Zeitschrift: Commentarii Mathematici Helvetici

Band: 62 (1987)

Artikel: Generalized Weierstrass ...-functions and KP flows in affine spaces.

Autor: Previato, Emma

DOI: https://doi.org/10.5169/seals-47350

Comment. Math. Helvetici 62 (1987) 292-310 0010-2571/87/020292-19$01.50 + 0.20/0

Generalized Weierstrass $\wp$-functions and KP flows in affine space

EMMA PREVIATO

The surprising discovery that the flows of the KP equation:

$$\tfrac{3}{4}u_{yy} - (u_t - \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x)_x = 0$$

linearize on Jacobians of curves ([7], [11]) was based on the (forgotten and newly proved) theory of Burchnall and Chaundy: if a curve C is viewed as the common "spectrum" (cf. [3]) of a commutative ring $\mathscr{A}$ of ordinary differential operators, then an affine subset of its Jacobian can be parametrized by isospectral deformations of $\mathscr{A}$. The flows of the KP hierarchy are an example of isospectral deformations of $\mathscr{A}$ and they correspond to (translation) invariant vector fields on Jac C: in fact they span the tangent bundle.

In the case of the KdV equation, the "2-reduction" of KP (cf. §3), the flows were made explicit [12] by the use of an algebraic parametrization of Jac $C\backslash\Theta$ (where Θ is a theta divisor) that dates back to Jacobi; indeed the corresponding curve is hyperelliptic, so after removing a branchpoint P_∞ it has an affine model $\mu^2 = \prod_{i=1}^{2g+1}(\lambda - e_i) = F(\lambda)$ and a nonspecial divisor $D = \sum_{i=1}^{g} P_i$ on $C\backslash P_\infty$ is equivalently given by a pair of polynomials in λ;

$$U_D(\lambda) = \prod_{i=1}^{g}(\lambda - \lambda(P_i)), \qquad V_D(\lambda) = \sum_{i=1}^{g} \frac{\mu(P_i)U_D(\lambda)}{U'_D(\lambda(P_i))(\lambda - \lambda(P_i))}$$

(with dashes indicating λ-derivatives), obviously modified if $\lambda(P_i)$ is a multiple root of U. The KdV evolutions are described by algebraic equations in the coefficients of U, V, which moreover play the role of "hyperelliptic $\wp$-functions"; this term was coined in [12], to signify the following analogy: if C is an elliptic curve, the Weierstrass $\wp$-function is a meromorphic function on Jac C which gives, together with its derivative, an affine embedding for Jac C: a cubic equation $\Phi(\wp, \wp') = 0$ defines Jac $C\backslash\Theta$ (here Θ is a point) in $\mathbf{C}^2$. Now we let C be hyperelliptic and associate to the divisor D the triple of polynomials (U, V, W) where U and V are as above and W is determined by U, V, F through euclidean division so that $UW + V^2 = F(\lambda)$; then the *coefficients* of U, V, W are meromorphic functions on Jac C having poles, of various order, exactly on Θ and they provide an embedding of $\operatorname{Jac} C\backslash\Theta$ in $\mathbf{C}^{3g+1}$; indeed, $\operatorname{Jac} C\backslash\Theta$ was shown by Mumford ([12]) to be isomorphic to the affine subvariety of $\mathbf{C}^{3g+1}$ given by the following equations on the coefficients of U, V, W : U, W are monic of degrees g, $g+1$ resp., $\deg V \leq g-1$ and $UW+V^2=F(\lambda)$.

The question this paper addresses is, what do the polynomials U, V, W become for a general curve? We find an answer by spotlighting the link between the ring $\mathscr{A}$ of ODO and the divisor D in $\operatorname{Jac} C\backslash\Theta$. Our main tool is a matrix, which we call the *BC matrix* after Burchnall and Chaundy. We sketch its definition here (for details cf. §1) so as to illustrate our results.

Say $\mathscr{A}$ is generated by (monic) elements $L_1, \ldots, L_s$, of orders $m_1, \ldots, m_s$. Then $L_j-\lambda_j$, $j=2, \ldots, s$, act on the solution space of $L_1-\lambda_1$; if we choose a basis for that space, these actions are recorded by the "BC matrix" E_0^*, whose coefficients are polynomials in the parameters $\lambda_1, \ldots, \lambda_s$; the matrix provides affine equations for C minus one point P_∞ in the coordinates $\lambda_1, \ldots, \lambda_s$; affine coordinates for $\operatorname{Jac} C\backslash\Theta$ are given by the coefficients of all $(m_1-1)\times(m_1-1)$ minors of E_0^*. The fact that the curve and the divisor should be thus linked to the ring $\mathscr{A}$ is an easy consequence of the Burchnall–Chaundy–Krichever–Mumford theory ([3], [7], [11]); our contribution consists in the simple but useful observation that Jacobi's polynomials (U, V, W) are given by the 1×1 minors of the BC matrix in the hyperelliptic case (§2). We then generalize both the $\wp$-functions and the Jacobi polynomials:

(I) To generalize $\wp$ we show that on the Jacobian of any curve C of genus g we can find $g-1$ invariant vector fields $\partial_1, \ldots, \partial_{g-1}$ so that the map $(\partial_1^k\partial_\alpha \log\vartheta)_{\alpha,k}=J$: $\operatorname{Jac} C\backslash\Theta\to\mathbf{C}^N$ is an embedding, i.e. the functions $\partial_1^k\partial_\alpha\log\vartheta$ generate the function ring of $\operatorname{Jac} C\backslash\Theta$, where $1\leq k\leq N_\alpha(g)$, a linear polynomial in g (1.9 and following Note).

For comparison, we recall that the classical means of embedding *all of* $\operatorname{Jac} C$ in $\mathbf{P}^M$ requires $M=3^g$, $g=\text{genus } C$.

(II) The appropriate generalization of Jacobi's polynomials are the $m_1\binom{(s-1)m_1}{m_1-1}$ minors of the matrix E_0^* that are determinants of the $(m_1-1)\times(m_1-1)$ submatrices. For computational rather than conceptual ease we concentrate on the case in which the ring $\mathscr{A}$ can be generated by $s=2$ elements; this corresponds to a special class of curves, which we regard as a generalization of the hyperelliptic: we say that a curve has the *plane model property* (with respect to the point P_∞) when the ring $R_\infty=\{$meromorphic functions on C regular outside $P_\infty\}$ can be generated by two elements. Say r is the lower of their pole orders at P_∞.

To mimick Jacobi's construction, given a nonspecial divisor D on $C\backslash\{P_\infty\}$, we associate to it r^2 polynomials that obey suitable constraints; we prove that these give the adjoint of the BC matrix associated to D through the ring $\mathscr{A}_D$ (2.4), and observe that they lend themselves to a generalization of the geometrical addition rule for elliptic curves (2.7). We also prove that there is an inverse morphism from the set of complex polynomial matrices that obey those constraints to $\operatorname{Jac} C \setminus \Theta$. The upshot is a one-to-one map, in fact a morphism in both directions, between $\operatorname{Jac} C \setminus \Theta$ and the $r \times r$ complex polynomial matrices in two variables that drop in rank exactly on the curve C and whose entries satisfy a given set of constraints. Unfortunately, this doesn't give us an explicit affine model of $\operatorname{Jac} C \setminus \Theta$ as in the hyperelliptic case because we don't have a way for defining the constraints in general, but only an existence proof. Each case of fixed r and $g = \text{genus } C$ can of course be worked out to obtain affine equations for $\operatorname{Jac} C \setminus \Theta$ and we do so for two examples: $r = 3$, $g = 1$, 3 (cf. §3). These were chosen because $r = 3$ corresponds to 3-sheeted coverings of $\mathbf{P}^1$ (trigonal curves) as opposed to 2-sheeted (hyperelliptic); on the other hand, for $g = 1$ the curve is (hyper)elliptic, so one may compare our equations with those found for the hyperelliptic case by McKean and van Moerbeke ([9]) as solution to a variational problem.

