Nonexistence of spatially localized free vibrations for a class of nonlinear wave equations†

P.-A. Vuillermot

Abstract. We prove the nonexistence of free vibrations of arbitrary period with polynomially decreasing profiles for a large class of nonlinear wave equations in one space dimension Our class of admissible models includes examples of non integrable wave equations with certain polynomial nonlinearities, as well as examples of completely integrable ones with exponential nonlinearities related to Mikhailov's equations. Our result thus proves a particular case of a conjecture first formulated by Eleonskii, Kulagin, Novozhilova and Silin, and dispels some confusion regarding the relationship between the existence of so-called breather-solutions and the complete integrability of the wave equation. Our class of admissible nonlinearities also contains a particular instance of the nonlinear scalar Higgs' equation, but does not contain the Sine-Gordon equation which is known to possess a 2π-periodic solution in time with exponential fall-off in the spatial direction. Our results may be considered as complementary to recent results by Coron and Weinstein. Our arguments are entirely global, and rest upon methods from the calculus of variations.

1. Introduction, statement of the main results and discussion of some examples

There has been much interest and controversy recently concerning the existence of spatially localized periodic solutions to nonlinear Klein–Gordon equations of the form

$$u_{tt}(x; t) = u_{xx}(x; t) - g(u(x; t)) \tag{1.1}$$

where $(x; t) \in \mathbb{R}^2$. While the Sine–Gordon equation

$$u_{tt}(x; t) = u_{xx}(x; t) - 2 \sin (u(x; t)) \tag{1.2}$$

is known to possess the particular breather-solution

$$u_{SGB}(x; t) = 4 \arctan \left\{ \frac{\sin (t)}{\cosh (x)} \right\} \tag{1.3}$$

(see for instance [1] and [2]), it has been recently suggested that the wave equation

$$u_{tt}(x;t) = u_{xx}(x;t) + u(x;t) - u^3(x;t) \qquad (1.4)$$

should also possess spatially localized free vibrations converging to the trivial solutions $u_0 = \mp 1$ as $|x| \to \infty$ [3]. The case of convergence to $u_0 = 0$ is ruled out by virtue of a recent result by Coron [4], whose one-sided decay conditions imply that every spatially localized periodic solution to (1.4) is independent of time (see however, Example 1.3 below and the remark following it). Coron's theorem asserts that as long as $g \in \mathscr{C}^{(2)}(\mathbb{R}; \mathbb{R})$ and $g(0) = 0$, classical solutions to equation (1.1) cannot simultaneously be nontrivial T-periodic in time and exhibit a sufficiently fast spatial decay at infinity, unless the period of vibration is chosen sufficiently large – specifically, unless $g'(0) > (2\pi/T)^2$. It has been further conjectured by Eleonskii, Kulagin, Novozhilova and Silin in [5] and [6] that equation (1.1) possesses no nontrivial spatially localized periodic solutions whenever g is a polynomial, and also speculated by Brézis in [7] that a behaviour similar to (1.3) for the solutions to (1.1) should rather be an intrinsic feature of equation (1.2). However, Weinstein recently proved that if $g \in \mathscr{C}^{(2)}(\mathbb{R}; \mathbb{R})$, $g(0) = 0$ and $g'(0) > (2\pi/T)^2$, there do exist exponentially localized nontrivial T-periodic solutions to equation (1.1), under the crucial additional assumption that (1.1) be considered on a half-plane instead of $\mathbb{R}^2$ [8]. In this paper, we go one step further in proving the nonexistence of free vibrations with polynomially decreasing profiles for equation (1.1) on $\mathbb{R}^2$, for a broad class of nonlinearities and regardless of the value of the preassigned period T. In view of the applications, our class of admissible wave equations must allow for both polynomial and exponential nonlinearities in (1.1); according to the physical picture, our class of admissible spatially localized free vibrations must contain solutions to (1.1) which converge to constant solutions $u = u_0 \in \mathbb{R}$ as $|x| \to \infty$. These remarks motivate the following definitions.

DEFINITION 1.1. Let $u_0 \in \mathbb{R}$; the function $g : \mathbb{R} \to \mathbb{R}$ is said to be an admissible nonlinearity for equation (1.1) if the following hypotheses hold:

(H₁) $g \in \mathscr{C}(\mathbb{R}; \mathbb{R})$

(H₂) $\limsup_{|u| \to \infty} \dfrac{e^{-(u-u_0)^2}\, |g(u)|}{|u - u_0|} < \infty$

(H₃) $\limsup_{u \to u_0} \dfrac{|g(u)|}{|u - u_0|} < \infty$

We note that hypotheses (H$_1$) and (H$_3$) imply $g(u_0) = 0$, hence that $u = u_0$ solves (1.1). For u smooth on $\mathbb{R}^2$, we now define $(u)_j = u, u_x, u_t$ for $j = 0, 1, 2$ respectively, and write $u_{ij} = ((u)_i)_j$. We still need the following.

