Bounded domains with prescribed group of automorphisms.

Comment. Math. Helvetici 62 (1987) 561–572 0010-2571/87/040561-12$01.50 + 0.20/0

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§ 0. Introduction

By an automorphism of a complex manifold Ω we mean a biholomorphic mapping $f:\Omega\to\Omega$. A classical result of *H.* Cartan (see [9]) states that for a bounded domain $\Omega\subset\mathbb{C}^n$ Aut (Ω) has the structure of a Lie group. This is also the case if Ω is a relatively compact domain in a Stein manifold.

Let $\Omega\Subset\tilde{\Omega}$ be a relatively compact domain in a Stein manifold with a C^2, strongly pseudoconvex boundary. It is known [11] that if Aut (Ω) is not compact then Ω is biholomorphic to an open unit ball in $\mathbb{C}^n$. Thus the automorphism group of such a domain is either $SU(n, 1)$ or a compact Lie group. It is natural to ask whether every compact group can appear as the automorphism group of such Ω. For the case of the trivial group $G=\{\text{id}\}$, there are triply connected domains in $\mathbb{C}$ with smooth boundary but with no nontrivial automorphisms. Finding contractible strongly pseudoconvex domains $\Omega\subset\mathbb{C}^n$, $n\geq 2$ with Aut $(\Omega)=\{\text{id}\}$ is less easy, but it is possible to take Ω to be a small, smooth perturbation of the ball B^n (see [4]). The next simplest case is $G=T^1$, the circle group. There is no smoothly bounded Riemann surface M with Aut $(M)=T^1$, but an appropriate domain may be constructed in $\mathbb{C}^2$ (Proposition 1.3). In this paper (Theorems 1, 2) we show how to construct a smoothly bounded domain Ω in $\mathbb{C}^n$ whose group of biholomorphisms is any prescribed compact group G. If G is connected our construction (§ 3) is quite explicit:

THEOREM 1. *Let G be a connected compact Lie Group and $G_{\mathbb{C}}$ its complexification. Then there exists a strongly pseudoconvex domain $\Omega\subset G_{\mathbb{C}}$ (or $\Omega\subset G_{\mathbb{C}}\times\mathbb{C}$ in case the center of G is one dimensional) with real analytic boundary so that* Aut $(\Omega)=G$, *acting by left translations.*

The object in constructing $\Omega\subset G_{\mathbb{C}}$ is to keep it invariant under left translations by G while ruling out additional symmetries. If G acts on a complex

* Supported in part by NSF grants MCS 82-122273 and MCS 81-01635.

manifold $M \neq G_{\mathbb{C}}$ by biholomorphisms it may happen that no such $\Omega \subset M$ exists (see example 3.0).

The following two theorems were first obtained by Saerens and Zame [12] independently of our Theorem 1, but the proofs we give in § 4 are shorter and more elementary in nature.

THEOREM 2. *Let G be any compact Lie group. Then there is a strongly pseudoconvex domain* $\Omega \Subset \mathbb{C}^n$ *with real analytic boundary such that* $\mathrm{Aut}(\Omega) = G$.

THEOREM 3. *Let G be a compact Lie group. Then there exists a surface* $\Sigma \subset \mathbb{R}^n$ *which is an arbitrarily small smooth perturbation of the unit sphere* $S^{n-1} \subset \mathbb{R}^n$ *whose group of isometries is linear and isomorphic to G. Moreover, if for some affine map T of* $\mathbb{R}^n$ $T\Sigma = \Sigma$ *then* $T \in G \subset O(n)$

Remark. The dimension n in the above two theorems may be taken to be $n = k^2$ if G has a faithful action on $\mathbb{R}^k$.

Note that while Theorem 2 applies to disconnected Lie groups it only gives existence of required domains Ω (typically with $\dim \Omega \gg \dim G$). Its proof cannot be used to actually construct Ω without prohibitive calculations.

Acknowledgement

The authors would like to thank David Barrett for many helpful comments and suggestions regarding this work.