The next two points might prove useful in studying explicit solutions of the KP flows. Indeed, while so far we were concerned with parametrizing $\operatorname{Jac} C \setminus \Theta$, the BC matrix as a function on $\operatorname{Jac} C$ also undergoes the hierarchy of KP evolutions (cf. §3). Notice that the "x-evolution" is already represented by the x-dependent BC matrix E^*, because it is given by x-translation in the ODO-ring $\mathscr{A}$, hence it corresponds to representing the $L_l - \lambda_l$ action on a different basis of the solution space for $L_1 - \lambda_1$, say a fundamental set normalized at x instead of 0. Obviously, this doesn't change the spectral curve of E_0^*.

(III) If C has the plane-model property, the (moving) divisor D is given by the intersection of $r \leqslant g$ moving algebraic curves. One of these curves yields the solution of the KP equation that corresponds to C in terms of algebraic functions (3.3); this may have the application that qualitative properties of the wave can be deduced from the dynamical behavior of the corresponding divisor. It should be interesting to study for which configurations of ovals of the given (real) curve the (real) solutions may be deformed into solitons: the picture is already very rich and complicated (cf. [8]) for the Boussinesq equation ($r = 3$ case). We have not yet pursued this direction.

(IV) In [12], Mumford writes the (algebraic) equations for the KdV flows in Jacobi's coordinates; to the same end, but in a different spirit, we write the KP equations on the generalized Jacobi polynomals; indeed, we translate the KP hierarchy into Lax-pair equations for the BC matrix, as well as an "infinitesimal generator" (3.2). We have thus converted the Lax-pair equations on coadjoint orbits of the Lie algebra of formal pseudo-differential operators (cf. [2]) into

Lax-pair equations for matrices in the formal loop algebra $\tilde{\mathfrak{g}}(r) = \mathfrak{gl}\ (r, \mathbf{C}) \otimes \mathbf{C}((\lambda^{-1}))$: this observation has the advantage that the x-variable is no longer singled out, as is remarked in [5] where the $\tilde{\mathfrak{g}}(r)$ model is given for $(r \times r)$-matrix (as opposed to scalar) hierarchies. Moreover, it is in this context that a generalization of the "Neumann system" is to be found; this system gives a way of interpreting Jacobi's hyperelliptic polynomials as functions on the phase space of a completely integrable system, whose flows in particular preserve the spectrum of a $(g + 1) \times (g + 1)$ matrix. In [1], we generalize that model to r-gonal curves. Through the BC matrix these systems can be reconciled with the KP flows; examples for $r = 3$ are to be found in [1] and [13].

A final comment: the presentation of a curve as an r-sheeted covering of $\mathbf{P}^1$ through a function with r fold pole at P_∞ and regular elsewhere is far from canonical; the general curve of genus g is r-gonal in the above sense for $r = g$ and no less, but the sublocus of, say, hyperelliptic curves will also be r-gonal for $r = 2$. If equations could be given on the $g \times g$ BC matrix to determine whether there is an alternative presentation of the curve for $r < g$, then these would be equations on the theta functions and derivatives along $\partial_1, \ldots, \partial_{g-1}$ to define that sublocus of special curves.

This work profited very much from the kind attention and suggestions of: B. Dwork, A. Mayer, H. McKean, and G. Wilson.

§1. The BC matrix as a function on the Jacobian; the generalized Weierstrass ℘-functions

In this section we recall how every curve can be described as the (compactification of the) joint spectrum of a ring of commuting operators; we write the defining equations for the joint spectrum in order to give affine equations for both the curve and the isospectral class of such rings, modulo conjugation by a function.

1.1. DEFINITION. Let $\mathcal{R}$ be the ring of differential operators with coefficients in $\mathbf{C}[[x]]$ (formal power series); the product is composition of operators and is denoted by "$\circ$"; $\mathcal{R} = \mathbf{C}[[X]][\partial]$ where $\partial = d/dx$, so that $\partial \circ u(x) = \dot{u}(x) + u(x)\partial$. For any pair of elements of $\mathcal{R}$ of the following type:

$$B = \partial^r + u_{r-1}\partial^{r-1} + \cdots + u_0$$

$$L = \partial^n + v_{n-1}\partial^{n-1} + \cdots + v_0$$

$(u_j, v_j \in \mathbf{C}[[x]])$, we form the "Burchnall–Chaundy" (BC) matrix $\Lambda(L; B)$ as follows. By "euclidean division" we expand $\partial^{i-1}\circ L$ in powers of B,

$$\begin{aligned} L &= S^{(1)}_{[n/r]}B^{[n/r]} + S^{(1)}_{[n/r]-1}B^{[n/r]-1} + \cdots + S^{(1)}_0 \\ \partial\circ L &= S^{(2)}_{[(n+1)/r]}B^{[(n+1)/r]} + \cdots + S^{(2)}_0 \\ &\vdots \\ \partial^{r-1}\circ L &= S^{(r)}_{[(n+r-1)/r]}B^{[(n+r-1)/r]} + \cdots + S^{(r)}_0 \end{aligned}$$

where $S^{(i)}_k$ is a differential operator of order $<r$. The i, j element of Λ is the polynomial

$$\sum_{0\leq k\leq[(n+i-1)/r]} (\text{coeff. of } \partial^{j-1} \text{ in } S^{(i)}_k)\lambda^k$$

Note that Λ is the matrix such that $\Lambda\begin{bmatrix} y \\ y^{\cdot} \\ \vdots \\ y^{(r-1)} \end{bmatrix} = \begin{bmatrix} Ly \\ (Ly)^{\cdot} \\ \vdots \\ (Ly)^{(r-1)} \end{bmatrix}$ for all $y(x)$ such that $By = \lambda y$.

1.2. PROPOSITION (Burchnall–Chaundy, cf. [3]). *If the operators B and L (as in Definition* 1.1) *commute, then the polynomial* $\det(\Lambda-\mu)=\phi(\lambda,\mu)$ *is independent of x and the set of its zeros* (λ,μ) *is the joint spectrum in the sense that there exists a* (*formal power series*) $y(x)$ *such that* $By=\lambda y$, $Ly=\mu y$.

1.3 *Remark.* Burchnall and Chaundy also proved that B, L satisfy the equation $\phi(B, L)=0$. They did not define Λ using the euclidean algorithm, but rather via an "elimination matrix" E for two operators, which we now extend to the case of any finite number of operators. Note the analogy with the elimination matrix for two polynomials f, g in one variable, which has zero determinant (Sylvester's resultant) if and only if f, g have a common root. Let $L_1,\ldots,L_s (s\geq 2)$ be differential operators with leading term $=1$ and orders $m_1,\ldots,m_s$ with $m_1\leq m_2\leq\cdots\leq m_s$. We define a matrix E with $(s-1)m_1+m_s$ rows and m_1+m_s columns, by putting along the rows the coefficients of $1, \partial, \partial^2, \ldots, \partial^{m_1+m_s-1}$ in the operators: $L_1-\lambda_1$, $\partial\circ(L_1-\lambda_1)\cdots\partial^{m_s-1}\circ(L_1-\lambda_1)$; $L_2-\lambda_2, \partial\circ(L_2-\lambda_2),\ldots,\partial^{m_1-1}\circ(L_2-\lambda_2);\ldots;L_s-\lambda_s,\ldots,\partial^{m_1-1}\circ(L_s-\lambda_s)$.

