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Autor(en): **Pfister, A. / Stolz, S.**

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On the level of projective spaces

A. Pfister and S. Stolz

Introduction

In this note we provide estimates for the level of projective spaces. Parts of these results are known [V], but we have not seen complete proofs in the literature. We hope to stimulate interest in the question of calculating the exact level which seems to be both attractive and difficult.

By definition (see [DL]), the level of a topological space X with a fixed point free involution i is the number

$$s(X, i) = \min \{n : \text{there exists a } \mathbb{Z}/2\text{-equivariant map } f : X \to S^{n-1}\}.$$

Here we think of X and the standard $(n-1)$-sphere S^{n-1} as spaces with a $\mathbb{Z}/2$-action given by the involution i resp. the antipodal involution.

A continuous map $f : Y \to Z$ between two spaces with G-action is called G-equivariant if $f(gy) = gf(y)$ for all $g \in G$, $y \in Y$. The Borsuk–Ulam theorem implies $s(S^{n-1}, -) = n$ for any $n \geq 1$. Further results on the level, in particular for Stiefel manifolds, are to be found in [DL]. In the earlier topological literature (see e.g. [CF]) the number $s - 1$ was used under the name "coindex".

In this paper we consider real (resp. complex) projective spaces $\mathbb{R}P^n$ (resp. $\mathbb{C}P^n$). For n even there are no fixed point free involutions on $\mathbb{R}P^n$ and $\mathbb{C}P^n$ since their Euler characteristic is odd, but the Euler characteristic of a space with a fixed point free involution is twice the Euler characteristic of its quotient space (see for example [S, p. 481, Th. 1]). On $\mathbb{R}P^{2m-1}$ (resp. $\mathbb{C}P^{2m-1}$) we have the following fixed point free involution i (resp. j): To define i write $\mathbb{R}P^{2m-1}$ as $\mathbb{R}P^{2m-1} = S^{2m-1}/\mathbb{Z}/2$ where $\mathbb{Z}/2$ acts by the antipodal involution. Multiplication with the complex number i gives a $\mathbb{Z}/4$-action on $S^{2m-1} \subset \mathbb{C}^m$ which induces a $\mathbb{Z}/2$-action on the quotient space $\mathbb{R}P^{2m-1}$. Analogously left multiplication by the quaternion j gives a $\mathbb{Z}/4$-action on $S^{4m-1} \subset \mathbb{H}^m$ ($\mathbb{H}$ = quaternions), which induces a $\mathbb{Z}/2$-action on the quotient space $\mathbb{C}P^{2m-1} = S^{4m-1}/S^1$.

THEOREM. *$m + 1 \leq s(\mathbb{R}P^{2m-1}, i) \leq \frac{1}{2}(3m + 1)$ and $2m + 1 \leq s(\mathbb{C}P^{2m-1}, j) \leq 3m$ for all $m \in \mathbb{N}$.*

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1. The lower estimate for $s(\mathbb{R}P^{2m-1}, i)$

We obtain restrictions to the existence of a $\mathbb{Z}/2$-equivariant map $f:\mathbb{R}P^{2m-1}\to S^{n-1}$ by analyzing the corresponding map between orbit spaces $\bar{f}:\mathbb{R}P^{2m-1}/\mathbb{Z}/2\to S^{n-1}/\mathbb{Z}/2$ and its induced map in K-theory. This gives the lower estimate for m even. Finally the case m odd is reduced to the case m even by taking the join of two equivariant maps.