Notation

Let G be a compact group and $\mathfrak{g}$ its Lie algebra. We choose a faithful imbedding of G into some unitary group $U(n)$. Thus $\mathfrak{g} \subset \mathcal{U}(n)$ is a subalgebra of skew Hermitian matrices. We set $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ and $G_{\mathbb{C}} \subset GL(n, \mathbb{C})$ the connected Lie group corresponding to $\mathfrak{g}_{\mathbb{C}}$. If G itself is connected then $G \subset G_{\mathbb{C}}$ as a totally real submanifold of the Stein manifold $G_{\mathbb{C}}$. If $\omega \subset \mathfrak{g}$ is a small neighborhood of $0 \in \mathfrak{g}$ then $\Omega = G \cdot \exp i\omega \approx G \times \omega$ is a tubular neighborhood of G in $G_{\mathbb{C}}$. Here exp is the matrix exponential function. The groups $L(G)$, $R(G) \subset \mathrm{Aut}(G_{\mathbb{C}})$ are the groups of left and right translations by G. If $g \in G$, X, $Y \in \mathfrak{g}$ we, as usual, define, $\mathrm{Ad}(g)X = gXg^{-1}$ and $\mathrm{ad}(X)(Y) = XY - YX$ (matrix multiplication).

§ 1. Tori

In this section we give examples of domains whose automorphism groups are T^n. First we consider a Reinhardt domain $\Omega \subset \mathbb{C}^n$, i.e. Ω is invariant under $(z_1, \ldots, z_n) \to (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n)$ for $\theta_1, \ldots, \theta_n \in \mathbb{R}$. It is obvious that T^n is contained in the automorphism group of a Reinhardt domain. The logarithmic image of Ω is

$$\omega = \mathrm{Log}(\Omega) = \{(\xi_1, \ldots, \xi_n) : (e^{\xi_1}, \ldots, e^{\xi_n}) \in \Omega\}. \tag{1}$$

The automorphism group for certain Ω is given as follows (see [1]).

THEOREM 1.1. *If Ω is a Reinhardt domain, and if* $\mathrm{Log}(\Omega)$ *is a bounded convex domain in $\mathbb{R}^n$, then* $\mathrm{Aut}(\Omega)$ *consists of transformations of the form*

$$(z_1, \ldots, z_n) \to (c_1 z^{m_1}, \ldots, c_n z^{m_n}) \tag{2}$$

where the matrix M with rows $m_1, \ldots, m_n$ belongs to $GL(n, Z)$.

It is evident that a mapping of the form (2) will map Ω to Ω if and only if $T\xi = M\xi + \log|c|$ is an affine self-mapping of ω.

COROLLARY 1.2. *Let $\omega \subset \mathbb{R}^n$ be a bounded convex domain, and let $\Omega \subset \mathbb{C}^n$ be the Reinhardt domain with* $\mathrm{Log}(\Omega) = \omega$. *If ω has no nontrivial affine self-mappings, then* $\mathrm{Aut}(\Omega) = T^n$.

If $n \geq 2$, then it is clear that a "generic" domain ω in $\mathbb{R}^n$ has no affine self-mappings. This is not true for $n = 1$, since every interval in $\mathbb{R}$ has an (affine) inversion.

Let $D \subset \mathbb{C}$ be a smoothly bounded triply connected domain with $\mathrm{Aut}(D) = id$. Let $0 < r_1(z) < r_2(z)$ be continuous functions on $\bar{D}$ and set

$$\Omega = \{(z, w) \in D \times \mathbb{C} : r_1(z) < |w| < r_2(z)\}.$$

PROPOSITION 1.3. *Let $D \subset \mathbb{C}$ be a smoothly bounded triply connected domain with* $\mathrm{Aut}(D) = id$. *If we choose $r_1(z)$, $r_2(z)$ such that $r_1(z) r_2(z)$ is not the modulus of an analytic function on D, then with Ω as above,* $\mathrm{Aut}(\Omega) = T^1$.

It is clear that we may arrange for Ω to have real analytic, strongly pseudoconvex boundary.

For the proof we will use invariant 2-forms, as in [2]. We may choose a_1,

$a_2 \in \mathbb{C}$ such that

$$T_j = \frac{dz \wedge dw}{(z - a_j)w}, \qquad j = 1, 2$$

are linearly independent cohomology classes. If $[T_j]$ is the set of holomorphic 2-forms cohomologous to T_j, then there exists a unique ω_{T_j} which minimizes the L^2-norm $\|\omega\| = |\int_\Omega \omega \wedge \bar{\omega}|^{1/2}$ over $[T_j]$. We may write

$$\omega_{T_j} = \sum_{k=-\infty}^{\infty} f_j^k(z) w^k \, dz \wedge dw.$$

Since T_j is independent of the rotation $(z, w) \to (z, e^{i\theta}w)$, so is ω_{T_j}. Thus

$$\omega_{T_j} = f_j(z) w^{-1} \, dz \wedge dw.$$

If $f \in \operatorname{Aut}(\Omega)$, then

$$\frac{\omega_{f^*T_1}}{\omega_{f^*T_2}} = f^*\left(\frac{\omega_{T_1}}{\omega_{T_2}}\right). \tag{3}$$