Let us first observe the link of E with Λ in the case $s=2$; if we partition E into blocks, $\begin{array}{cc} & \begin{matrix} r & n \end{matrix} \\ \begin{matrix} n \\ r \end{matrix} & \begin{bmatrix} E_1 & F_1 \\ E_2 & F_2 \end{bmatrix} \end{array}$ and set $\lambda_2=\mu$, then $\Lambda-\mu=E_2-F_2F_1^{-1}E_1$, as can be checked on the r-dimensional space of solutions of $By=\lambda y$. If B, L commute then L determines an endomorphism of that space represented by $\Lambda^T|_{x=0}$ on a fundamental set of solutions normalized at $x=0$ (cf. [15], Proof of 5.1). In general, we partition

$$E = \begin{array}{cc} \begin{bmatrix} E_1 & F_1 \\ E_2 & F_2 \\ \vdots & \\ E_s & F_s \end{bmatrix} & \begin{array}{c} m_s \\ m_1 \\ \vdots \\ m_1 \end{array} \\ \begin{array}{cc} m_1 & m_s \end{array} & \end{array}$$

If $y(x)$ is a solution of $L_1 y = \lambda_1 y$ then $[((L_k - \lambda_k)y)^{(\alpha)}]_{0 \le \alpha \le m_1 - 1} = [E_k - F_k F_1^{-1} E_1][y^{(\alpha)}]$, $k = 2, \ldots, s$. Finally, we denote by E^* the matrix

$$\begin{bmatrix} E_2 - F_2 F_1^{-1} E_1 \\ \vdots \\ E_s - F_s F_1^{-1} E_1 \end{bmatrix}$$

1.4 PROPOSITION. *If $L_1, \ldots, L_s$ (as in 1.3) commute pairwise, then the joint spectrum is the set of $(\lambda_1, \ldots, \lambda_s)$ for which the matrix $E_0^* = E^*|_{x=0}$ has rank $< m_1$, thus is given by $\binom{(s-1)m_1}{m_1}$ polynomial equations.*

Proof. The action of L_2 on the m_1-dimensional kernel W_1 of $L_1 = \lambda_1$ will determine a λ_2-eigenspace W_2; L_3 acts on W_2 and will have a λ_3-eigenspace W_3 and so on. A function $y(x)$ is in the intersection of these spaces if and only if the vector $[y^{(\alpha)}]_{0 \le \alpha \le \sum m_j - (s-1)}$ is in the kernel of E. But because the operators commute, the action of $L_k - \lambda_k$ on W_1 is represented by the matrices $(E_k - F_k F_1^{-1} E_1)^T|_{x=0}$, as in the case of 2 operators. Thus the condition defining the joint spectrum is the same as the condition that the matrix fails to have maximal rank. QED

Next we assemble some known results in order to describe the spectral curve and its Jacobian. First we enlarge the ring $\mathcal{R}$ to a ring of (formal) pseudodifferential operators $\mathfrak{g} = \mathbf{C}[[x]]((\partial^{-1})) = \{\sum_{-\infty}^{finite} u_j(x)\partial^j,\ u_j(x)$ a formal power series$\}$ with the (associative) product:

$$\partial \circ u = \dot{u} + u\partial,\ \partial^{-1} \circ u = u\partial^{-1} - \dot{u}\partial^{-2} + \ddot{u}\partial^{-3} - \cdots$$

$\mathfrak{g}$ is also a Lie algebra via commutators $[B, L] = BL - LB$, which as a vector space we view as the direct sum of two subalgebras: $\mathfrak{g}^+ \oplus \mathfrak{g}^-$ with projections: $X_+ = \sum_0^N u_j\partial^j$, $X_- = \sum_{-\infty}^{-1} u_j\partial^j$ for all $X = \sum_{-\infty}^N u_j\partial^j \in \mathfrak{g}$.

1.5 PROPOSITION. *Let C be a smooth irreducible curve with a fixed point P_∞ and a local parameter z centered at P_∞. We let $z^{-1} = \kappa$. Let $D = P_1 + \cdots + P_g$ be a divisor on C such that $D - P_\infty$ is nonspecial.*

(i) (Krichever, [7]) *Associated to the above data there is a unique function ψ of $P \in C$ and $x \in \mathbf{C}$ such that: for fixed x with $|x|$ small ψ is meromorphic in P outside P_∞ with pole divisor $\leqslant D$ and $\psi e^{-\kappa x} = \chi(x, \kappa^{-1})$ is holomorphic in κ^{-1} near P_∞; for fixed $P \neq P_\infty$, P_j and $|x|$ small ψ is holomorphic in x and $\psi(0, P) = 1$.*

(ii) *$\psi(x, P)$ determines an element $\mathfrak{L}_D = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \ldots$ of $\mathfrak{g}$ so that $\mathfrak{L}_D\psi = \kappa\psi$ (formally in x and for κ in an open domain of the complex plane) with the convection $\mathfrak{L}_D\psi = (\mathfrak{L}_D \circ \chi)e^{\kappa x}$ and $\partial^{-1}e^{\kappa x} = \kappa^{-1}e^{\kappa x}$.*

(iii) (Schur [14]). *The ring of differential operators that commute with $\mathfrak{L}_D$ is a maximal commutative subalgebra $\mathcal{A}_D$ of $\mathcal{R}$ and $\mathcal{A}_D = \{X = \sum_{-\infty}^{finite} c_j\mathfrak{L}^j$ s.t. $X_+ = X$, where $c_j \in \mathbf{C}\}$.*

(iv) *$\mathcal{A}_D$ is isomorphic to the ring R_∞ of meromorphic functions on C regular outside P_∞ via the map: $B \in \mathcal{A}_D$, $B \mapsto f_B$ where $B\psi = f_B(P)\psi$; the order of B is the order of pole of f_B at P_∞.*

(v) (Mumford, [11]; Wilson, [15]). *The affine curve $C_0 = \operatorname{Spec} \mathcal{A}_D$ has a one-point smooth compactification, isomorphic to the curve C. Each point P of $C \backslash P_\infty$ corresponds to the homomorphism $\mathcal{A}_D \to \mathbf{C}$, $B \mapsto f_B(P)$. If $L_1, \ldots, L_s$ is a set of generators of $\mathcal{A}_D$, then C_0 is the joint spectrum of $L_1, \ldots, L_s$, i.e. the set $(\lambda_1, \ldots, \lambda_s) \in \mathbf{C}^s$ such that $L_j\psi = \lambda_j(P)\psi$.*

If $L_1, \ldots, L_s (s \geqslant 2)$ generate a smaller ring $\mathcal{B} \neq \mathcal{A}_D$, the corresponding affine curve Γ_0 in $\mathbf{C}^s$ is the image of C_0 under a morphism, which is generically of degree 1 provided the g.c.d. of the orders of the elements of $\mathcal{B}$ is 1; thus Γ_0 is singular.

(vi) ([11]) *The space of common solutions of $\mathcal{A}_D$ is one dimensional at each point of C_0; the dual of such spaces can be glued into a line bundle whose extension over P_∞ corresponds to the divisor $D - P_\infty$.*

(vii) ([11]) *$\psi_D(x, P)$ is a global section of $\mathcal{M}_x \otimes \mathcal{O}(D)$, where $\mathcal{M}_x$ is the analytic line bundle given by glueing data $e^{x\kappa}$ on a punctured neighborhood of P_∞. $\mathcal{M}_x$ has zero Chern class; the divisor corresponding to $\mathcal{M}_x \otimes \mathcal{O}(D)$ is $D_x - D$, D_x nonspecial, and it moves linearly with x on* Jac C.

