On simple fibered knots in S5 and the existence of decomposable algebraic 3-knots.

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On simple fibered knots in S^5 and the existence of decomposable algebraic 3-knots

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§1. Introduction

Let f be an analytic function on some neighborhood of the origin 0 in $\mathbb{C}^{n+1}$ such that $f(0) = 0$. Suppose that f has an isolated critical point at the origin. The *algebraic knot* associated with f is the isotopy class of the codimension 2 smooth closed oriented submanifold of S_ε^{2n+1} given by $f^{-1}(0) \cap S_\varepsilon^{2n+1}$ for $\varepsilon > 0$ sufficiently small. (As a general reference for this see [20].) More generally, a $(2n-1)$-*knot* in the sphere S^{2n+1} is the isotopy class of a codimension 2 smooth closed oriented submanifold of S^{2n+1}. We say that a knot is *decomposable* if it is the connected sum of two non-trivial knots.

In the case of classical knots ($n = 1$), algebraic 1-knots are always indecomposable by a theorem of H. Schubert [26]. A. Durfee asks whether an algebraic knot is always indecomposable ([6, Problem 4]). In 1982, F. Michel and C. Weber showed that for any $n \geqq 3$ there exist decomposable algebraic $(2n-1)$-knots in S^{2n+1} ([19]). This result is obtained by using the classification theorem of simple fibered knots by their Seifert matrices ([5], [10]). However, for $n = 2$ this classification breaks down and the problem above has been open until now. The main purpose of this paper is to prove the existence of infinitely many algebraic 3-knots which are the connected sum of two non-trivial simple fibered 3-knots (§5).

We also prove that for a given closed orientable 3-manifold K there exists a simple fibered knot (S^5, K') such that K' is diffeomorphic to K (§6).

Throughout the paper, all manifolds and maps are C^∞, the symbol $\approx$ denotes diffeomorphism between manifolds, and the symbol $\simeq$ denotes congruence over $\mathbb{Z}$ between integral square matrices.

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§2. Preliminaries

In this section we recall the definitions of simple fibered knots and their Seifert matrices.

DEFINITION 2.1. A *fibered knot* is a knot (S^m, K^{m-2}) together with a smooth fiber bundle $\phi: S^m - K \to S^1$ that has the following property:

There exist a tubular neighborhood T of K and a bundle equivalence α of T to the trivial bundle $K \times D^2$ so that the diagram commutes, where p is the obvious

$$\begin{array}{ccc} T - K & \xrightarrow{\alpha|(T-K)} & K \times (D^2 - \{0\}) \\ & {\scriptstyle \phi|(T-K)} \searrow \quad \swarrow {\scriptstyle p} & \\ & S^1 & \end{array}$$

projection. In other words, K is the binding of an open book decomposition of S^m.

DEFINITION 2.2. A fibered knot (S^{2n+1}, K^{2n-1}) $(n \geqq 1)$ is *simple* if the manifold K is $(n-2)$-connected and the fiber of ϕ is $(n-1)$-connected.

Set $F = \phi^{-1}(1) - \mathrm{Int}\, T$. Then if $n \geqq 3$, the above condition is equivalent to that F has a handlebody decomposition consisting of one 0-handle and some n-handles which are attached to the 0-handle simultaneously. In this case we say that F has a *special handlebody decomposition.*

Remark 2.3. An algebraic knot is always a simple fibered knot ([20]). Furthermore, even when $n = 2$, F has a special handlebody decomposition ([16]).

Let (S^{2n+1}, K^{2n-1}) be a simple fibered knot. For $\theta \in \mathbb{R}$, let $F_\theta = \phi^{-1}(e^{i\theta}) - \mathrm{Int}\, T$. Then we have the homology isomorphism $h_\theta: H_n(F) \to H_n(F_\theta)$ induced by the path $\omega: I \to S^1$ defined by $\omega(t) = e^{i\theta t}$, where $I = [0, 1]$. (We always assume that the homology is with integer coefficient, unless otherwise indicated.)

DEFINITION 2.4. The *Seifert form* of a simple fibered knot (S^{2n+1}, K^{2n-1}) is the bilinear form $\Gamma: H_n(F) \times H_n(F) \to \mathbb{Z}$ defined by $\Gamma(x, y) = \mathrm{link}\,(h_\pi x, y)$ where $\mathrm{link}\,(h_\pi x, y)$ is the linking number of $h_\pi x$ and y in S^{2n+1}.

When we fix a basis $\{a_i\}$ of $H_n(F)$, we can identify Γ with the square matrix $L = (\Gamma(a_i, a_j))$. We call L a *Seifert matrix.* By the Alexander duality, we see easily that L is always unimodular.

For $n \geqq 3$, one has the following classification theorem.

THEOREM A ([5], [10]). *For $n \geqq 3$, the map*

$$\Phi_n: \left\{\begin{array}{l}\text{isotopy classes of} \\ \text{simple fibered } (2n-1)\text{-} \\ \text{knots in } S^{2n+1}\end{array}\right\} \to \left\{\begin{array}{l}\text{congruence classes of} \\ \text{integral unimodular} \\ \text{matrices}\end{array}\right\}$$

which associates with each knot its Seifert matrix is well-defined and bijective.