By the arguments above, $\omega_{T_1}/\omega_{T_2} = m(z)$ is a nonconstant meromorphic function, and the left hand side of (3) is another meromorphic function, $\tilde{m}(z)$. Thus writing $f(z, w) = (f_1(z, w), f_2(z, w))$ we have

$$\tilde{m}(z) = m(f_1(z, w))$$

and so $f_1(z, w)$ depends on the variable z alone. We conclude, then, that f induces a mapping of the vertical fibers $\Omega_{z_0} = \{(z, w) \in \Omega : z = z_0)$ of Ω. Thus f_1 is an automorphism of D, and therefore $f_1(z, w) = z$.

We conclude from this, that $f_2(z, w)$ must be an automorphism of the fiber

$$\Omega_z = \{w \in \mathbb{C} : r_1(z) < |w| < r_2(z)\}.$$

Therefore either

$$f_2(z, w) = e^{i\theta(z)} w$$

or

$$f_2(z, w) = c(z)/w.$$

In the first case, $\theta(z) = \theta_0$ is constant, since it must be real-valued and holomorphic. In the second case, however, $c(z)$ is a holomorphic function and

$$r_1(z) = |c(z)|/r_2(z).$$

which is a contradiction.

§ 2. Simple and connected groups

Given a simple compact group G we construct (in Lemma 2.3) a domain $\Omega \subset G_{\mathbb{C}}$ in the complexification of G and prove (Proposition 2.6) that the connected component of the identity $\mathrm{Aut}(\Omega)_0 = G$. At first we shall assume that G is compact connected and semi-simple. Simplicity of G is necessary only in Proposition 2.6.

The Killing form $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y)$ is negative definite on $\mathfrak{g}$, the Lie algebra of G. In this section we shall use $-\kappa$ as the inner product on $\mathfrak{g}$. This inner product on the left invariant vector fields gives a biinvariant metric on G.

Let $\mathrm{Aut}(\mathfrak{g})$ be the Lie group of Lie algebra automorphisms of $\mathfrak{g}$, and $\mathrm{Ad}(G) \subset \mathrm{Aut}(\mathfrak{g})$ be the image of G under the adjoint representation i.e. the inner automorphisms of $\mathfrak{g}$. Recall [7] that $\mathrm{Aut}(\mathfrak{g})/\mathrm{Ad}(G)$ is a finite group (of order ≤ 6 if $\mathfrak{g}$ is simple) and that each $\sigma \in \mathrm{Aut}(\mathfrak{g})$ preserves the Killing form which we will write as $\mathrm{Aut}(\mathfrak{g}) \subset O(\mathfrak{g})$, the orthogonal transformations on $\mathfrak{g}$. If $I \in O(\mathfrak{g})$ is the identity map we readily observe:

LEMMA 2.1. $-I \notin \mathrm{Aut}(\mathfrak{g})$.

We proceed to construct the domain $\Omega \subset G_{\mathbb{C}}$. Let $\{e_1, \ldots, e_d\}$ be an orthonormal basis for $\mathfrak{g}$ and let $x_1, \ldots, x_d$ be coordinates with respect to this basis.

LEMMA 2.2. *On $\mathfrak{g} \backslash \{0\}$ we may find a smooth function $\psi(x)$ with the following properties:*

i) $\psi(\lambda x) = |\lambda|\, \psi(x)$ *for all* $\lambda \in \mathbb{R}$

ii) $\psi \circ g = \psi$, $g \in O(\mathfrak{g})$ *implies* $g = \pm I$.

Proof. Let $\psi_0(x) = \sum_{i=1}^{d} x_i^4$. Note that the maxima of ψ_0 on the unit sphere $S^{d-1} \subset \mathfrak{g}$ are at $\pm e_i$ $i = 1, 2, \ldots, d$. Set $v = (1, 2, 3, \ldots, d)$ and

$$\psi_\epsilon(x) = \frac{\psi_0(x) + \epsilon \langle x, v\rangle^4}{|x|^3}.$$

For small enough $\epsilon > 0$ there will be local maxima of ψ_ϵ at $\pm \tilde{e}_i$ with $\tilde{e}_i$ very close to e_i, and with

$$\langle \tilde{e}_i, \tilde{e}_j \rangle > 0 \quad \text{for all } i, j. \tag{4}$$

Further, it is clear that for small enough ϵ we will have

$$\psi_\epsilon(\tilde{e}_i) \neq \psi_\epsilon(\tilde{e}_j), \qquad \text{if } i \neq j. \tag{5}$$

Thus if we fix $\epsilon > 0$ with above properties and assume that $\psi_\epsilon \circ g = \psi_\epsilon$, $g \in O(\mathfrak{g})$ we obtain, using (5), that $g(\tilde{e}_i) = \pm\tilde{e}_i$. Finally, using (4), we must have $g(\tilde{e}_i) = \tilde{e}_i$ or $g(\tilde{e}_i) = -\tilde{e}_i$ for all $i = 1, 2, \ldots, d$.