DEFINITION 1.2. Let $T > 0$, $u_0 \in \mathbb{R}$. We denote by $B(T; u_0)$ the set of all breather-solutions with polynomial decay to equation (1.1), that is the set of all $u \in \mathscr{C}^{(2)}(\mathbb{R}^2; \mathbb{R})$ such that the following conditions hold:

(C$_1$) $\quad$ u is a time-periodic solution to (1.1) of period T, that is
$\quad$ $u(x; t + T) = u(x, t)$ for each $(x, t) \in \mathbb{R}^2$.

(C$_2$) $\qquad\qquad\qquad\qquad$ $u(x; 0) = u_0 \quad$ for each $\quad x \in \mathbb{R}$

(C$_3$) $\qquad\qquad$ $\displaystyle\sup_{x \in \mathbb{R}} \left| x^m \max_{t \in [0, T]} |(u(x; t) - u_0)_{ij}| \right| < \infty$

for each $m \in \{0; 1; 2\}$, each $i \in (0; 1)$ and each $j \in \{0; 1; 2\}$ with $j \geq i$. We note that if g is an admissible nonlinearity for (1.1), then $u = u_0 \in B(T; u_0)$; $B(T; u_0)$ is therefore not empty. We also observe that with $u_0 = 0$, the Sine–Gordon nonlinearity is admissible, while the breather-solution u_{SGB} given by (1.3) satisfies (C$_1$) with $T = 2\pi$, (C$_2$) with $u_0 = 0$ and (C$_3$) for each $m \in \mathbb{N}$ (exponential fall-off). Thus, the breather-class $B_{SG}(2\pi; 0)$ has at least two elements. But we also have $B_{SG}(T; 0) = \{0\}$ for each $T \in (0; \sqrt{2}\pi)$, which follows from Coron's result in [4] since our condition (C$_3$) implies his assumptions of decay at infinity; in this specific case, the actual number of elements in $B(T; u_0)$ thus crucially depends on the value of the preassigned period T. What the main result of this paper does is to exhibit a set of admissible nonlinearities for which the breather-class $B(T; u_0)$ reduces to the singleton $\{u_0\}$ for each $T > 0$. Its statement requires the following

DEFINITION 1.3. Let $G_{u_0}(u) = \int_{u_0}^{u} d\xi g(\xi)$; G_{u_0} is said to be convex in $(u - u_0)^2$ if there exists a function $H \in \mathscr{C}^1(\mathbb{R}_0^+; \mathbb{R})$ such that $G_{u_0}(u) = H(y)$ with $y = (u - u_0)^2$ and H convex in y.

The above notion was introduced in [9] and used in [10], [11], [12] and [13] for investigations concerning nonlinear elliptic eigenvalue problems. Its relevance to the above hyperbolic problem is exhibited in the following

THEOREM 1.1. *Let $T > 0$ and $u_0 \in \mathbb{R}$; assume that g is an admissible nonlinearity for equation* (1.1). *If one of the following statements holds*:

(S$_1$) $G_{u_0}(u) \leq 0$ *for each $u \in \mathbb{R}$*
(S$_2$) G_{u_0} *is convex in $(u - u_0)^2$*
(S$_3$) $G_{u_0}(u) + \frac{1}{2}g(u)(u - u_0) \leq 0$ *for each $u \in \mathbb{R}$*
then $B(T; u_0) = \{u_0\}$.

Remarks. (1) The fact that our result holds for each $T > 0$ is in sharp contrast with Weinstein's results on the half-space $\mathbb{R}_0^+ \times \mathbb{R}$; this will be illustrated by several examples.

(2) Theorem (1.1) dispels some confusion regarding the relationship between the existence of breathers and the complete integrability of the wave equation (1.1); indeed, our examples will show that the class of nonlinearities for which Theorem (1.1) holds contains models for which (1.1) is nonintegrable, as well as nonlinearities for which (1.1) is completely integrable. It also contains certain polynomial nonlinearities, which proves a part of the Eleonskii–Kulagin–Novozhilova–Silin conjecture mentioned above.

(3) Coron and Weinstein assume $g \in \mathscr{C}^{(2)}(\mathbb{R}; \mathbb{R})$ and $g(0) = 0$, but have no other significant restrictions besides the period conditions; we only assume $g \in \mathscr{C}(\mathbb{R}; \mathbb{R})$ and have no period condition, but this is at the expense of having the growth conditions (H_2) and (H_3) at infinity and around u_0 (see Proposition 2.1 below).