1.6 COROLLARY. *Let the data (C, P_∞, κ, D) and the notation be as in* 1.3. *If $L_1, \ldots, L_s$ (of degree $m_1, \ldots, m_s$ and leading coefficient 1) generate $\mathcal{A}_D$, then:*

(i) *equations for the curve C_0 are given by the rank condition in* 1.4;

(ii) *the divisor D is defined by the vanishing of all $(m_1 - 1) \times (m_1 - 1)$ minors of the matrix $E^*|_{x=0}$ that are adjoint to the first column.*

Proof. (i) is a consequence of 1.4 and 1.5(v). (ii): by 1.5(vi), the space of common solutions of $\mathcal{A}_D$ is one dimensional at each point $(\lambda_1, \ldots, \lambda_s)$;

equivalently, the kernel of $E^*|_{x=0}$ is one dimensional, thus the cofactors $(v_1, \ldots, v_{m_1})$ of a row in an appropriate $m_1 \times m_1$ submatrix of $E^*|_{x=0}$ give a nonzero eigenvector. Finally, if $y_1(x), \ldots, y_{m_1}(x)$ is a fundamental set of solutions for $L_1 y = \lambda_1 y$ at $x = 0$, then $\psi_D = y_1 + (v_2/v_1)y_2 \cdots + (v_{m_1}/v_1)y_{m_1}$, so the poles of ψ_D on C_0 are given by the equation $v_1(\lambda_1, \ldots, \lambda_s) = 0$. QED

Because of its construction (1.3), the BC matrix E_0^* of a ring $\mathscr{A}_D$ can be viewed as a function on $\operatorname{Jac} C \setminus \Theta$, thus the entries can be expressed in terms of ϑ-functions; our next calculation gives a sufficient number of parameters for recovering the entries.

1.7 PROPOSITION. *Let $L_1 = B_r$ be an element of minimal order $m_1 = r$ in $\mathscr{A}_D$ and let $\mathfrak{L}_D = B_r^{1/r}$ (we can always assume this to be the case by a suitable choice of the local parameter κ). Let $\mathfrak{L}_D = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \ldots$, $B_r = \partial^r + u_{r-2}\partial^{r-2} + \cdots + u_0$ and let $L_k = \sum_{-\infty}^{m_k} c_j^{(k)} \mathfrak{L}^j$; we may also assume that $c_j^{(k)} = 0$ for j a multiple of r.*

(i) *The coefficients of L_k are differential polynomials (in the variable x) of the coefficients of B_r. The coefficient of $\partial^{m_k - j}$ in L_k is a universal polynomial in u_{r-2}, $\dot u_{r-2}, \ldots, u_{r-2}^{(j-2)}$; u_{r-3}, $\dot u_{r-3}, \ldots, u_{r-3}^{(j-3)}, \ldots, u_{r-j}$ (if $j > r$ the sequence stops with $u_0^{(j-r)}$) involving the $[m_k/r](m_k - 1) + l - 1$ constants $c_j^{(k)}$ where $0 \leqslant l \leqslant r - 1$ and $m_k \equiv l \pmod r$)*

(ii) *The entries of the elimination matrix E are linear in the coefficients of L_k and their derivatives; the highest derivatives involved in the block $[E_1\ F_1]$ are determined as follows: the (ij) entry involves the coefficient of $\partial^{m_1 - \alpha}$ for $\alpha > m_1 - j$ up to its $(m_1 + i - \alpha - j)$th derivative when this number is nonnegative (otherwise that coefficient is not involved). The blocks $[E_k\ F_k]$, $k > 1$, if we use* (i) *to express the coefficients of L_k, involve at most the same number of derivatives as the block $[E_1\ F_1]$.*

(iii) (Baker, Akhiezer, Krichever; cf. [7]). *There exist $r - 1$ invariant vector fields on* Jac *C, $\partial_1, \ldots, \partial_{r-1}$ such that the coefficient of $\partial^{m_1 - j}$ in B_r is a polynomial in $\partial_1^{n(\alpha)}\partial_\alpha \log \vartheta(P_\infty - D + \Delta + \sum_1^{r-1} t_j U_j)$ where $1 \leqslant n(\alpha) \leqslant j - \alpha$, $\partial_\alpha = \partial/\partial t_\alpha$ and $t_1 = x$; the notation $P_\infty - D$ is an abbreviation for the image of $P_\infty - D$ under the Abel map, which is determined up to coordinate change (immaterial here) by a choice of homology basis for C and of base point P_0 and Δ is a consequently determined universal constant; U_j is the g vector obtained by expanding a (normalized) basis of holomorphic differentials $\omega_1, \ldots, \omega_g$ in powers of κ^{-1} and taking the $(j-1)$st coefficient of the expansion with negative sign.*

Proof. (i) is straightforward, by first expressing the coefficients of B_r in terms of $u_{-1}, \ldots, u_{-r+1}$ and then solving for u_{-r-l}, $l \geqslant 0$, in terms of $u_{-1}, \ldots, u_{-r+1}$, which is possible because $\mathfrak{L}^r = \mathfrak{L}^r_+$; finally, $L_k = \sum c_j^{(k)} \mathfrak{L}^j$ (a consequence of 1.5(iii)) gives the statement. (ii) is likewise straightforward: by Leibnitz' rule, the coefficient of ∂^{j-1} in $\partial^{i-1} \circ L_1$ is $\sum_{1 \leq \alpha \leq \min(j, i-1)} \binom{i-1}{\alpha-1} u_{j-\alpha}^{(i-\alpha)}$. The statement on the blocks $[E_k \; F_k]$ follows from the fact that $\partial^{m_1-1} \circ L_k$ and $\partial^{m_k-1} \circ L_1$ involve the same highest power of $\mathfrak{L}_D$, and $m_k \leq m_s$. QED

1.8 DEFINITION. We say that the generalized Weierstrass $\wp$-functions for C are the functions $\partial_1 \partial_\alpha \log \vartheta(z)$, $1 \leq \alpha \leq r-1$, in the notation of 1.7(iii).

1.9 PROPOSITION. *The morphism* J:

$$\operatorname{Jac} C \setminus \Theta \rightarrow \mathbf{C}^N, \; N = (r-1)m_s + \frac{(r-1)(r-2)}{2}$$

given by $z \mapsto (\partial_1^{N(\alpha)} \partial_\alpha \log \vartheta(z))_{1 \leq \alpha \leq r-1,\, 1 \leq N(\alpha) \leq m_s + r - (\alpha+1)}$ *is an embedding; equivalently, the coordinate functions following the arrow generate the function ring of* $\operatorname{Jac} C \setminus \Theta$.

Proof. The given functions separate points because they allow us to write the matrix E which defines the divisor D corresponding to $P_\infty + \Delta - z$ (1.6 and 1.7(iii)). Also, no derivatives on Jac C can annihilate all the given functions, because the tangent space to Jac C at any point can be spanned by the KP flows ∂_{t_j} (cf. §3) and the effect of ∂_{t_j} on the BC matrix can only be trivial if ∂_{t_j} is the trivial flow, as we will see in (3.2). QED

Note. For the general curve the smallest m_1 (as in 1.5) is g (cf [6], Chapter 2, §4) $s = g$ with $L_1, \ldots, L_g$ of orders g, $g+2$, $g+3, \ldots, 2g-1$, $2g+1$. By the construction of the BC matrix we can write the equation of the curve C_0 using a suitable element of $\mathbf{C}^N$, $N = 5(g(g-1)/2)$.

