a *homeomorphism* $\bar{f}: \bar{B}^n \to \bar{B}^n$ of the closed ball $\bar{B}^n$. We also give some consequences of this statement.

Turn to the precise formulations.

Let D and D' be regions in euclidean space R^n and $f: D \to D'$ a homeomorphism. The number

$$k(x, f) = \limsup_{t \to 0} \frac{\max\limits_{|y-x|=t} |f(y) - f(x)|}{\min\limits_{|y-x|=t} |f(y) - f(x)|}$$

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will be called the coefficient of quasiconformality of f at $x \in D$. If D is the unit ball B^n let

$$K_f(r) = K(r) = \sup_{|x| \leq r} k(x, f).$$

In connection with the sequel recall that the coefficient of quasiconformality of a homeomorphism is a Borel measurable function (cf. [8]).

The rest of the notation and terminology that we use here is generally the same as in [8].

LEMMA 1. *Let $f: B^n \to B^n$ be a homeomorphism with $\int^1 K^{n-1}(r)\,dr < \infty$. Then $g = f^{-1}$ has a continuous extension $\bar{g}: \bar{B}^n \to \bar{B}^n$ of the closed ball $\bar{B}^n$ into itself.*

Proof. Since $k(x, f)$ is bounded in every ball $B^n(r)$ of radius $0 < r < 1$, it follows that $k(y, g)$ is locally bounded and f is in the Sobolev space $W^1_{n,\mathrm{loc}}(B^n)$, i.e. ACL^n in the sense of [8], (cf. [8], 32.3). So, for coordinate functions g^i, $1 \leq i \leq n$, of g we have (cf. [5], [6]):

$$\int_{B^n} |\nabla g^i|^n\,dy \leq \int_{B^n} k^{n-1}(y, g) J(y, g)\,dy \leq \int_{B^n} k^{n-1}(x, f)\,dx$$
$$\leq \int_{S^{n-1}} d\omega_{n-1} \int_0^1 r^{n-1} K^{n-1}(r)\,dr \leq \omega_{n-1} \int_0^1 K^{n-1}(r)\,dr < \infty.$$

By the standard argument (for example in the same way as in proof of theorem 10.1 in [3]), one concludes the proof of the lemma.

LEMMA 2 (*fundamental lemma*). *Let F be a compact subset of the unit ball B^n, $b \in S^{n-1} = \partial B^n$ and Γ the family of all curves γ in B^n such that γ has a common point with F and contains b in its closure. Let $f: B^n \to D$ be a homeomorphism such that*

$$\int^1 \frac{dr}{(1-r)K(r)} = \infty, \tag{a}$$

and for some $m > 1$

$$\int_{1-t^m}^1 K^{n-1}(r)\,dr = 0(t) \quad \text{when} \quad t \to 0,\ (t > 0). \tag{b}$$

Then $M(\Gamma') = 0$, where $\Gamma' = f(\Gamma)$.

Proof. Let (r_k) be an increasing sequence in $[0, 1)$ such that $r_k \to 1$ when $k \to \infty$ and $F \subset B^n(r_0)$. Let Γ_k be the family whose elements are subcurves of elements of Γ that connect through the spherical ring $R_k = \{x \in R^n : 1 - r_k < |x - b| < 1 - r_{k-1}\}$ its boundary spheres $S^{n-1}(b, 1 - r_k)$ and $S^{n-1}(b, 1 - r_{k-1})$. The condition (a) (as well as (b)) implies $K(r) < \infty$ for $0 \leq r < 1$ and by theorem 32.3 in [8] a homeomorphism f is in the class $W^1_{n,\mathrm{loc}}(B^n)$. Consequently, families $\Gamma'_k = f(\Gamma_k)$ are separate and $\Gamma' > \Gamma'_k$ (cf. [8]). Therefore [2]

$$\frac{1}{M^{1-n}}(\Gamma') \geq \sum_{k=0}^{\infty} \frac{1}{M^{1-n}}(\Gamma'_k). \tag{2}$$

Standard arguments yield (cf. [4], Lemma 1)

$$M(\Gamma'_k) \leq \int_{R_k \cap B^n} \rho^n(x) k^{n-1}(x, f)\, dx,$$

for every ρ admissible for Γ_k. If for ρ we choose the extremal function of the ring R_k then we obtain

$$M(\Gamma'_k) \leq \frac{1}{\left(\log \dfrac{1 - r_{k-1}}{1 - r_k}\right)^n} \int_{R_k \cap B^n} \frac{k^{n-1}(x, f)}{|x - b|^n}\, dx. \tag{3}$$