(4) G_{u_0} need not be convex in $(u - u_0)$ in order to be convex in $(u - u_0)^2$ (see Example 1.3 below).

In the following examples, we simply write $G_0 = G$ and $B(T; 0) = B(T)$ if $u_0 = 0$; we begin with the following

EXAMPLE 1.1. Consider the wave equation

$$u_{tt}(x; t) = u_{xx}(x; t) \pm \sum_{j=1}^{N} c_j u^{2j-1}(x; t) \tag{1.5}$$

on $\mathbb{R}^2$, where $c_j \geq 0$ for each j; here, $g(u) = \mp \sum_{j=1}^{N} c_j u^{2j-1}$, $G(u) = \mp \sum_{j=1}^{N} \hat{c}_j u^{2j}$ with $\hat{c}_j \geq 0$. With the minus sign, we have $G(u) \leq 0$ for each u; otherwise, $G(u)$ is convex in u^2. In either case, $B(T) = \{0\}$ for each $T > 0$, by statements (S_1) and (S_2) of Theorem 1.1.

The preceding result contrasts sharply with the following, which is a direct consequence of Weinstein's arguments in [8].

EXAMPLE 1.2. Consider the equation

$$u_{tt}(x; t) = u_{xx}(x; t) - \sum_{j=1}^{N} c_j u^{2j-1}(x; t) \tag{1.6}$$

on the half-plane $\mathbb{R}_0^+ \times \mathbb{R}$, where $c_1 > 0$ and the c_j's are arbitrary for $j \in \{2, \ldots, N\}$; let $T > 2\pi c_1^{-1/2}$; then there do exist nontrivial solutions to equation (1.6), T-periodic in time, which decay exponentially fast along the x-direction.

Remark. If $c_1 < 0$ in (1.5) taken with the minus sign, then $B(T) = \{0\}$ as well for arbitrary c_j's, $j \in \{2, \ldots, N\}$. This follows from the fact that $c_1 < 0$ implies $g'(0) < 0$ and that condition (C_3) with $u_0 = 0$ implies Coron's in [4]. The above results thus prove the Eleonskii–Kulagin–Novozhilova–Silin conjecture in a significant number of cases.

Our next example is concerned with a translated version of equation (1.4), the so-called scalar Higgs equation; it illustrates the use of Theorem (1.1) for $u_0 \neq 0$ and also suggests that the Eleonskii–Kulagin–Novozhilova–Silin conjecture might not be true in general.

EXAMPLE 1.3. Consider the equation

$$u_{tt}(x; t) = u_{xx}(x; t) - 2u(x; t) + 3u^2(x; t) - u^3(x; t) \tag{1.7}$$

on $\mathbb{R}^2$; we have $g(u) = 2u - 3u^2 + u^3 = (u - 1)^3 - (u - 1)$; in this case, $u_0 = 1$ is an admissible trivial breather-solution to (1.7) and we have

$$G_1(u) = \frac{(u - 1)^4}{4} - \frac{(u - 1)^2}{2}$$

which is convex in $(u - 1)^2$ (but not in $u - 1$). Theorem (1.1) then implies that $B(T; 1) = \{1\}$ for each $T > 0$; by translation, this is of course the same as saying that $B(T) = \{0\}$ for equation (1.4), or that $B(T; -1) = \{-1\}$ for

$$u_{tt}(x; t) = u_{xx}(x; t) - 2u(x; t) - 3u^2(x; t) - u^3(x; t) \tag{1.8}$$

on $\mathbb{R}^2$.

Remark. The above argument does not preclude equation (1.4) from having nontrivial breather-solutions which converge to $u_0 = \mp 1$ as $|x| \to \infty$. This is because the potential $G_{+1}(u) = u^4/4 - u^2/2 + \frac{1}{4} = \frac{1}{4}(u - 1)^2(u + 1)^2$ is neither convex in $(u - 1)^2$ nor convex in $(u + 1)^2$. It is perhaps such solutions which might explain the existence of breather-solutions in field theory which is suggested by the numerical and asymptotic evidence exhibited by Campbell et al in [3]; the polynomial $g(u) = u^3 - u$ might thus provide a counterexample to the Eleonskii–Kulagin–Novozhilova–Silin conjecture.

Our next example is one with exponential nonlinearities, which appears in Mikhailov's study regarding the integrability properties of two-dimensional generalizations of the Toda chain [14]. It shows that there are integrable hyperbolic equations which possess no breather-solutions.