§2. The plane-model case: geometric addition law

In the previous section we introduced the analog of the Weierstrass $\wp$-function in the following sense: a set of functions on Jac C with poles on Θ which, together with a number of derivatives along a given direction, can be taken to be affine coordinates for Jac $C \setminus \Theta$. In this section we show how to generalize the geometric construction of the addition law on the elliptic curve, under a "speciality" assumption. Under the same assumption, we give a characterization of nonspecial divisors that allows us to invert the map on the set of "BC matrices" and prove that the map in 1.9 has an inverse on that set.

2.1 DEFINITION. We say that the curve C has the plane-model property if there is a point $P_\infty \in C$ such that the ring R_∞ can be generated by two elements.

Note: 1. Unless $g = 1$, such a point must be a Weierstrass point, namely the lowest order r of a function $f \in R_\infty$ at P_∞ must be $\leq g$; indeed, g is the number of gaps at P_∞ and R_∞ is generated by two elements, one of which has order r. 2. As a consequence of the plane-model property we can represent the curve $C \backslash P_\infty$ as a smooth plane curve, by the equation $\det(\Lambda(B; L) - \mu) = \phi(\lambda, \mu) = 0$ where B, L are generators of R_∞ (of orders, say, r and m resp.)

2.2 PROPOSITION. *If C has the plane-model property, a divisor $D = \sum_{i=1}^{g} P_i$ on C_0 is such that $D - P_\infty$ is nonspecial if and only if $H^0((2g + i)P_\infty - D)$ has a basis $\{f_{j+1}^i; j = 0, \ldots i\}$ such that the order of pole of f_{j+1}^i at P is $2g + j$, for some $i \geq 0$.*

Proof. We note that $(2g - 2)P_\infty$ is a canonical divisor; indeed, $\dfrac{d\lambda}{\phi_\mu}\left(= -\dfrac{d\mu}{\phi_\lambda}\right)$ is a holomorphic differential on C with no zeroes on C_0. By Riemann–Roch, $D - P_\infty$ is special if and only if $\dim H^0((2g - 1)P_\infty - D)$ is nonzero. If there is a function $f \in H^0((2g - 1)P_\infty - D)$, then $f \in H^0((2g + i)P_\infty - D)$ for any $i \geq -1$, but this contradicts the assumption that for some $i \geq 0$ there exists a basis whose elements have order of pole at P_∞ strictly increasing from $2g$. Conversely, if $D - P_\infty$ is nonspecial, then by Riemann–Roch the dimension of $H^0((2g + i)P_\infty - D)$ is $i + 1$ for all $i \geq -1$, so a basis with the stated property can be found by induction, for all $i \geq 0$ in fact. QED

We can finally emphasize a property of the BC matrix which is quite intriguing and motivates our looking at the plane-model case.

2.3 PROPOSITION. *If the divisor $D - P_\infty$ (as in 2.2) is nonspecial and $\Lambda_0 = \Lambda(L; B)|_{x=0}$ is the BC matrix determined by D, then the sequence of $(r - 1) \times (r - 1)$ minors adjoint to the first column of $\Lambda_0 - \mu$, which define D, is a sequence of functions $f_{j+i} \in H^0((2g + j)P_\infty - D)$, normalized by the condition that the monomial of f_{j+1} in λ, μ with highest-order pole at P_∞ has coefficient ± 1; if the minors adjoint to the first column are ordered from top to bottom, then the corresponding sequence is $f_r, f_{r-1}, \ldots, f_1$.*

Proof. Let $[v_{ij}(\lambda, \mu)]$ be the adjoint matrix of $\Lambda_0 - \mu$, i.e. the one made up with the $(r-1)\times(r-1)$ minors. The r functions $v_{i1}(\lambda, \mu)$, $1 \leq i \leq r$, define D (1.6(ii)) and are contained in $H^0((2g+r-1)P_\infty - D)$ as an easy calculation shows; indeed, by definition (cf. 1.1), $\Lambda - \mu$ has the following top-weight, monic terms (the "weight" is the order of pole at P_∞):

$$
\begin{array}{ccc}
 & \begin{array}{cc} \longleftarrow l \longrightarrow \qquad\qquad & \qquad\qquad \longleftarrow r-l \longrightarrow \end{array} & \\
\begin{array}{c} \\ \\ \\ \\ \uparrow \\ l \\ \downarrow \end{array} &
\left[\begin{array}{cccc|cccc}
-\mu & & & & \lambda^M & & & \\
 & -\mu & & & & \cdot & & \\
 & & \cdot & & & & \cdot & \\
 & & & \cdot & & & & \lambda^M \\
\hline
\lambda^{M+1} & & & & \cdot & & & \\
 & \cdot & & & & & & \\
 & & \cdot & & & & & \\
 & & & \lambda^{M+1} & & & & -\mu
\end{array}\right]
& \begin{array}{c} \uparrow \\ r-l \\ \downarrow \\ \\ \\ \\ \\ \end{array}
\end{array}
$$

where the order of L is $Mr + l$, l is a fixed number between 1 and $r-1$ and prime with r. Notice that the genus $g = \dfrac{(Mr+l-1)(r-1)}{2}$ (cf. [3]). Thus $\{v_{i1}\}_{1\leq i\leq r}$ must be a basis of $H^0((2g+r-1)P_\infty - D)$. On the other hand, each $\sum_{j=1}^r v_{ij}y_j$, $i = 1, \ldots, r$ is a common eigenfunction of B and L, where y_j is a fundamental system for B at $x=0$; by the same reason why 1.5 (vi), (vii) hold, the function $\psi_i = (\sum_{j=1}^r v_{ij}y_j)/v_{ii}$ has a pole divisor of degree $g+i-1$ on C_0 (notice that the normalizing property of the section ψ_i is that $\psi_i^{(i-1)}|_{x=0} = 1$) and v_{i1}/v_{ii} has order of pole $i-1$ at P_∞; on the other hand, by inspection v_{ii} has highest-weight term μ^{r-1}, with weight $(r-1)(Mr+l) = 2g+r-1$, so the weight of v_{i1} is $2g+r-i$, as claimed, and the leading coefficient is ± 1, by inspection. The sequence of leading terms is $\mu^\gamma\lambda^{M\alpha}\lambda^{(M+1)\beta}$ with $0\leq\gamma\leq r-1$, but γ not necessarily in decreasing order: this depends on what l is. QED

In the proof of the previous Proposition, we noticed that the adjoint $[v_{ij}]$ of the BC matrix is linked to various boundary conditions for $\mathscr{A}_D$; in fact, the jth column defines the divisor D_j, with $D_1 = D$, attached to the eigenfunction ψ_j of $\mathscr{A}_D$ such that $\psi_j^{(j-1)}|_{x=0} \doteq 1$; if we picture the zeroes of v_{ij},

$$
\begin{bmatrix}
D_1 + \tilde{D}_r & D_2 + \tilde{D}_r & \cdots & D_r + \tilde{D}_r \\
D_1 + \tilde{D}_{r-1} & & & \\
D_1 + \tilde{D}_1 & D_2 + \tilde{D}_1 & \cdots & D_r + \tilde{D}_1
\end{bmatrix}
$$

The $\tilde{D}_j$ are the corresponding divisors for the (formally) "dual" (cf. [4]) ring. The v_{ij} are our generalized Jacobi polynomials because, as we'll prove next, they can be determined from D *algebraically*, namely without recourse to the ring $\mathscr{A}_D$ or, which is the same, the theta function. This determination takes place by picking out one curve v_{ij} of weight $2g+j-1+r-i$ in each $H^0((2g+j-1+r-i)P_\infty - D_j)$ through a set of constraints which, moreover, ensure uniqueness for a polynomial matrix $[w_{ij}]$ that defines a given divisor D. If $r=2$, the constraint is satisfied by choosing the curve $V(\lambda)-\mu$ through D_1 and the curve $-V(\lambda)-\mu$ through $\tilde{D}_1$ whose monomial of weight $2g$ has zero coefficient; the BC matrix $\Lambda_0-\mu$ is given by Jacobi's polynomials:

$$\Lambda_0 - \mu = \begin{bmatrix} -V(\lambda)-\mu & U(\lambda) \\ W(\lambda) & V(\lambda)-\mu \end{bmatrix}$$

If the ring $\mathscr{A}_D$ happens to be periodic in x, the divisors D_1, D_2 correspond to the Dirichlet, resp. Neumann boundary conditions.