Let $(t, \omega) \overset{P}{\mapsto} x$, $\omega \in S^{n-1}(b, 1)$ be the spherical coordinate system with origin in b. Let τ_m be the hypersurface defined by $x \in \tau_m$ if and only if $|x| = 1 - t^m$, where $m > 1$ is such that the condition (b) is satisfied. Denote by A_t the central projection from b of the set $S^{n-1}(b, t) \cap \bar{B}^n$ onto the unit sphere $S^{n-1}(b, 1)$, by $A'_t \subset A_t$ the projection of that part of the set $S^{n-1}(b, t) \cap B^n$ which lies inside of the surface τ_m and by A''_t the difference $A_t - A'_t$. Then, taking into account that $k^{n-1}(P(t, \omega)) \leq K^{n-1}(|P(t, \omega)|)$, we get

$$\begin{aligned} \int_{R_k \cap B^n} \frac{k^{n-1}(x, f)}{|x - b|^n}\, dx &\leq \int_{1 - r_k}^{1 - r_{k-1}} \frac{dt}{t} \int_{A_t \subset S^{n-1}(b,1)} k^{n-1}(P(t, \omega))\, dS^{n-1} \\ &\leq \int_{1 - r_k}^{1 - r_{k-1}} \frac{dt}{t} \int_{A_t} K^{n-1}(|P(t, \omega)|)\, dS^{n-1}. \end{aligned} \tag{4}$$

Further, for $1 - r_k \leq t \leq 1 - r_{k-1}$

$$\int_{A'_t} K^{n-1}(|P(t, \omega)|)\, dS^{n-1} \leq \omega_{n-1} K^{n-1}(1 - t_k^m), \qquad (t_k = 1 - r_k), \tag{5}$$

and for $0<t\leq 1-r_0$

$$\int_{A'_{t'}} K^{n-1}(|P(t,\omega)|)\,dS^{n-1}=\int_{S^{n-2}} dS^{n-2}\int_{\theta_\tau}^{\theta_S} K^{n-1}(r(t,\theta))\,d\theta,$$

where θ is the angle between the vectors $x-b$ and $-b$, $r(t,\theta)=|x|$ and θ_τ and θ_S correspond to these points of $S^{n-1}(b,t)$ that lie on τ_m and S^{n-1} respectively. It is easy to see that for $t\neq 0$

$$d\theta=\frac{1-2t\cos\theta+t^2}{rt\sin\theta}\,dr.$$

Consequently, there exist $0<t'<1$ and a constant $c>0$ such that

$$d\theta\leq c\frac{dr}{t}\quad\text{for}\quad 0<t\leq t',\qquad \theta_\tau<\theta<\theta_S(\theta_S<\pi).$$

So we have for $0<t\leq t'$

$$\int_{A'_{t'}} K^{n-1}(|P(t,\omega)|)\,dS^{n-1}\leq\frac{c\omega_{n-2}}{t}\int_{r(t,\theta_\tau)}^{1} K^{n-1}(r)\,dr, \tag{6}$$

with $r(t,\theta_\tau)=1-t^m$. According to (b) there exist $0<t''<1$ and $c_1>0$ such that

$$\int_{1-t^m}^{1} K^{n-1}(r)\,dr\leq c_1 t\quad\text{for}\quad 0<t\leq t''. \tag{7}$$

Let $t_0=\min\{t',t''\}$. Then from (6) and (7) it follows that

$$\int_{A''_t} K^{n-1}(|P(t,\omega)|)\,dS^{n-1}\leq\tilde{c}\omega_{n-2}, \tag{8}$$

for $0<t\leq t_0$ and some $\tilde{c}>0$. From (3), (4), (5) and (8) it follows that there exist $C>0$ and $0\leq R_0<1$ such that

$$M(\Gamma'_k)\leq C\frac{K^{n-1}(1-(1-r_k)^m)}{\left(\log\dfrac{1-r_{k-1}}{1-r_k}\right)^{n-1}} \tag{9}$$

whenever $r_{k-1} \geqslant R_0$. From (2) and (9) one gets

$$M^{1/1-n}(\Gamma') \geqslant C^{1/1-n} \sum_{r_{k-1} \geqslant R_0} \frac{\ln(1-r_{k-1}) - \ln(1-r_k)}{K(1-(1-r_k)^m)}$$

for every increasing sequence (r_k), $r_k \to 1$. It follows that

$$M^{1/1-n}(\Gamma') \geqslant M \int_R^1 \frac{dr}{(1-r)K(1-(1-r)^m)},$$

for some $M > 0$ and $R \geqslant 0$. Changing variable by $1-(1-r)^m = u$ we finally have

$$M^{1/1-n}(\Gamma') \geqslant \frac{M}{m} \int^1 \frac{dr}{(1-r)K(r)}. \tag{10}$$

If $M(\Gamma') > 0$ it follows from (10) that the integral in (a) converges. This yields a contradiction and the proof of the lemma is complete.