EXAMPLE 1.4. Consider the equation

$$u_{tt}(x;t) = u_{xx}(x;t) \pm \sinh\left(u(x;t)\right) \tag{1.9}$$

on $\mathbb{R}^2$; here, $g(u) = \mp \sinh(u)$, $G(u) = \mp(\cosh(u) - 1)$ so that either $G(u) \leq 0$ for each u or else $G(u)$ is convex in u^2; therefore $B(T) = \{0\}$ for each $T > 0$, as in Example (1.1).

Similar arguments also apply to the following examples; in each case, Weinstein's result still predicts nontrivial exponentially localized breather-solutions on $\mathbb{R}_0^+ \times \mathbb{R}$ for sufficiently large periods.

EXAMPLE 1.5. Consider the equation

$$u_{tt}(x;t) = u_{xx}(x;t) \pm 2u(x;t) \exp\left[u^2(x;t)\right] \tag{1.10}$$

on $\mathbb{R}^2$; then $B(T) = \{0\}$ for each $T > 0$. Here, Weinstein's arguments apply for $T > \sqrt{2}\,\pi$ if (1.10) is taken with the minus sign.

EXAMPLE 1.6. A similar conclusion holds for the equation

$$u_{tt}(x;t) = u_{xx}(x;t) \pm \sum_{j=1}^{N} |u(x;t)|^{q_j - 2} |u(x;t)| \tag{1.11}$$

on $\mathbb{R}^2$, where $q_j \geq 2$ for each j; for equation (1.11) taken with the minus sign, Weinstein's argument can be applied if $q_j = 2$ for at least one j.

Remark. The class of nonlinear Klein–Gordon equations recently investigated by Zhiber and Shabat [15] cannot be discussed using the above method; the problem of the existence of breather-solutions for these equations remains open. Their class contains in particular Liouville's equation $u_{tt} = u_{xx} - \exp[u]$ and another Mikhailov's equation, namely $u_{tt} = u_{xx} + \exp[2u] - \exp[-u]$.

While the proof of Coron's theorem rests on an astute application of Wirtinger's inequality, Weinstein's argument amounts to applying the stable manifold theorem to an infinite-dimensional dynamical system equivalent to (1.1) for which $x \in \mathbb{R}_0^+$ plays the role of time. Our proof of Theorem 1.1 rests upon methods from the calculus of variations, and the rest of this paper will accordingly be organized as follows: in Section 2, we associate with equation (1.1) a variational problem on the infinite strip $\Omega_T = \mathbb{R} \times (0, T)$. We then establish two integral identities valid for each $u \in B(T; u_0)$ and finally prove Theorem (1.1). We end section 2 by some remarks. For a short announcement of the results, see [24].

2. A smooth variational problem associated with equation (1.1) and proof of Theorem (1.1)

Let $T > 0$ and $u_0 \in \mathbb{R}$; write $\Omega_T = \mathbb{R} \times (0; T)$ and consider the Sobolev space $\mathring{H}^1(\Omega_T; \mathbb{R})$. Define $g_{u_0}: \mathbb{R} \to \mathbb{R}$ by $g_{u_0}(u) = g(u + u_0)$ and write $\hat{G}_{u_0}(u) = \int_0^u d\xi \, g_{u_0}(\xi)$ (note that $\hat{G}_{u_0}$ is different from the G_{u_0} of Theorem 1.1). On $\mathring{H}^1(\Omega_T; \mathbb{R})$, define the functional

$$S_{\Omega_T, u_0}[u] = \int_{\Omega_T} dx \, dt \{ \tfrac{1}{2} u_t^2(x; t) - \tfrac{1}{2} u_x^2(x; t) - \hat{G}_{u_0}(u(x; t)) \} \tag{2.1}$$

Our first result is the following

PROPOSITION 2.1. *Assume that g is an admissible nonlinearity for equation (1.1); then S_{Ω_T, u_0} is a real-valued, differentiable functional on $\mathring{H}^1(\Omega_T; \mathbb{R})$, with Fréchet derivative $S'_{\Omega_T, u_0}[u] \in H^{-1}(\Omega_T; \mathbb{R})$ for each $u \in \mathring{H}^1(\Omega_T; \mathbb{R})$ and*

$$S'_{\Omega_T, u_0}[u](v) = \int_{\Omega_T} dx \, dt \{ u_t(x; t) v_t(x; t) - u_x(x; t) v_x(x; t) - g_{u_0}(u(x; t)) v(x; t) \} \tag{2.2}$$

Proof. The only part of S_{Ω_T, u_0} which requires attention is

$$s_{\Omega_T, u_0}[u] = \int_{\Omega_T} dx \, dt \, \hat{G}_{u_0}(u(x; t)) \tag{2.3}$$

and we subdivide the proof into three steps for convenience (see also Appendix A).

Step 1. Finiteness of s_{Ω_T, u_0} on $\mathring{H}^1(\Omega_T; \mathbb{R})$.