LEMMA 2.3. *There exists a domain $\omega \subset \mathfrak{g}$ such that*

i) $\omega = -\omega$

ii) $\Omega = G \cdot \exp(i\omega)G_{\mathbb{C}}$ *is strongly pseudoconvex and smoothly bounded.*

iii) *If $\sigma \in \operatorname{Aut}(\mathfrak{g})$ and $\sigma(\omega) = \omega$, then $\sigma = I$.*

Proof. Let $\psi(x)$ be as in Lemma 2.2 and set

$$\omega_{\epsilon,\delta} = \left\{x \in \mathfrak{g} : \sum_{i=1}^{d} x_i^2 + \delta\psi(x) < \epsilon^2\right\}.$$

From Lemmas 2.1 and 2.2 it follows that for ϵ, $\delta > 0$ properties i) and iii) hold. For $\epsilon \gg \delta > 0$ sufficiently small property ii) holds as well.

For the next lemma we observe that any group automorphism h of G extends to (a holomorphic) automorphism of $G_{\mathbb{C}}$, here denoted also by h.

LEMMA 2.4. *Let Ω be as in Lemma 2.3. Suppose h, an automorphism of G, and $X \in \omega$ have the property that*

$$R_{\exp iX} \circ h(\Omega) = \Omega.$$

Then $X = 0$ and h is the identity automorphism.

Proof. The differential $dh \in \operatorname{Aut}(\mathfrak{g}) \subset \operatorname{Aut}(\mathfrak{g}_{\mathbb{C}})$. We thus have

$$h(\exp iY) = \exp(i\, dh(Y)), \qquad Y \in \mathfrak{g}.$$

In $G_{\mathbb{C}}$ consider the curve $\gamma(t) = \exp itX$. Since $dh(\omega) = -dh(\omega)$ it follows that

$$\{t \in \mathbb{R} : \gamma(t) \in \Omega\} = (-a, a), \qquad a > 0$$

is a symmetric interval. Next we observe that if $X \neq 0$ the set

$$\begin{aligned}\{t \in \mathbb{R} : \gamma(t) \in R_{\exp iX} \circ h(\Omega)\} &= \{t \in \mathbb{R} : \gamma(t)\exp(-tX) \in h(\Omega)\} \\ &= \{t \in \mathbb{R} : (t-1)X \in dh(\omega)\}\end{aligned}$$

is of the form $(-b+1, b+1)$ and thus not symmetric. This contradiction shows that $X = 0$ and Lemma 2.3 then forces $dh = I$.

COROLLARY 2.5. *Suppose $R_z(\Omega) = \Omega$, $z \in G_{\mathbb{C}}$. Then $z \in Z(G_{\mathbb{C}})$, the center of $G_{\mathbb{C}}$.*

Proof. Write $z = g \exp iX$, $g \in G$, $X \in \mathfrak{g}$. Since $L_{g^{-1}}(\Omega) = \Omega$ we have that $R_{\exp iX} \circ h(\Omega) = \Omega$, where $h(x) = g^{-1}xg$ is an inner automorphism of G. Lemma 1.4 then implies that $X = 0$ and $gx = xg$ for all $x \in G$ and thus, by extending holomorphically, for all $x \in G_{\mathbb{C}}$. □

PROPOSITION 2.6. *Let G be a simple connected Lie group and $\Omega \subset G_{\mathbb{C}}$ as constructed in Lemma* 2.3. *Then the connected component of the identity is* $\operatorname{Aut}(\Omega) = L(G)$.