LEMMA 3. *Let $f: B^n \to D$ be a homeomorphism such that $\int^1 K^{n-1}(r)\,dr < \infty$. Then D is a proper subset of R^n.*

Proof. Suppose on the contrary that $D = R^n$. Let p, q be two different points of the unit sphere $S^{n-1} = \partial B^n$, let s be a fixed element of $(0, 1)$ and Γ the family of curves which through B^n join the segments $[sp, p)$ and $[sq, q)$. Let a be the distance between the points sp and sq. Then the function $x \mapsto \rho(x) = 1/a$ is admissible for Γ. Let $\Gamma' = f(\Gamma)$. Then we have

$$\begin{aligned} M(\Gamma') &\leqslant \int_{B^n} \rho^n(x) k^{n-1}(x, f)\,dx \leqslant \frac{1}{a^n} \int_{B^n} k^{n-1}(x, f)\,dx \\ &\leqslant \frac{\omega_{n-1}}{a^n} \int_0^1 K^{n-1}(r)\,dr < \infty. \end{aligned}$$

On the other side, since the modulus of curve family is a conformal invariant, we can suppose that Γ' is the family of curves which join two arcs that begin in the same point of R^n. This implies $M(\Gamma') \geqslant c \log(b/t)$ for a fixed b and each $0 < t \leqslant b$ (cf. [8], 10.12). This is a contradiction.

THEOREM 1. *If $f: B^n \to B^n$ is a homeomorphism with $K(r) = O(\log(1/1-r))$ than f can be extended to a homeomorphism $\bar{f}: \bar{B}^n \to \bar{B}^n$ of the closed ball $\bar{B}^n$.*

Proof. Let $b \in S^{n-1} = \partial B^n$. Let $C(f, b)$ be the cluster set of f at b. Let F, Γ, Γ' be as in Lemma 2, F being connected and having more than one point. Since the family $\Delta(f(F), C(f, b), B^n)$ of all curves that join $f(F)$ and $C(f, b)$ through B^n is a subfamily of Γ', because of the monotonicity of the modulus and Lemma 2, we obtain $M(\Delta(f(F), C(f, b), B^n) = 0$. (If $K(r) = O(\log(1/1-r))$ then the conditions (a) and (b) of Lemma 2 are satisfied). Since $C(f, b)$ is connected this means that $C(f, b)$ has exactly one point. It follows that f has a continuous extension $\bar{f}: \bar{B}^n \to \bar{B}^n$. On the base of Lemma 1 we conclude that f is a homeomorphism.

Remarks. 1) Theorem 1 was in fact proved under the hypothesis (a) and (b) of Lemma 2. But the condition (b) is slightly stronger than the condition $\int^1 K^{n-1}(r)\,dr < \infty$ in [10]. 2) If $K(r)$ increases faster than $\log(1/1-r)$ then Theorem 1 does not hold. According to [10] for every nondecreasing function h such that

$$\int^1 \frac{dr}{(1-r)h(r)} < \infty \quad \text{or} \quad \int^1 \frac{dr}{(1-r)h(r)} = \infty \quad \text{and} \quad \int^1 h^{n-1}(r)\,dr = \infty$$

there exists a diffeomorphism $f: B^n \to B^n$ with $K(r) \leq h(r)$ having no continuous extension from $\bar{B}^n$ into itself. 3) Theorem 1 also holds (under the conditions (a) and (b)) if we replace the ball B^n in the range by a region D which has property P_2 on the boundary (cf. [8], 17.5 and 17.15).

It was pointed out in [10] that the question about boundary behavior of different classes of homeomorphisms in the plane is reduced, from the metrical-point of view, to the study of boundary behavior of automorphisms of a disc B^2. (This is a consequence of the Riemann mapping theorem and the Caratheodory theory of prime ends).

THEOREM 2. *For the class of locally quasiconformal mappings* $f: B^2 \to D$ *which satisfy the condition* $K(r) = O(\log(1/1-r))$ *the Caratheodory theory of prime ends holds.*

Proof. It is enough to show that a region D is conformally equivalent to the unit disc B^2. But that is a consequence of Lemma 3.

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