Hypotheses (H_1), (H_2), (H_3) imply $g_{u_0} \in \mathscr{C}(\mathbb{R}; \mathbb{R})$ and

$$\limsup_{|\xi| \to \infty} |\xi|^{-1} e^{-\xi^2} |g_{u_0}(\xi)| < \infty \tag{2.4}$$

$$\limsup_{|\xi| \to 0} |\xi|^{-1} |g_{u_0}(\xi)| < \infty \tag{2.5}$$

Hence, $|g_{u_0}(\xi)| \le c(|\xi| + |\xi| e^{\xi^2})$ by interpolation, for each $\xi \in \mathbb{R}$ and for some

$c > 0$ (depending on u_0 in general). Let $\Phi(t) = e^{t^2} - 1$; Φ is a Young function in the sense of [10]. Let $K_\Phi(\Omega_T)$ be the convex balanced set of all (equivalence classes of) real-valued measurable functions u on Ω_T such that $\exp[u^2] - 1 \in L^1(\Omega_T; \mathbb{R})$; let $L_\Phi(\Omega_T; \mathbb{R})$ be the Orlicz space associated with Φ, namely the real Banach space obtained by taking the linear hull of $K_\Phi(\Omega_T)$ endowed with the norm

$$\|u\|_{\Phi, \Omega_T} = \inf\left\{k > 0 : \int_{\Omega_T} dxdt\, \Phi\left(\frac{u(x; t)}{k}\right) \le 1\right\} \tag{2.6}$$

Finally, let $E_\Phi(\Omega_T; \mathbb{R})$ be the Banach space obtained by closing the set of all bounded functions with bounded support in Ω_T with respect to (2.6). From Adams' improvement [16] to Trudinger's embedding theorem [17], we have the continuous embedding $\mathring{H}^1(\Omega_T; \mathbb{R}) \to L_\Phi(\Omega_T; \mathbb{R})$; hence if $u \in \mathring{H}^1(\Omega_T; \mathbb{R})$, there exists $(u_n)_{n=1}^\infty \subset \mathscr{C}_0^\infty(\Omega_T; \mathbb{R})$ with $u_n \to u$ strongly in $L_\Phi(\Omega_T; \mathbb{R})$ as $n \to \infty$; since $(u_n)_{n=1}^\infty \subset E_\Phi(\Omega_T; \mathbb{R})$, we in fact have $u \in E_\Phi(\Omega_T; \mathbb{R})$, so that the chain of continuous embeddings/inclusions

$$\mathring{H}^1(\Omega_T; \mathbb{R}) \to E_\Phi(\Omega_T; \mathbb{R}) \subset K_\Phi(\Omega_T) \subset L_\Phi(\Omega_T; \mathbb{R}) \to L^2(\Omega_T; \mathbb{R}) \tag{2.7}$$

holds. From (2.3) and the interpolated estimate for g_{u_0}, we therefore get

$$|s_{\Omega_T, u_0}[u]| \le \frac{c}{2} \int_{\Omega_T} dxdt\{u^2(x; t) + \exp[u^2(x; t)] - 1\} < \infty \tag{2.8}$$

so that s_{Ω_T, u_0} is real-valued on $\mathring{H}^1(\Omega_T; \mathbb{R})$.

Step 2. Membership of $s'_{\Omega_T, u_0}[u]$ in $H^{-1}(\Omega_T; \mathbb{R})$.

On $\mathring{H}^1(\Omega_T; \mathbb{R})$, consider the functional

$$s'_{\Omega_T, u_0}[u](v) = \int_{\Omega_T} dxdt g_{u_0}(u(x; t))v(x; t) \tag{2.9}$$

First, note that Φ is a Young function convex in t^2 in the sense of [10]; hence $u \to \Phi' \circ u$ maps $E_\Phi(\Omega_T; \mathbb{R})$ continuously into the strong dual $E_{\tilde\Phi}(\Omega_T; \mathbb{R})$ by Lemma 2.1 of [10], where $\tilde\Phi : s \to \max_{t \ge 0}\{|s| t - \Phi(t)\}$ denotes the Legendre transform of Φ. We therefore have $2u^2 \exp[u^2] = u(\Phi' \circ u) \in L^1(\Omega_T; \mathbb{R})$ by Young's inequality, for all $u \in \mathring{H}^1(\Omega_T; \mathbb{R})$. By the interpolated inequality for g_{u_0},

we infer that

$$\int_{\Omega_T} dx\, dt\, |g_{u_0}(u(x;t))|^2 \le c^2 \int_{\Omega_T} dx\, dt\{u^2(x;t) + 2u^2(x;t)e^{u^2(x;t)}$$

$$+ u^2(x;t)e^{2u^2(x;t)}\} < \infty.$$

Thus $g_{u_0} \circ u \in L^2(\Omega_T; \mathbb{R})$, hence $s'_{\Omega_T}[u] \in H^{-1}(\Omega_T; \mathbb{R})$ for each $u \in \overset{\circ}{H}{}^1(\Omega_T; \mathbb{R})$, from (2.9) and Schwarz inequality.