2.4 PROPOSITION. *Given a curve C with the plane-model property, let r, $m=Mr+l$, $\phi(\lambda,\mu)=\det(\Lambda_0-\mu)$ be associated to (C, P_∞, κ, D) as in* (2.1), (2.3). *To N parameters $(\gamma_1,\ldots,\gamma_N)$ in $\mathbf{C}^N$, $N=\dfrac{(r-2)(r-1)}{2}+m(r-1)$ we can associate in a one-to-one fashion a matrix E_0 as in* 1.7(ii) *by letting these parameters play the role of the coefficients of $B=\partial^r+u_{r-2}\partial^{r-1}+\cdots+u_0$, $L=\partial^m+v_{m-1}\partial^{m-1}+\cdots+v_0$ and their derivatives at $x=0$. If E_0 satisfies the equation* $\det E_0=\phi(\lambda,\mu)$, *then the corresponding matrix Λ_0 defines a point D in* Jac $C\backslash\Theta$ *and the composition of this morphism from the Λ_0-matrix to D with the morphism J* (*cf.* 1.9) *is the identity.*

Proof. By construction the highest-weight monic terms of Λ_0 are as in the sketch within the proof of 2.3. This implies that the minors $f_1,\ldots,f_r$ adjoint to the first column have weights $2g+r-1$, $2g+r-2,\ldots 2g$ at P_∞. On the other hand, they define a divisor; indeed at every point (λ,μ) of C_0 there is a one-dimensional eigenspace of Λ_0 with eigenvalue μ (if the dimension were higher, the polynomial $\phi(\mu,\lambda)$ would be singular at that point). We take the extension of this line bundle across P_∞. The minors $f_1,\ldots,f_r$ are a basis for $H^0((2g+r-1)P_\infty-D)$, where D is the corresponding divisor (cf. [11]), so D is nonspecial by 2.2, of degree g. This map from $\mathbf{C}^N$ to Jac $C\backslash\Theta$ is a morphism, because D is given as the common zeroes of r functions polynomial in the parameters (γ) as well as the entries of $\Lambda_0-\mu$. To prove that J is the inverse map, we show that two matrices Λ, Λ' that are constructed as in 1.7(ii) and define the same divisor D must coincide. This follows from two remarks: (I) It is enough to show that $\operatorname{adj} \Lambda = \operatorname{adj} \Lambda'$ since the determinant $\phi(\lambda, \mu)$ is the same. Let $\operatorname{adj} \Lambda = (v_{ij})$. Each function v_{i1} belongs to $H^0((2g+r-1)P_\infty - D)$, is monic up to sign and has weight $2g+r-i$ (cf. 2.3). Likewise, each function v_{rj} belongs to $H^0((2g+r-1)P_\infty - E)$, is monic up to sign and has weight $2g+j-1$, where $E+D$ is the zero divisor of v_{r1}. (II) For a matrix Λ that is constructed through the map J of 1.9 and 1.7 (ii), the function $(\sum_{j=1}^r v_{ij}y_j)/v_{i1}$ is the eigenfunction ψ of 1.5(i), for all $i=1, \ldots, r$, thus it has an expansion around P_∞ $\psi(x, \kappa) = e^{\kappa x}(1+\xi_1(x)\kappa^{-1}+\xi_2(x)\kappa^{-2}+\cdots)$. We use this fact to show that for any matrix Λ' constructed through 1.7(ii) using N numbers $\gamma_1, \ldots, \gamma_N$, the polynomial entries v'_{ij} of adj Λ' are determined by D algebraically. Let us fix $i=r$, the argument for the other rows being similar. Notice that $v_{r2}/v_{r1} = \dot{\psi}(0, \kappa) = \kappa + \dot{\xi}_1(0)\kappa^{-1} + \dot{\xi}_2(0)\kappa^{-2} + \cdots$ so that the two top coefficients of the function v_{r2}, which belongs to the 2-dimensional space $H^0((2g+1)P_\infty - E)$ depend only on E and the form of the matrix, so v'_{r2} is determined. Next, $v_{r3}/v_{r1} = \ddot{\psi}(0, \kappa) = \kappa^2 + 2(\xi_1(0) + \xi_2(0)\kappa^{-1} + \cdots) + \ddot{\xi}_1(0)\kappa^{-1} + \cdots$ so that the three top coefficients of the function v_{r3}, which belongs to the 3-dimensional space $H^0((2g+2)P_\infty - E)$ depend only on E and the form of the matrix (indeed $\dot{\xi}_1(0)$ appears among the coefficients of v_{r2}), so v'_{r3} is determined, and so on inductively. QED

2.5 *Remark*. 2.4 does not give equations for the image of J, because the map from the E to the Λ matrices is not 1:1 (cf. §3). If we could extract from the $\mathfrak{p}$-functions (1.8) a least number N' of parameters to determine Λ in a 1:1 fashion, we would have equations for an affine subvariety of $\mathbf{C}^{N'}$ isomorphic to Jac $C \setminus \Theta$.

We conclude this section by pointing out the analog of the geometric construction for the addition law on elliptic curves. Recall that for a plane elliptic curve with a branchpoint P_∞ at infinity, the sum of two points, $P+Q$ "=" R, unless $P+Q$ is a special divisor is found by taking the third (finite) intersection of C with the line through P and Q and the corresponding point, R, under the "sheet exchange" ι. Thus, $P+Q+\iota R \sim 3P_\infty$ or $P+Q \sim P_\infty + R$; P_∞ is taken to be the zero of the group. For a curve C with the plane-model property, the construction is the following:

(2.6) Assume that the divisors $P_1+\cdots+P_g$ and $P_1+\cdots P_g+Q$ on C_0 are nonspecial. Then the sum $\sum_{i=1}^g R_i$ such that $\sum_{i=1}^g P_i + Q \sim \sum_{i=1}^g R_i + P_\infty$ is found as follows: let $Q_1+\cdots+Q_g$ be the residual intersection with C of the unique curve (cf. 2.2) of weight $2g+1$ that goes through $\sum P_i + Q$; then $R_1+\cdots+R_g$ is the residual intersection with C of the unique curve of weight $2g$ that goes through $Q_1+\cdots+Q_g$.

As a corollary, we state an analog of the following property of the Weierstrass

$\wp$-function: (cf. [6]):

$$\begin{vmatrix} 1 & \wp(z_1) & \wp'(z_1) \\ 1 & \wp(z_2) & \wp'(z_2) \\ 1 & \wp(z_3) & \wp'(z_3) \end{vmatrix} = 0 \quad \text{if and only if}$$

$z_1 + z_2 + z_3 =$ a lattice point; the determinantal condition says that the points P, Q, ιR with coordinates $\wp(z_i)$, $\wp'(z_i)$, $i = 1, 2, 3$ (resp.) lie on a straight line; the equivalent condition is the image of the statement $P + Q + \iota R \sim 3P_\infty$ under the Abel map. Thus,

(2.7) COROLLARY. *For two nonspecial divisors $\sum_1^g P_i$, $\sum_1^g Q_i$, the sum $A(\sum P_i) + A(\sum Q_i)$ is a lattice point (where A is the Abel map with base-point P_∞) if and only if the matrix of (λ, μ)-coefficients of the functions $f_{\sum P_i}$, $f_{\sum Q_i}$ which go through $\sum P_i$, $\sum Q_i$ (resp.) and have weight $2g$ has rank < 2.*

Note. This condition can be expressed in terms of generalized $\wp$-functions (1.8) since the functions $f_{\sum P_i}$, $f_{\sum Q_i}$ are the minors adjoint to the $(r, 1)$ entry of $\Lambda_{\sum P_i}$, $\Lambda_{\sum Q_i}$. Likewise, we can express the conditions $\sum_1^g P_i + Q$ "$=$" $- \sum_1^g Q_i$ by saying that the rank of the coefficient matrix of $f_{\sum P_i + Q}$, $f_{\sum Q_i + Q}$ is < 2, etc.