Proof. Recall that $d = \dim G$. Since $\Omega \subset G_{\mathbb{C}}$ is a small tubular neighborhood of G, we have $H_d(\Omega, \mathbb{Z}) = \mathbb{Z}$. By Lemma 2.3 of [3] there exists an orbit of $\operatorname{Aut}(\Omega)$ in Ω whose dimension is at most d. Since $L(G) \subset \operatorname{Aut}(\Omega)$ that orbit must be a finite union of G orbits, and any of these are stable under $\operatorname{Aut}(\Omega)_0$. So suppose $G \cdot x_0$ is $\operatorname{Aut}(\Omega)_0$ stable for some $x_0 \in \Omega$. Restricting the Bergmann metric ds^2 to the mainfold $G \cdot x_0 \simeq G$ we see that $\operatorname{Aut}(\Omega)_0$ is naturally a subgroup of the connected component $I_0(G, ds^2)$ of the isometry group. By Theorem 1 of [10] it now follows that any $f \in \operatorname{Aut}(\Omega)_0$ is of the form

$$f(g \cdot x_0) = agb \cdot x_0, \qquad g \in G$$

for some a, $b \in G$. Extending holomorphically to $g \in G_{\mathbb{C}}$ we see that $f = L_a \circ R_{x_0^{-1}bx_0}$, and hence $R_{x_0^{-1}bx_0}(\Omega) = \Omega$. By Corollary 2.5 $b \in Z(G_{\mathbb{C}})$, and so $f = L_{ab} \in L(G)$. □

§ 3. Connected groups and proof of Theorem 1

PROPOSITION 3.1. *Let G be a compact, connected Lie group. Then there exists a piecewise strongly pseudoconvex domain $\Omega \subset G_{\mathbb{C}}$,* (or *$\Omega \subset G_{\mathbb{C}} \times \mathbb{C}$ in case the center of G is one dimensional*) *such that* $G = \operatorname{Aut}(\Omega)$.

One may contemplate constructing such ω domain Ω inside other complex manifolds that posses a natural G action. The following example shows that achieving $G = \operatorname{Aut}(\Omega)$ may be impossible.

EXAMPLE 3.0. Let $G = SO(3)$ act on the complex sphere

$$\Sigma = \{z \in \mathbb{C}^3 \mid \sum z_i^2 = 1\}$$

Every G orbit on Σ intersects the curve

$$\alpha(t) = (\cosh t, i \sinh t, 0)$$

at $\alpha(\pm s)$ for some s. Consequently the only G invariant pseudoconvex domains in Σ are $\Omega_R = \{z \in \Sigma \mid |z| < R\}$. These domains are also $O(3)$ invariant.

Proof of 3.1. Any compact connected Lie group G is of the form

$$G = T^l \times G_1 \times \cdots \times G_k/H \quad \text{where} \quad G_1, \ldots, G_k \text{ are simple,}$$

1-connected and connected, $H \subset Z(T^l \times G_1 \times \cdots \times G_k)$ is finite, and $H \cap T^l = \{e\}$.

In the following we will denote $G_1 \times \cdots \times G_k$ by G_s. Let Ω^0 be a domain with Aut $(\Omega^0) = T^l$, as constructed in Section 1. For each simple factor G_j let Ω^j the domain constructed in Lemma 2.3. Moreover, we may arrange our choice of ω_i's so that Ω_i is not biholomorphically equivalent to Ω_j if $i \neq j$. To see this, we need only to note that if we shrink ω_j, then we obtain a biholomorphically inequivalent Ω_j (see, for instance, Theorem 3.3 of [2]). Now set

$$D = \Omega_0 \times \Omega^1 \times \cdots \times \Omega^k.$$

We note that D is biholomorphic to a domain in $(T^l \times G_1 \times \cdots \times G_k)_{\mathbb{C}}$ of the form $(T^l \times G_s) \cdot \exp\{i(\omega^0 \times \omega^1 \times \cdots \times \omega^k)\}$. By our choice of D and theorem of *H*. Cartan [9]

$$\operatorname{Aut}(D) = T^l \times \operatorname{Aut}(\Omega^1) \times \operatorname{Aut}(\Omega^2) \times \cdots \times \operatorname{Aut}(\Omega^k). \tag{6}$$

Next set $\Omega = D/L(H)$. Again, biholomorphically

$$\Omega = G \cdot \exp\{i(\omega^0 \times \omega^1 \times \cdots \times \omega^k\} \subset G_{\mathbb{C}}.$$