Step 3. Differentiability of S_{Ω_T, u_0} *on* $\overset{\circ}{H}{}^1(\Omega_T; \mathbb{R})$.

It remains to show that s_{Ω_T, u_0} has Fréchet derivative $s'_{\Omega_T, u_0}[u]$ for each $u \in \overset{\circ}{H}{}^1(\Omega_T; \mathbb{R})$. By the conclusion of Appendix A, it is sufficient to show that $s'_{\Omega_T, u_0}[u]$ is the Gâteaux derivative of s_{Ω_T, u_0}; using the continuity of g_{u_0} again, we have

$$\lambda^{-1}\{\hat{G}_{u_0} \circ (u + \lambda v) - \hat{G}_{u_0} \circ u\} \to v(g_{u_0} \circ u) \qquad (2.10)$$

as $\lambda \to 0$, almost everywhere on Ω_T and for each $v \in \overset{\circ}{H}{}^1(\Omega_T; \mathbb{R})$. By the interpolated inequality for g_{u_0} and for each $|\lambda| \in (0; 1]$, we have, almost everywhere on Ω_T,

$$|\lambda|^{-1} |\hat{G}_{u_0} \circ (u + \lambda v) - \hat{G}_{u_0} \circ u| \le c\, |v|\{|u| + |v|\}$$

$$+ (|u| + |v|) \exp\,[(|u| + |v|)^2]\} \in L^1(\Omega_T; \mathbb{R})$$

uniformly in λ, since $(|u| + |v|)(1 + \exp\,[(|u| + |v|)^2]) \in L^2(\Omega_T; \mathbb{R})$ from the method outlined in Steps 1 and 2. Hence

$$\int_{\Omega_T} dx\, dt \lambda^{-1}(\hat{G}_{u_0}(u(x;t) + \lambda v(x;t)) - \hat{G}_{u_0}(u(x;t))) \to \int_{\Omega_T} dx\, dt g_{u_0}(u(x;t))v(x;t)$$

as $\lambda \to 0$, by dominated convergence. ∎

We now can prove the following

PROPOSITION 2.2. *Assume that* g *is an admissible nonlinearity for equation* (1.1) *and let* $u \in B(T; u_0)$. *Then* $u - u_0 \in \overset{\circ}{H}{}^1(\Omega_T; \mathbb{R})$, $xu_x \in \overset{\circ}{H}{}^1(\Omega_T, \mathbb{R})$ *and* $u - u_0$ *is a critical point of* S_{Ω_T, u_0}.

Proof. Condition (C$_3$) readily implies $u - u_0 \in H^1(\Omega_T; \mathbb{R})$ and $xu_x \in H^1(\Omega_T; \mathbb{R})$. Furthermore, conditions (C$_1$) and (C$_2$) imply that $u - u_0 = 0$ on the lines $t = 0$ and $t = T$, so that by a slight generalization of the trace theorem for bounded regions in $\mathbb{R}^N$ [18], we get $u - u_0 \in \mathring{H}^1(\Omega_T; \mathbb{R})$; a similar argument applies to xu_x. Finally, $\hat{u} \equiv u - u_0$ solves the wave equation $\hat{u}_{tt} = \hat{u}_{xx} - g_{u_0}(\hat{u})$; using standard considerations in the calculus of variations, we then get

$$S'_{\Omega_T, u_0}[u - u_0](v) = 0$$

for each $v \in \mathscr{C}_0^\infty(\Omega_T; \mathbb{R})$, hence $S'_{\Omega_T, u_0}[u - u_0] = 0$ on $\mathring{H}^1(\Omega_T; \mathbb{R})$ from the continuity result of Proposition 2.1. ∎

Having shown that $S'_{\Omega_T, u_0}[u - u_0](v) = 0$ for each $v \in \mathring{H}^1(\Omega_T; \mathbb{R})$, we now can prove Theorem (1.1). The main idea is to exploit the freedom on the available test functions to establish two integral identities for $u \in B(T; u_0)$; the method is reminiscent of the celebrated Pohožaev technique in nonlinear elliptic theory [19].