We use the following notation: Given a $\mathbb{Z}/2$-equivariant map $f:\mathbb{R}P^{2m-1}\to S^{n-1}$, we denote by $\tilde{f}:S^{2m-1}\to S^{n-1}$ the composition of f with the projection map $S^{2m-1}\to\mathbb{R}P^{2m-1}$. The map $\tilde{f}$ is $\mathbb{Z}/4$-equivariant with respect to the action of $\mathbb{Z}/4$ on S^{2m-1} (resp. S^{n-1}) given by multiplication by i (resp. -1). We denote by $\bar{f}:S^{2m-1}/\mathbb{Z}/4\to S^{n-1}/\mathbb{Z}/4=\mathbb{R}P^{n-1}$ the map induced by $\tilde{f}$ on the orbit spaces. The orbit space $S^{2m-1}/\mathbb{Z}/4$ is an example of a lens space and is usually denoted by $L^{2m-1}(4)$. More generally the lens space $L^{2m-1}(k)$, $k\in\mathbb{N}$, is defined to be the orbit space of the $\mathbb{Z}/k$-action on $S^{2m-1}\subset\mathbb{C}^m$ given by multiplication by the k-th root of unity $e^{2\pi i/k}\in\mathbb{C}$. Note that for $n=2t$, $\mathbb{R}P^{n-1}$ is the lens space $L^{2t-1}(2)$. Let $H\searrow\mathbb{C}P^{m-1}$ be the Hopf bundle over the complex projective space and let $H_k\searrow L^{2m-1}(k)$ be the pull back of H be the natural projection map $L^{2m-1}(k)=S^{2m-1}/\mathbb{Z}/k\to\mathbb{C}P^{m-1}=S^{2m-1}/S^1$. Denote by $\eta\in K(\mathbb{C}P^{m-1})$ (resp. $\eta_k\in K(L^{2m-1}(k))$) the elements represented by H (resp. H_k).

The complex K-theory of lens spaces is well-known (see for example [K, p. 192, 2.12]):
As a ring $K(L^{2m-1}(k))=\mathbb{Z}[\sigma_k]/(1-(1+\sigma_k)^k, \sigma_k^m)$, *where* $\sigma_k=\eta_k-1$.

LEMMA. *Let* $f:\mathbb{R}P^{2m-1}\to S^{2t-1}$ *be a* $\mathbb{Z}/2$-*equivariant map and let* $\bar{f}:L^{2m-1}(4)\to L^{2t-1}(2)$ *be the induced map of orbit spaces. Then* $\bar{f}^*(\eta_2)=\eta_4\cdot\eta_4$.

Proof. We have to construct a bundle map $\bar{F}:H_4\otimes H_4\to H_2$ covering $\bar{f}$. To do this we use the explicit description of $H_k\searrow L^{2m-1}(k)$ and $H\searrow\mathbb{C}P^{m-1}$ as "associated vector bundles": let G be the group $\mathbb{Z}/k$ or S^1 and let $\rho:G\times V\to V$ be a representation of G on some complex vector space V. Then the projection map $S^{2m-1}\times V\to S^{2m-1}$ is G-equivariant with respect to the diagonal action on $S^{2m-1}\times V$. The induced map on the orbit spaces $(S^{2m-1}\times V)/G\to S^{2m-1}/G$ is the projection map of a complex vector bundle called the vector bundle associated to the representation ρ. With this terminology, the Hopf bundle H over $\mathbb{C}P^{m-1}=S^{2m-1}/S^1$ is the bundle associated to the standard 1-dimensional representation $\alpha:S^1\times\mathbb{C}\to\mathbb{C}$, $(z,v)\mapsto z\cdot v$, and $H_k\searrow L^{2m-1}(k)=S^{2m-1}/\mathbb{Z}/k$ is obtained by restricting α to a representation α_k of $\mathbb{Z}/k \subset S^1$. The above construction is compatible with tensor products. In particular $H_4 \otimes H_4 \searrow L^{2m-1}(4)$ is the vector bundle associated to the representation $\alpha_4 \otimes \alpha_4 : \mathbb{Z}/4 \times \mathbb{C} \otimes \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$. It is easy to check that

$$\begin{array}{llll} \bar{F}: H_4 \otimes H_4 = (S^{2m-1} \times \mathbb{C} \otimes \mathbb{C})/\mathbb{Z}/4 & \to & H_2 = (S^{2t-1} \times \mathbb{C})/\mathbb{Z}/2 \\ \qquad [\ x, \quad z \otimes z'] & \mapsto & [\bar{f}(x), \quad z \cdot z'] \end{array}$$

is a well defined bundle map covering f. Q.E.D.