If $f \in \operatorname{Aut}(\Omega)_0$ then it is homotopic to the identity and may thus be lifted to $\tilde f \in \operatorname{Aut}(D)_0$. By (6) and Proposition 2.6 $\tilde f = L_{\tilde g}$, $\tilde g \in T^l \times G^1 \times \cdots \times G^k$ and thus $f = L_g$ for some $g \in G$. Hence $L(G) = \operatorname{Aut}(\Omega)_0$ is a normal subgroup of Aut (Ω). Therefore if $h \in \operatorname{Aut}(\Omega)$

$$hL_g h^{-1} = L_{\chi(g)}, \quad \chi \in \operatorname{Aut}(G),$$

that is for any $x \in \Omega$

$$h(g \cdot x) = \chi(g) \cdot h(x).$$

Setting $x = e \in G$, $h = R_{h(e)} \circ \chi$ on $G \subset \Omega$ which gives $h = R_{h(e)} \circ \chi$ on Ω after extending holomorphically to $\chi \in \operatorname{Aut}(G_{\mathbb{C}})$. By composing h with a suitable left translation L_g, $g \in G$ we may assume that $h(e) = \exp iX$, $X \in \mathfrak{g}$. We write the Lie algebra of $\mathfrak{g}$ as $\mathfrak{g}_0 + \mathfrak{g}_s$ where $\mathfrak{g}_s$ is the Lie algebra of G_s and $\mathfrak{g}_0$ is the center of $\mathfrak{g}$, i.e. the Lie algebra of T^l. The differential of χ must preserve this decomposition, so $d\chi = d\chi_0 \circ d\chi_s = d\chi_s \circ d\chi_0$. Similarly we can write $X = X_0 + X_s$ so $\exp iX_0 \exp iX_s = \exp iX_s \exp iX_0$. We conclude that translation by X_0 followed by $d\chi_0$ preserves ω^0 and thus by assumption on ω^0 $X_0 = 0$ and $d\chi_0 = I$. Finally as in the proof of Lemma 2.4 we must have $X_s = 0$ and $d\chi_s(\omega^1 \times \cdots \times \omega^s) = \omega^1 \times \cdots \times \omega^2$. Our choice of these domains forces first $d\chi_s(\omega^j) = \omega^j$ and then $d\chi_s = I$. □

To complete the proof of Theorem 1 we now apply the semicontinuity theorem of Greene and Krantz [6] to smoothen the domain Ω. Let $r(z)$ be a G invariant strongly plurisubharmonic exhaustion function of Ω. For large λ

$$\Omega_\lambda = \{z \in \Omega : r(z) < \lambda\}$$

is strongly pseudoconvex with smooth real analytic boundary. Evidently $G \subset \mathrm{Aut}(\Omega_\lambda)$. Lemma 3.2 below shows that $\mathrm{Aut}(\Omega_\lambda)$ is a normal family of groups in the sense of Greene and Krantz, and thus by their semicontinuity theorem $\mathrm{Aut}(\Omega_\lambda) \subset G$ for λ sufficiently large. The proof of the theorem is now complete.

LEMMA 3.2. *Let (λ_j) be a sequence converging to $+\infty$ and let $\varphi_j \in \mathrm{Aut}(\Omega_{\lambda_j})$. Then there exists a subsequence $\{\varphi_{j_k}\}$ converging uniformly on compact sets to an element $\varphi \in \mathrm{Aut}(\Omega)$.*

Proof. Since Ω is bounded we may assume: (by extracting a subsequence) that $\{\varphi_j\}$ converges uniformly on compact subsets to a holomorphic $\psi \in \Omega \to \bar{\Omega}$. By a theorem of H. Cartan [9] either $\psi \in \mathrm{Aut}(\Omega)$ or $\psi(\Omega) \subset \partial\Omega$. We now show, arguing as in [3], that the latter case is impossible. Recall that by construction Ω is covered by a product of bounded domains. By lifting our maps we may assume that Ω itself is a product

$$\Omega = \Omega^0 \times \Omega^1 \times \cdots \times \Omega^k.$$

Suppose $\psi(\Omega) \cap \partial\Omega^0 \times \Omega^1 \times \cdots \times \Omega^k \neq 0$. Then, since $\partial\Omega^0$ is strongly pseudoconvex, $\psi(\Omega) \subset \{p_0\} \times \Omega^1 \times \cdots \times \Omega^k$ for some $p_0 \in \partial\Omega^0$. Now let U be a contractible neighborhood of p_0 in Ω^0, and let T be a compactly supported cycle representing a nontrivial class in $H_q(\Omega)$, where $q = \dim_{\mathbb{C}} \Omega$. For large j we have $\varphi(T) \subset U \times \Omega^1 \times \cdots \times \Omega^k$ which is homologically trivial in dimension q. On the other hand φ_j is a diffeomorphism and hence $\psi_j(T)$ cannot be a boundary in Ω for j large.

§ 4. Existence proofs

In this section we prove Theorems 2 and 3.