Proof of Theorem 1.1. Choose $v = u - u_0$ in $S'_{\Omega_T, u_0}[u - u_0](v) = 0$; then

$$\int_{\Omega_T} dx dt (u_t^2(x; t) - u_x^2(x; t)) = \int_{\Omega_T} dx dt g(u(x; t))(u(x; t) - u_0) \tag{2.11}$$

Now, choose $v = xu_x$; we get

$$\int_{\Omega_T} dx dt x [\tfrac{1}{2} u_t^2(x; t) - \tfrac{1}{2} u_x^2(x; t) - G_{u_0}(u(x; t))]_x = \int_{\Omega_T} dx dt u_x^2(x; t)$$

which, upon integrating by parts and using (C$_3$), leads to

$$\int_{\Omega_T} dx dt (u_t^2(x; t) + u_x^2(x; t)) = 2 \int_{\Omega_T} dx dt G_{u_0}(u(x; t)) \tag{2.12}$$

Note that in (2.12), G_{u_0} is now the potential of Theorem (1.1) and not the $\hat{G}_0$ of expression (2.1). Adding and subtracting (2.11) and (2.12) we get

$$\int_{\Omega_T} dx dt u_x^2(x; t) = \int_{\Omega_T} dx dt (G_{u_0}(u(x; t)) - \tfrac{1}{2} g(u(x; t))(u(x; t) - u_0)) \tag{2.13}$$

and

$$\int_{\Omega_T} dx dt u_t^2(x; t) = \int_{\Omega_T} dx dt (G_{u_0}(u(x; t)) + \tfrac{1}{2} g(u(x; t))(u(x; t) - u_0)) \tag{2.14}$$

If statement (S_1) holds, then $u_t = u_x = 0$ on Ω_T, from identity (2.12). Hence $u = u_0$ on $\mathbb{R}^2$ by (C_1) and (C_2). If statement (S_2) holds, then $G_{u_0}(u) - \frac{1}{2}g(u)(u - u_0) = H(y) - yH'(y) \leq 0$ by the convexity of H, so that $u_x = 0$ on Ω_T by (2.13), which again implies $u = u_0$ on $\mathbb{R}^2$ by conditions (C_1) and (C_3). If statement (S_3) holds, then $u_t = 0$ on Ω_T by (2.14), so that $u = u_0$ on $\mathbb{R}^2$ by (C_1) and (C_2). ∎

Remarks. (1) If one requires (C_3) to hold for each $m \in \mathbb{N}$ along with (C_1) and (C_2), one gets a subclass $B_{\exp}(T; u_0) \subset B(T; u_0)$ consisting of breather-solutions to (1.1) with exponential decay, for which Theorem (1.1) obviously holds. The open question is whether the requirement of exponential fall-off along the spatial direction is sufficient to prove that $B_{\exp}(T; u_0)$ reduces to the singleton $\{u_0\}$ for "almost all" admissible nonlinearities, with the exception of the Sine–Gordon case $g(u) = 2 \sin(u)$ and trivial perturbations thereof. The proof of such a result clearly requires a detailed study of the geometry of the Sine–Gordon breather (1.3) considered as a critical point of $S_{\Omega_{2\pi}}$ on $\mathring{H}^1(\Omega_{2\pi}; \mathbb{R})$, and cannot follow from entirely global arguments. Such a study is currently being carried out in [20], and also has its dynamical system counterpart in [21].

(2) The sharp contrast between Weinstein's results on $\mathbb{R}_0^+ \times \mathbb{R}$ and ours on $\mathbb{R}^2$ can also be explained in two different ways. On the one hand, if one associates with (1.1) a dynamical system for which $x \geq 0$ plays the role of time, the problem is that of the intersection of the stable manifold with the unstable one ([8], [21]). This intersection is a very unlikely event in the corresponding phase space, which prevents one to carry over Weinstein's results to the whole plane. On the other hand, if one attempts to define a class of breather-solutions to (1.1) on $\mathbb{R}_0^+ \times \mathbb{R}$ in replacing condition (C_3) by

$$(C_3^+) \qquad \sup_{x \in \mathbb{R}_0^+} \left(x^m \max_{t \in [0,T]} |(u(x; t) - u_0)_{ij}| \right) < \infty$$

our argument breaks down since membership of $u - u_0$ in $\mathring{H}^1(\mathbb{R}_0^+ \times (0; T); \mathbb{R})$ does not hold. This is because condition (C_3^+) does not necessarily imply $u(0; t) - u_0 = 0$ for $t \in [0, T]$.

(3) If condition (C_3) is replaced by

$$(C_3^w) \qquad \sup_{x \in \mathbb{R}} \left| x^m \int_0^T dt |(u(x; t) - u_0)_{ij}| \right| < \infty$$

conditions (C_1), (C_2) and (C_3^w) then provide a possible definition of Segur's wobbling kink-solutions [22]. However, since condition (C_3) implies (C_3^w), the class of wobbling kinks is larger than the class of breather-solutions, and the question whether equation (1.1) possesses such solutions for admissible non-linearities other than $g(u) = \sin(u)$ and $g(u) = u^3 - u$ remains open.