For abbreviation write $s(m) = s(\mathbb{R}P^{2m-1}, i)$. We now proceed to the proof of $s(m) > m$ for m even: Let $f: \mathbb{R}P^{2m-1} \to S^{2t-1}$ be a $\mathbb{Z}/2$-equivariant map, where t is as small as possible, i.e. $2t = s(m)$ resp. $s(m) + 1$ for $s(m)$ even resp. odd. The lemma implies $\bar{f}^*(\sigma_2) = \bar{f}^*(\eta_2 - 1) = \eta_4 \cdot \eta_4 - 1 = (1 + \sigma_4)^2 - 1$ and hence the (additive) order of $(1 + \sigma_4)^2 - 1 \in \mathbb{Z}[\sigma_4]/(1 - (1 + \sigma_4)^4, \sigma_4^m)$ is a divisor of the order of $\sigma_2 \in \mathbb{Z}[\sigma_2]/(1 - (1 + \sigma_2)^2, \sigma_2^t)$. The following computation of the orders of $(1 + \sigma_4)^2 - 1$ (resp. σ_2) then implies $t - 1 \geq [m/2]$ and hence $s(m) \geq 2t - 1 \geq m + 1$ for m even (for a real number a, $[a]$ is defined as $\max\{n \in \mathbb{Z} \mid n \leq a\}$).

LEMMA. i) *The additive order of* $(1 + \sigma_4)^2 - 1 \in \mathbb{Z}[\sigma_4]/(1 - (1 + \sigma_4)^4, \sigma_4^m)$ *is* $2^{[m/2]}$

ii) *The additive order of* $\sigma_2 \in \mathbb{Z}[\sigma_2]/(1 - (1 + \sigma_2)^2, \sigma_2^t)$ *is* 2^{t-1}.

Proof. i) To evaluate the order of an element of $\mathbb{Z}[\sigma_4]/(1 - (1 + \sigma_4)^4, \sigma_4^m)$ represented by a polynomial $p \in \mathbb{Z}[\sigma_4]$ with vanishing constant term we expand the quotient $p/(1 - (1 + \sigma_4)^4)$ as a power series $\sum_{i=0}^{\infty} a_i \sigma_4^i$, $a_i \in \mathbb{Q}$. Then p is contained in the ideal generated by $1 - (1 + \sigma_4)^4$ and σ_4^m if and only if $a_0, \ldots, a_{m-2}$ are integers. Hence the order of p is given by l.c.m. {denominators of $a_0, \ldots, a_{m-2}$}.

In particular for $p = (1 + \sigma_4)^2 - 1$ we obtain:

$$\begin{aligned} \frac{(1+\sigma_4)^2 - 1}{1 - (1+\sigma_4)^4} &= \frac{-1}{1 + (1+\sigma_4)^2} = \frac{-1}{2 + 2\sigma_4 + \sigma_4^2} = \frac{-(2 - 2\sigma_4 + \sigma_4^2)}{(2 + \sigma_4^2)^2 - 4\sigma_4^2} \\ &= \tfrac{1}{4}(-2 + 2\sigma_4 - \sigma_4^2) \frac{1}{1 + \frac{\sigma_4^4}{4}} = \tfrac{1}{4}(-2 + 2\sigma_4 - \sigma_4^2) \sum_{k=0}^{\infty} (-1)^k \frac{1}{2^{2k}} \sigma_4^{4k}, \end{aligned}$$

$$\text{order of } (1+\sigma_4)^2 - 1 = \begin{cases} 2^{2k+1} & \text{for } m - 2 = 4k,\ 4k+1 \\ 2^{2k+2} & \text{for } m - 2 = 4k+2,\ 4k+3 \end{cases} = 2^{[m/2]}$$

ii)

$$\frac{\sigma_2}{1-(1+\sigma_2)^2}=\frac{-1}{2+\sigma_2}=-\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\frac{1}{2^k}\sigma_2^k,$$

order of $\sigma_2 = \text{l.c.m.}\ \{2^{k+1}, 0 \le k \le t-2\} = 2^{t-1}$. Q.E.D.