§3. Applications: The infinitesimal generator for the KP flows; the solutions of the KP equation; equations for Jac $C \backslash \Theta$ in two examples

As we said in the introduction, the Burchnall–Chaundy isospectral theory can be used as a tool for solving the KP equation. We give a definition for the equation in our setting.

3.1 DEFINITION. The equations $\partial_{t_n}\mathfrak{L} = [(\mathfrak{L}^n)_+, \mathfrak{L}]$ are constrained commuting Hamiltonians on the Lie algebra $\mathfrak{g}$ (cf. [2]). Therefore, there exists a formal solution which can be regarded as a function of infinitely many variables t_n, provided only a finite number is nonzero: $(t) \in \mathbf{C}_\infty = \varinjlim \sum t_n\mathbf{C}$; we let $x = t_1$. If we let $B_n = (\mathfrak{L}^n)_+$, then the system of equations $\partial_{t_n}B_m - \partial_{t_m}B_n = [B_n, B_m]$ is by definition the KP hierarchy. By further letting $t_2 = y$, $t_3 = t$ and eliminating from the (two) equations on the coefficients of the operator:

$$\partial_{t_2}B_3 - \partial_{t_3}B_2 = [B_2, B_3]$$

we obtain, respectively: the KP equation; the KdV equation ($u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x =$

0), and the Boussinesq equation ($\frac{3}{4}u_{yy} + (\frac{1}{4}u_{xxx} + \frac{3}{2}uu_x)_x = 0$) for $u = 2u_{-1}$, in the: general case; case when $(\mathfrak{L}^2)_+ = \mathfrak{L}^2$, and case when $(\mathfrak{L}^3)_+ = \mathfrak{L}^3$. More generally, if the initial condition $\mathfrak{L}(x)$ satisfies the ODE $(\mathfrak{L}^r)_+ = \mathfrak{L}^r$, we say that the resulting hierarchy is the r-reduction of KP; notice that in this case the coefficients of $\mathfrak{L}(t)$ are independent of t_r, for the flow ∂_{t_r} is trivial.

Suppose we have an initial condition $\mathfrak{L}$ for the r-reduction of KP. Since the operators B_n correspond to the matrices $\Lambda(B_n, B_r)$ and the commutators of two such matrices to the action of the commutator of operators, we address the question of writing a matrix that corresponds to $\mathfrak{L}$, the infinitesimal generator of KP.

(3.2) For $\mathfrak{L}$ such that $\mathfrak{L}^r = (\mathfrak{L}^r)_+ = B_r$, we define the $r \times r$ matrix $\tilde{\Lambda}$ as follows: let

$$\begin{aligned} \mathfrak{L} &= S_0^{(1)} + S_{-1}^{(1)} B_r^{-1} + S_{-2}^{(1)} B_r^{-2} + \cdots \\ \partial \circ \mathfrak{L} &= S_0^{(2)} + S_{-1}^{(2)} B_r^{-1} + \cdots \\ &\vdots \\ \partial^{r-1} \circ \mathfrak{L} &= S_1^{(r)} B_r + S_0^{(r)} + S_{-1}^{(r)} B_r^{-1} + \cdots \end{aligned}$$

with $S_j^{(i)}$ differential operators of order $<r$; the ij entry of $\tilde{\Lambda}$ is the negative Laurent series $\sum_k$ (coefficient of ∂^{j-1} in $S_k^{(i)})\lambda^k$. Then, the KP hierarchy is equivalent to the following evolution equations:

$$\partial_{t_n} \tilde{\Lambda} = [\Lambda(B_n, B_r), \tilde{\Lambda}], \text{ as well as:}$$

$$\partial_{t_n} \Lambda_m - \partial_{t_m} \Lambda_n = [\Lambda_n, \Lambda_m]$$

where we abbreviated $\Lambda(B_n, B_r) = \Lambda_n$.

(3.2) can be checked on a basis of eigenfunctions for B_r. We point out this way of interpreting the flows in $\tilde{\mathfrak{g}}(r) = \mathfrak{gl}\,(r, \mathbf{C}) \otimes \mathbf{C}((\lambda^{-1}))$ because in [1] we give a construction of flows in $\tilde{\mathfrak{g}}(r)^*$ that are Hamiltonian in the classical sense, by considering the isospectral deformations of a matrix $N(\lambda)$, the image of the moment map for a suitable infinitesimal $\tilde{\mathfrak{g}}(r)^+$ action. In some examples, we show that $N(\lambda)$ equals a linear combination of powers of $\tilde{\Lambda}$, translated by a constant matrix and multiplied by a fixed meromorphic function on C.

KP, KdV and Boussinesq are nonlinear approximations of equations for waves in shallow water and the qualitative properties of the solution u (reality for suitable x, y, t; blow-up in finite time; soliton behavior) are of interest for the applications. We now give an expression for u in terms of the coordinates of the divisor $\sum P_i$ on the affine curve: such a formula proved very useful in the hyperelliptic case (corresponding to KdV) for understanding the motion. Let the data (C, P_∞, κ, D) determine $\mathfrak{L}_D$ (as in 1.5); by Krichever's theory ([7]), $\mathfrak{L}_D$ satisfies the KP hierarchy if the flows on Jac C are given by $\vec{U}_n) \cdot (\partial/\partial z_j)$, where U_n is $(-1) \times$ the vector of $(n-1)$st coefficients in the κ^{-1}-expansion of a normalized basis of holomorphic differentials, (z_j) are the corresponding abelian coordinates.

3.3 PROPOSITION. *If C has the plane-model property, then the function u_{-1} (as a function of the divisor D) equals the second-from-the-top coefficient in the polynomial giving the normalized curve of weight $2g$ through D ($=$ the minor adjoint to the $(r, 1)$ entry of the* BC *matrix attached to D), up to a multiplicative constant depending on r, m and an additive constant depending on the curve.*

Proof. The highest possible weight after $2g$ for a monomial in λ, μ is $2g-2$ (cf. [3]); in fact, such a monomial *must* appear in the expression of a curve of weight $2g$ that goes through g (variable) points of C in general position, by a dimension count (there are only g possible weights $\leq 2g$ cf. [3]). By definition of the BC matrix (cf. 1.1), and because $B = \partial^r + ru_{-1}\partial^{r-2} + \ldots$, $L = \partial^m + const.\ \partial^{m-1} + mu_{-1}\partial^{m-2} + \ldots$, in each row the top coefficient of $S_k^{(i)}$ (largest k) is 1, but the next nonconstant one, either in $S_k^{(i)}$ or $S_{k-1}^{(i)}$, is a multiple of u_{-1}. Finally, such a coefficient contributes to the monomial of weight $2g-2$ because the weight of the monomial equals the total number of "∂'s" corresponding to the variables. QED

Examples: Let's denote here by f_1 the "monic" curve of weight $2g$ through D. When $r=2$, $m=2g+1$, $u_{-1}=$ the coefficient of λ^{g-1} in f_1 minus $c_{2g-1}^{(2)}$ (cf. 1.7 for the meaning of $c_j^{(k)}$); when $r=3$, $m=3M+1$, $u_{-1}=$ the coefficient of $\mu\lambda^{M-1}$ in f_1 minus $c_{3M-1}^{(2)}$.