PROPOSITION 4.1. *Let G be a compact Lie group. Then there exists an* *orthogonal action of G on $\mathbb{R}^n$ with the following properties*

(i) *If $H \subset O(n)$ is a subgroup such that $Hx = Gx$ for all $x \in \mathbb{R}^n$, then $H = G$.*

(ii) *There exists a set $F \subset \mathbb{R}^n$ consisting of finitely many G-orbits such that if $g \in O(n)$ and $gF = F$, then $g \in G$.*

Proof. Let G be faithfully imbedded in $O(k)$, and let G act diagonally on

$$\mathbb{R}^n = \mathbb{R}^k \oplus \cdots \oplus \mathbb{R}^k \quad (k \text{ times}). \tag{$*$}$$

First we show that (i) is satisfied for this action. By assumption, the decomposition of $\mathbb{R}^n$ in ($*$) is also H-invariant. For $v_1, \ldots, v_k \in \mathbb{R}^k$, we write $v = (v_1, \ldots, v_k) \in \mathbb{R}^n$ by the decomposition ($*$). For $h \in H$, and $u = (v, \ldots, v) \in \mathbb{R}^n$, there exists by assumption $g \in G$ such that

$$hu = gu = (w, \ldots, w).$$

Thus we conclude that H acts diagonally on the decomposition ($*$). Finally, if $\{e_1, \ldots, e_k\}$ is a basis of $\mathbb{R}^k$, we set $v = (e_1, \ldots, e_k)$ to see that $hv = gv$ implies that $h = g$.

For part (ii), we construct a sequence of sets F_j, $j = 1, 2, 3, \ldots,$ with the following properties:

1. F_j is the union of j G-orbits.
2. If $H_j = \{g \in O(n): gF_j = F_j\}$, and if $H_j \neq G$, then $H_j \supsetneqq H_{j+1}$.

Since the H_j are compact and each contains G, we must have $H_l = G$ form some l. Indeed, at each step either the dimension or the number of components must decrease.

Now fix $\epsilon > 0$, pick x_1 of length $1 + \epsilon$, and set $F_1 = Gx_1$. We proceed inductively, under the assumption that $H_j \neq G$. By part (i) there exists a point x_{j+1} such that $H_j x_{j+1} \neq Gx_{j+1}$. We then set

$$F_{j+1} = F_j \cup Gx_{j+1}.$$

Since we may take $\|x_{j+1}\| > \|x\|$ for all $x \in F_j$, we see that $H_{j+1}F_j \subset F_j$, and thus $H_{j+1} \subsetneqq H_j$.

PROPOSITION 4.2. *Let G be a compact Lie group. There exists an orthogonal action of G on $\mathbb{R}^n$ and a G invariant domain $\omega \subset \mathbb{R}^n$ which is a small, smooth perturbation of the unit ball with the property that $g\omega = \omega$ and g affine implies $g \in G$.*

Proof. Let $Gx_1 \cup \cdots \cup Gx_l$ denote the set obtained in (ii) of Proposition 1. For any $\delta > 0$, we may assume that $1 + \delta > |x_1| > |x_2| > \cdots > |x_l| > 1$. From S^{n-1} we remove a small tubular neighborhood V_j of $|x_j|^{-1} \cdot Gx_j$ such that the area of V_j is small and such that $\bar{V}_i \cap \bar{V}_j = \varnothing$ for $i \neq j$.

Now we may make a small smooth perturbation of S^{n-1} of the form

$$\Sigma = \{r(x)x : x \in S^{n-1}\}$$

where $r(x)$ is a smooth function on S^{n-1} with $r \geq 1$, and $r(x) = 1$ for $x \notin \bigcup_{j=1}^{l} V_j$. Let us write

$$\omega = \{x \in \mathbb{R}^n : |x| < r(x/|x|)\}.$$

Before we specify $r(x)$ more precisely, let us note that if h is an affine transformation of $\mathbb{R}^n$ with $h(\Sigma) = \Sigma$, then $h \in O(n)$. To see this, write

$$\omega_1 = \{x \in \mathbb{R}^n : |x| < 1, x/|x| \notin V_1 \cup \cdots \cup V_l\}.$$

Thus ω_1 is a conical subset of ω generated by the complement of $V_1 \cup \cdots \cup V_l$. Since $h(\omega) = \omega$, h must preserve volume. And since the volume of $\omega - \omega_1$ is small, $h(\omega_1) \cap \omega_1$ contains an open set. It follows, then, that $|h(x)| = 1$ for x in an open subset of S^{n-1}. We conclude, then, that $h \in O(n)$.