To complete the proof of $s(m) \ge m+1$ for m odd, recall the definition of the join $X * Y$ of two topological spaces $X, Y: X * Y$ is the quotient space of $X \times [0, 1] \times Y$ by the equivalence relation $(x, 0, y) \sim (x', 0, y)$, $(x, 1, y) \sim (x, 1, y')$ for all $x, x' \in X$, $y, y' \in Y$. Given two real (resp. complex) vector spaces V, W the join $S(V) * S(W)$ of the corresponding spheres can be identified with $S(V \oplus W)$. Moreover, this identification is compatible with the $\mathbb{Z}/2$-action given by multiplication by -1 (resp. the $\mathbb{Z}/k$-action given by multiplication by $e^{2\pi i/k}$). In particular, if there were a $\mathbb{Z}/2$-equivariant map $f:\mathbb{R}P^{2m-1} \to S^{m-1}$ we would obtain, by taking the join of the $\mathbb{Z}/4$-equivariant map $\tilde{f}: S^{2m-1} \to S^{m-1}$ with itself, a $\mathbb{Z}/4$-equivariant map $\tilde{f} * \tilde{f}: S^{2m-1} * S^{2m-1} = S^{4m-1} \to S^{m-1} * S^{m-1} = S^{2m-1}$ and hence by passing to the quotient a $\mathbb{Z}/2$-equivariant map $\mathbb{R}P^{4m-1} \to S^{2m-1}$ contradicting our estimate $s(2m) \ge 2m+1$.

2. The upper estimate for $s(\mathbb{R}P^{2m-1}, i)$

For this we use quadratic forms in order to construct an equivariant map $f:\mathbb{R}P^{2m-1} \to S^{r-1}$ with $r = 3t$ for $m = 2t$ and $r = 3t+2$ for $m = 2t+1$. We use the following notation:

$\varphi_1, \dots, \varphi_r$ are complex quadratic forms in the complex variables $z_1, \dots, z_m$. The imaginary part $\operatorname{Im}\varphi_k(z_1, \dots, z_m)$ is a real quadratic form in the real variables x_j, y_j $(j = 1, \dots, m)$, denoted by $q_k(z) = q_k(x, y)$ $(k = 1, \dots, r)$. Obviously, $q_k(iz) = -q_k(z)$ and $q_k(\lambda z) = \lambda^2 q_k(z)$ for $\lambda \in \mathbb{R}^*$. Suppose that the system $q = (q_1, \dots, q_r)$ is anisotropic, i.e. $q_1(z) = \cdots = q_r(z) = 0$ implies $z = 0$ resp. $x = y = 0$. Then q defines an equivariant map $f:\mathbb{R}P^{2m-1} \to S^{r-1}$ given by $f([z]) = q(z)/\|q(z)\|$. It remains to find appropriate forms $\varphi_1, \dots, \varphi_r$.

a) $m = 1$, $r = 2$: Take $\varphi_1(z) = z_1^2$, $\varphi_2(z) = iz_1^2$. Then $q_1(z) = q_2(z) = 0$ implies that φ_1 is both real and pure imaginary, hence $\varphi_1 = 0$ and $z_1 = 0$. Thus, the system $q = (q_1, q_2)$ is anisotropic.

b) $m = 2$, $r = 3$: Take $\varphi_1(z) = 2z_1z_2$, $\varphi_2(z) = z_1^2 - z_2^2$, $\varphi_3(z) = i(z_1^2 + z_2^2)$. These 3 forms satisfy the identity $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$. Therefore $q_1(z) = q_2(z) = q_3(z) = 0$ implies that $\varphi_1(z)$, $\varphi_2(z)$, $\varphi_3(z)$ are real and then $\varphi_1(z) = \varphi_2(z) = \varphi_3(z) = 0$ which gives $z_1 = z_2 = 0$. This shows that the system $q = (q_1, q_2, q_3)$ is anisotropic.

Remark. The induced equivariant map $\mathbb{R}P^3 \to S^2$ is related to (but not directly derived from) the well-known Hopf map $S^3 \to S^2$.

c) $m = 2t$, $r = 3t$: Take (for $k = 1, \ldots, t$)

$$\varphi_{3k-2}(z) = 2z_{2k-1}z_{2k},\ \varphi_{3k-1}(z) = z_{2k-1}^2 - z_{2k}^2,\ \varphi_{3k}(z) = i(z_{2k-1}^2 + z_{2k}^2)$$

Part b) shows that the induced real system $q = (q_1, \ldots, q_r)$ is anisotropic.

d) $m = 2t + 1$, $r = 3t + 2$: Take $\varphi_1, \ldots, \varphi_{3t}$ as in c) (depending only on the variables $z_1, \ldots, z_{2t}$) and take $\varphi_{3t+1}(z) = z_m^2$, $\varphi_{3t+2}(z) = iz_m^2$ as in a). Then clearly the induced system $q = (q_1, \ldots, q_r)$ is again anisotropic.