Appendix A. On the $\mathscr{C}^{(1)}$-differentiability of S_{Ω_T, u_0}

In this appendix, we prove that under the above admissibility conditions for g (hypotheses (H_1), (H_2) and (H_3) of section 1), S_{Ω_T, u_0} is in fact $\mathscr{C}^{(1)}$-Fréchet differentiable on $\overset{\circ}{H}{}^1(\Omega_T; \mathbb{R})$. This requires the following two extra steps.

Step A.1. An Almost Everywhere Pointwise Bound for Strongly Convergent Sequences in $E_\Phi(\Omega_T; \mathbb{R})$.

Let $u_n \to u$ strongly in $E_\Phi(\Omega_T; \mathbb{R})$; there exists a subsequence $\{u_{n_k}\}_{k=1}^\infty$ with $\|u_{n_{k+1}} - u_{n_k}\|_{\Phi, \Omega_T} \le 2^{-k}$ and $u_{n_k} \to u$ pointwise almost everywhere on Ω_T; define

$$v_N(x; t) = \sum_{k=1}^N |u_{n_{k+1}}(x; t) - u_{n_k}(x; t)|$$

From the above selection of $\{u_{n_k}\}_{k=1}^\infty$, it follows that $\{v_N\}_{N=1}^\infty$ is a Cauchy sequence in $E_\Phi(\Omega_T; \mathbb{R})$. Let $v \in E_\Phi(\Omega_T; \mathbb{R})$ its limit and define $\hat{u} = |u_{n_1}| + v$; then $\hat{u} \in E_\Phi(\Omega_T; \mathbb{R})$ and the inequalities $|u_{n_k}| \le \hat{u}$, $|u| \le \hat{u}$ hold pointwise almost everywhere on Ω_T.

Step A.2. Strong Convergence $s'_{\Omega_T, u_0}[u_n] \to s'_{\Omega_T, u_0}[u]$ in $H^{-1}(\Omega_T; \mathbb{R})$.

It remains to prove that if $u_n \to u$ strongly in $H^1(\Omega_T, \mathbb{R})$, then $s'_{\Omega_T, u_0}[u_n] \to s'_{\Omega_T, u_0}[u]$ strongly in $H^{-1}(\Omega_T; \mathbb{R})$; we already know that $g_{u_0} \circ u$, $g_{u_0} \circ u_n \in L^2(\Omega_T; \mathbb{R})$ so that by Schwarz inequality, we get

$$\|s'_{\Omega_T, u_0}[u_n] - s'_{\Omega_T, u_0}[u]\|_{H^{-1}(\Omega_T, \mathbb{R})} \le \|g_{u_0} \circ u_n - g_{u_0} \circ u\|_{L^2(\Omega_T, \mathbb{R})}$$

Thus, it remains to prove that $g_{u_0} \circ u_n \to g_{u_0} \circ u$ strongly in $L^2(\Omega_T; \mathbb{R})$; by the continuity of g_{u_0}, we may assume that $|g_{u_0} \circ u_n - g_{u_0} \circ u|^2 \to 0$, pointwise almost everywhere on Ω_T; since $|g_{u_0} \circ u_n - g_{u_0} \circ u| \le |g_{u_0} \circ u_n| + |g_{u_0} \circ u|$, it is sufficient to prove that $|g_{u_0} \circ u_n|$ possesses an $L^2(\Omega_T)$-bound uniform in n; by the interpolated inequality for g_{u_0} and the result of step A.1, we may write

$$|g_{u_0} \circ u_n| \le c(\hat{u} + \hat{u}e^{\hat{u}^2})$$

almost everywhere on Ω_T, since $u_n \to u$ strongly in $\overset{\circ}{H}{}^1(\Omega_T; \mathbb{R})$ implies $u_n \to u$ strongly in $E_\Phi(\Omega_T; \mathbb{R})$. Since $\hat{u} \in E_\Phi(\Omega_T; \mathbb{R})$, the fact that $\hat{u} + \hat{u}e^{\hat{u}^2} \in L^2(\Omega_T; \mathbb{R})$ follows from the same arguments as in step 2; the conclusion then follows from dominated convergence.

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Note added in proof: The results of this paper have now been considerably generalized in [23], where the authors deal with the problem of almost periodic breathers.

ETH-Forschungsinstitut für Mathematik, CH 8092 Zürich/Switzerland

Permanent address:
Dept. of Mathematics
The University of Texas at Arlington
Arlington, Texas 76019 (USA)