Let $\chi \in C^\infty(\mathbb{R})$ be monotone decreasing with $\chi(0) = 1$, $\chi'(0) < 0$ and $\chi = 0$ on $[1, \infty)$. We define

$$r(x) = 1 + (|x_j| - 1)\chi(M \operatorname{dist}^2(x, |x_j|^{-1} G x_j))$$

for $x \in V_j$ and $r = 1$ elsewhere on S^{n-1}. For M sufficiently large, r is smooth. Choosing $\delta > 0$ sufficiently small, we have r close to 1.

Now if $h \in O(n)$ and $h\Sigma = \Sigma$, then h must map Gx_j to a portion of Σ with distance $|x_j|$ to the origin. At the same time, h must map Gx_j to a portion of Σ where the distance to the origin takes a local maximum. Thus $h(Gx_j) \subset Gx_j$. We conclude from Proposition 1, then, that $h \in G$.

Proof of Theorem 3. We let ω be the domain obtained in Proposition 2, and let $\Sigma = \partial\omega$. If ω is sufficiently close to the unit ball, then Σ is positively curved. Thus Σ is rigid, and any isometry g of Σ extends to an isometry of $\mathbb{R}^n$ (cf. [8]). It follows that $g \in G$, and thus G is the group of isometries of Σ.

Proof of Theorem 2. Let $\omega \subset \mathbb{R}^n$ be the domain from Theorem 1, and let

$$\Omega = (\omega + i\mathbb{R}^n) - V,$$

where

$$V = \{z_1^2 + \cdots + z_{n+1}^2 = \tfrac{1}{2}\}.$$

We claim that $\operatorname{Aut}(\Omega) = G$. Since $G \subset O(n)$, it follows that $\operatorname{Aut}(\Omega) \supset G$. On the other hand, ω is contained in a proper cone, and thus is biholomorphic to a bounded domain. Thus any $f \in \operatorname{Aut}(\Omega)$ extends to a holomorphic mapping $f \in \operatorname{Aut}(\omega + i\mathbb{R}^n)$. By the Corollary to Theorem 1 of [5] or by [13] $f(z)$ is of the form

$$f(z) = Az + b + ic$$

where $A \in GL(n, \mathbb{R})$, and b, $c \in \mathbb{R}^n$. Since $Az + b$ maps ω to itself, it follows from Proposition 2 that $b = 0$ and A represents an orthogonal transformation in G. Thus A maps V to itself, but it is evident that $V \neq V + ic$ if $c \neq 0$. We conclude, then, that $f \in G$.

To complete the proof of Theorem 2, we now smoothen the domain Ω, as in the proof of Theorem 1. The only difference is that in the normal families argument we now use the fact that Ω cannot be retracted to V, since $H_n(V \cap \Omega, \mathbb{Z}) \neq 0$ but Ω is contractible. We can then apply the Semicontinuity theorem of Greene and Krantz [6] to smoothen Ω.

REFERENCES

[1] E. BEDFORD, *Holomorphic mappings of products of annuli in* $\mathbb{C}^n$, Pacific J. Math. *87* (1980), 271–281.
[2] ——, *Invariant forms on complex manifolds with application to holomorphic mappings*, Math. Ann. 265 (1983), 377–397.
[3] ——, *On the automorphism group of Stein manifold*, Math. Ann. 266 (1983), 215–227.
[4] D. BURNS, S. SHNIDER, and R. O. WELLS, *Deformations of strictly pseudoconvex domains*, Inventiones Math. 46 (1978).
[5] J. DADOK and P. YANG, *Automorphisms of tube domains and spherical hypersurfaces*, Amer. J. Math. (1985), 999–1013.
[6] R. GREENE and S. KRANTZ, *Normal families and the semicontinuity of isometry and automorphism groups*, Math. Z. 190 (1985), 455–467.
[7] S. HELGASON, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic press, New York, 1978.
[8] N. HICKS, *Notes on Differential Geometry*, Van Nostrand, New York, 1971.
[9] R. NARASHIMHAN, *Several Complex Variables*, U. of Chicago Press, 1971.
[10] T. Ochiai and T. Takahashi, *The group of isometries of a left invariant metric on a Lie group*, Math. Ann. 223 (1976).
[11] J.-P. ROSAY, *Sur une caracteristation de la boule parmi les domaines de* $\mathbb{C}^n$ *par son groupe d'automorphismes*, Ann. Inst. Fourier Grenoble, 29 (1979), 91–97.
[12] R. SAERENS and W. ZAME, *The isometry groups of manifolds and the automorphism groups of domains*, preprint.
[13] P. YANG, *Automorphisms of tube domains*, Amer. J. Math., 104 (1982), 1005–1024.

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Received July 15, 1985/May 14, 1986