This proves the upper estimate $s(m) \le \frac{1}{2}(3m + 1)$ for all $m \in \mathbb{N}$.

3. The estimate for $s(\mathbb{C}P^{2m-1}, j)$

Let $z = (z_1, z_2, \ldots, z_{2m-1}, z_{2m}) \in \mathbb{C}^{2m}$ and let $q(z) = (z_1\bar{z}_2, |z_1|^2 - |z_2|^2, \ldots) \in (\mathbb{C} \times \mathbb{R})^m = \mathbb{R}^{3m}$. q induces a map $f: \mathbb{C}P^{2m-1} \to S^{3m-1}$ defined by $f([z]) = q(z)/\|q(z)\|$. f is $\mathbb{Z}/2$-equivariant with respect to the involutions j resp. - since $j([z]) = [(-\bar{z}_2, \bar{z}_1, \ldots,)]$. This implies the upper estimate $s(\mathbb{C}P^{2m-1}, j) \le 3m$.

To prove the lower estimate denote by $m_h: S^{4m-1} \to S^{4m-1} \subset \mathbb{H}^m$ the map given by left multiplication by a quaternion $h \in \mathbb{H}$ of norm 1. For $h = (1 + ij)/\sqrt{2}$ the identity $jh = hi$ shows that the map

$$m_h: S^{4m-1} \to S^{4m-1}$$

is $\mathbb{Z}/4$-equivariant with respect to the $\mathbb{Z}/4$-action given by m_i on the domain and m_j on the range. If $f: \mathbb{C}P^{2m-1} \to S^{t-1}$ is any $\mathbb{Z}/2$-equivariant map with respect to j,- the composition

$$S^{4m-1} \xrightarrow{m_h} S^{4m-1} \xrightarrow{pr} \mathbb{C}P^{2m-1} \xrightarrow{f} S^{t-1}$$

is $\mathbb{Z}/4$-equivariant with respect to m_i,-. Dividing by the action of $\mathbb{Z}/2 \subset \mathbb{Z}/4$ we obtain a $\mathbb{Z}/2$-equivariant map

$$(\mathbb{R}P^{4m-1}, i) \to (S^{t-1}, \text{-})$$

which implies $t \ge s(\mathbb{R}P^{4m-1}, i) \ge 2m + 1$ and thus $s(\mathbb{C}P^{2m-1}, j) \ge 2m + 1$.

4. Additional remarks (to the real case)

1) Our estimates for $s(m) = s(\mathbb{R}P^{2m-1}, i)$ imply $s(1) = 2$, $s(2) = 3$, $4 \leq s(3) \leq 5$, $5 \leq s(4) \leq 6$. Conner–Floyd [CF] proved $s(3) = 5$. For $m > 3$ the exact value of $s(m)$ seems to be unknown.

2) One can define a purely algebraic invariant $r(m)$ as follows: Call a system $q = (q_1, \ldots, q_r)$ of real quadratic forms in an even number $2m$ of variables x_j, y_j $(j = 1, \ldots, m)$ "induced" if there exist complex quadratic forms $\varphi_1, \ldots, \varphi_r$ in the m variables $z_j = x_j + y_j i$ such that $q_k(x, y) = \operatorname{Im} \varphi_k(z)$ $(k = 1, \ldots, r)$. Define $r(m) = \min \{r : \text{there exists an induced anisotropic system } q = (q_1, \ldots, q_r) \text{ in } 2m \text{ variables}\}$. Then it can be proved by purely algebraic methods that

$$m + 1 \leq r(m) \leq \tfrac{1}{2}(3m + 1)$$

The upper estimate has been given in part 2, the lower estimate can easily be derived from Hilbert's nullstellensatz. (Compare [P] where the inequality $[2r/3] \leq m(r) < r$ for $m(r) = \frac{1}{2}u_r'(\mathbb{R})$ is equivalent to the above inequality for $r(m)$).

It is obvious that $s(m) \leq r(m)$ and it is tempting to conjecture $s(m) = r(m)$. However, we see no reason that the invariant $r(m)$ should be better accessible than the invariant $s(m)$.

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Fachbereich Mathematik
der Universität Mainz
Postfach 3980
D-6500 Mainz

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