Finally, we compute the equations for the image of Jac $C \setminus \Theta$, in two cases only, to emphasize the relation of the elimination matrix E to the BC matrix Λ. Since the case $r=2$ (hyperelliptic) has been treated exhaustively ([12]), we develop the $r=3$ model. Let $g=1$, $r=3$. Then: $m=2$, $M=0$, $l=2$ in the notation above and the generators of the ring $\mathcal{A}_D$ are taken to be: $B = \partial^3 + 3u_{-1}\partial + 3\dot{u}_{-1} + 3u_{-2}$, $L = \partial^2 + c\partial + 2u_{-1}$. We need 5 parameters (cf. 1.9) to write the E matrix: u_{-1}, $\dot{u}_{-1}$, $\ddot{u}_{-1}$, u_{-2}, $\dot{u}_{-2}$ (at $x=0$), but only 4 to write the Λ_0

matrix:

$$\Lambda_0 = \begin{bmatrix} 2\gamma_1 - \mu & c & 1 \\ \lambda - \gamma_2 & -\gamma_1 - \mu & c \\ c\lambda - \gamma_3 & \lambda - \gamma_4 & -\gamma_1 - \mu \end{bmatrix}$$

where $\gamma_1 = u_{-1}$, $\gamma_2 = \dot{u}_{-1} + 3u_{-2}$, $\gamma_3 = \ddot{u}_{-1} + 3\dot{u}_{-2} + 3c(\dot{u}_{-1} + u_{-2})$, $\gamma_4 = 2\dot{u}_{-1} + 3u_{-2} + 3cu_{-1}$. Thus the equation of C_0 is:

$$-\mu^3 + \mu(3c\lambda + 3\gamma_1^2 - c\gamma_2 - \gamma_3 - c\gamma_4) + \lambda^2 + \lambda(c^3 - \gamma_2 - \gamma_4) + 2\gamma_1^3 - c\gamma_1\gamma_2$$
$$-\gamma_1\gamma_3 + 2c\gamma_1\gamma_4 + \gamma_2\gamma_4 - \gamma_3 c^2 = -\mu^3 + \mu(3c\lambda + a) + \lambda^2 + b\lambda + d, \text{ fixed,}$$

and $\operatorname{Jac} C \setminus \Theta$ is isomorphic to the affine subvariety of $\mathbf{C}^4 = \{(\gamma_1, \gamma_2, \gamma_3, \gamma_4)\}$ given by the equations:

$$3\gamma_1^2 - c\gamma_2 - \gamma_3 - c\gamma_4 = a$$
$$c^3 - \gamma_2 - \gamma_4 = b$$
$$2\gamma_1^3 - c\gamma_1\gamma_2 - \gamma_1\gamma_3 + 2c\gamma_1\gamma_4 + \gamma_2\gamma_4 - \gamma_3 c^2 = d$$

Remark. In the above example, the map $E_0 \mapsto \Lambda_0$ has fibres of dimension 1. In particular, not all E_0-matrices satisfying the equation of the curve correspond to a divisor; equivalently, the initial conditions $\psi(0) = v_1, \ldots, \psi^{(m+r-1)}(0) = v_{m+r}$ for the joint eigenfunctions in general do not correspond to a solution.

For $g = 2$, we cannot have the plane-model property *corresponding to a point* P_∞ such that $H^0(3P_\infty) \setminus H^0(2P_\infty) \neq \varnothing$. To describe $\operatorname{Jac} C \setminus \Theta$ we need two BC matrices: 11 parameters are needed for writing E.

For $g = 3$, the general curve has Weierstrass number 1 ([6]) and does not have the plane-model property for any point P_∞: 15 parameters are needed for writing E_0. We work out the example of curves with the plane-model property; $r = 3$, $m = 4$, $M = 1$, $l = 1$, $B = \partial^3 + 3u_{-1}\partial + 3\dot{u}_{-1} + 3u_{-2}$, $L = \partial^4 + 4u_{-1}\partial^2 + (4u_{-2} + 6\dot{u}_{-1})\partial + 4\ddot{u}_{-1} + 6\dot{u}_{-2} + 4u_{-3} + 6u_{-1}^2 + c_2(\partial^2 + 2u_{-1}) + c_1\partial$, but we let $c_1 = c_2 = 0$ for simplicity. 9 parameters are needed for writing E_0, according to (1.9); indeed, we need $u_{-1}, \dot{u}_{-1}, \ldots, u_{-1}^{(4)}$; $u_{-2}, \ldots, u_{-2}^{(3)}$, $u_{-3}, \ldots, \ddot{u}_{-3}$ but $u_{-3} = -\frac{1}{3}(3\dot{u}_{-2} +$

$3u_{-1}^2 + \ddot{u}_{-1})$. In this case, in calculating the BC matrix we also need 9 parameters,

$$\Lambda_0 - \mu = \begin{bmatrix} \gamma_1 - \mu & \lambda + \gamma_2 & \gamma_3 \\ \gamma_3\lambda + \gamma_4 & \gamma_5 - \mu & \lambda + \gamma_6 \\ \lambda^2 + \gamma_7\lambda + \gamma_8 & -2\gamma_3\lambda + \gamma_9 & -(\gamma_1 + \gamma_5) - \mu \end{bmatrix} \text{(where } \gamma_3 = u_{-1}\text{)}$$

and the fixed curve $\det(\Lambda_0 - \mu)$ gives us 6 equations (the coefficient of μ^2 and $\lambda^2\mu$ are zero because of the relations holding among the entries of Λ_0), defining $\operatorname{Jac} C\setminus\Theta$:

$$\gamma_9 - 2\gamma_3\gamma_6 + \gamma_2\gamma_3 + \gamma_4 + \gamma_3\gamma_7 = a$$

$$(\gamma_1 + \gamma_5)^2 - \gamma_1\gamma_5 + \gamma_6\gamma_9 + \gamma_2\gamma_4 + \gamma_3\gamma_8 = b$$

$$\gamma_2 + \gamma_6 + \gamma_7 = c$$

$$3\gamma_1\gamma_3 + \gamma_2\gamma_6 + \gamma_2\gamma_7 + \gamma_6\gamma_7 + \gamma_8 - 2\gamma_3^3 = d$$

$$\gamma_1(-\gamma_9 + 2\gamma_3\gamma_6) + (\gamma_1 + \gamma_5)(\gamma_2\gamma_3 + \gamma_4) + \gamma_2\gamma_6\gamma_7 + \gamma_8(\gamma_2 + \gamma_6) + \gamma_3(\gamma_9\gamma_3 - 2\gamma_4\gamma_3) - \gamma_3\gamma_5\gamma_7 = e$$

$$(\gamma_1 + \gamma_5)(\gamma_2\gamma_4 - \gamma_1\gamma_5) - \gamma_1\gamma_6\gamma_9 + \gamma_2\gamma_6\gamma_8 + \gamma_3\gamma_4\gamma_9 - \gamma_3\gamma_5\gamma_8 = f$$

where $-\mu^3 + (a\lambda + b)\mu + \lambda^4 + c\lambda^3 + d\lambda^2 + e\lambda + f = 0$ is the equation of the curve.

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Emma Previato
Mittag-Leffler Institute
S-18262 Djursholm – Sweden
(Permanent address:
Mathematics Dept. – Boston University
Boston, MA 02215)

Received September 23, 1986