For $n=2$, Φ_2 is still well-defined. (For example, consider the 2-fold cyclic suspension of knots. For details see [22].) However, we shall see in §3 that Φ_2 is not injective. We do not know whether Φ_2 is surjective or not.

§3. Constructing simple fibered 3-knots

We first describe how to construct simple fibered knots. Our method is the open book construction.

Let F^{2n} be an $(n-1)$-connected compact smooth $2n$-manifold with boundary $\partial F = K^{2n-1}$ $(n-2)$-connected $(n \geqq 2)$. Let $h: F \to F$ be a diffeomorphism which is the identity map on the boundary. Then we define $N_h = (K \times D^2) \bigcup_\varphi E$ where

$$E = F \times I/(h(x), 0) \sim (x, 1)$$

and

$$\varphi: \partial E = \partial F \times I/(x, 0) \sim (x, 1) \to \partial(K \times D^2) = K \times S^1$$

is the map defined by $\varphi(x, t) = (x, e^{2\pi i t})$. It is easy to check that N_h is a smooth closed 1-connected $(2n+1)$-manifold. If $N_h \approx S^{2n+1}$, then of course $(N_h, K \times \{0\})$ is a simple fibered knot.

DEFINITION 3.1. The *variation map* of h is the homomorphism $\Delta_h: H_n(F, \partial F) \to H_n(F)$ induced by $(id_F)_* - h_*$. (Note that $h \mid \partial F = id_{\partial F}$.)

We recall the following lemma of L. Kauffman ([11]).

LEMMA B. *If $n \geqq 2$, then N_h is a homotopy sphere if and only if Δ_h is an isomorphism.*

From now on we shall confine ourselves to simple fibered 3-knots in S^5. Our aim in this section is to realize certain Seifert forms by simple fibered 3-knots.

Let

$$E_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

both of which are integral unimodular symmetric matrices. Our first result is the following.

THEOREM 3.2. *Let L be an integral unimodular matrix such that $L + {}^tL \simeq \alpha E_8 \oplus \beta U$ where α is even and $\beta \geqq \frac{3}{2}|\alpha| + 1$. (${}^tL$ is the transposed matrix of L.) Then there exists a simple fibered 3-knot (S^5, K) such that its Seifert matrix is L and $K \approx S^3$.*

Proof. Let $\alpha' = \alpha/2$, $\beta' = \beta - 3|\alpha'|$ and $Y = -\alpha' V_4 \# \beta'(S^2 \times S^2)$ where V_4 is diffeomorphic to a non-singular hypersurface of degree 4 in $\mathbb{C}P_3$ (see [18, pp. 23–24 and p. 33]). Let $F = Y^\circ$ $(= Y - \operatorname{Int} D^4)$. It is well-known that V_4 and $S^2 \times S^2$ have intersection matrices $-2E_8 \oplus 3U$ and U respectively. Therefore F has $\alpha E_8 \oplus \beta U$ $(\simeq L + {}^tL)$ as its intersection matrix.

Now set $H = -L^{-1} \cdot {}^tL$, then H is unimodular and ${}^tH(L + {}^tL)H$ is equal to $L + {}^tL$. Thus H is the matrix of an isometry of $H_2(F)$. The hypothesis on α and β implies that at least one $S^2 \times S^2$ is present. Thus, by [29] there exists a self-diffeomorphism h of F which induces H. Furthermore we may assume that $h \mid \partial F = id_{\partial F}$.

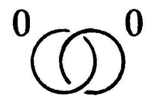
Since $\partial F \approx S^3$, $i_*: H_2(F) \to H_2(F, \partial F)$ is an isomorphism where i is the inclusion map. Hence, the variation map $\Delta_h: H_2(F, \partial F) \to H_2(F)$ is an isomorphism. By Lemma B, $N_h = (\partial F \times D^2) \cup E$ is a homotopy 5-sphere, so that $N_h \approx S^5$. Therefore $(N_h, \partial F \times \{0\})$ is a simple fibered knot.

If we let L_1 be the Seifert matrix of $(N_h, \partial F \times \{0\})$, then $L + {}^tL = L_1(I - H)$ by [5, p. 52], where I is the unit matrix. This implies that $L_1 = L$. This completes the proof.

Before stating the next theorem, we introduce some notations concerning framed links in S^3 (see [13]). Let λ be a framed link in S^3, then M_λ denotes the 4-manifold obtained by adding 2-handles to the 4-ball D^4 along the components of λ using their framings, and ∂M_λ denotes the boundary 3-manifold of M_λ.

By the argument given in the proof of Theorem 3.2, we can prove the following theorem.

THEOREM 3.3. *Let L be an integral unimodular matrix such that $L + {}^tL \simeq \alpha E_8 \oplus \beta U$ where α and β satisfy one of the following conditions:*
1) $\alpha = 0$
2) $|\alpha| = 1$ *and* $\beta \geqq 1$
3) $|\alpha| \geqq 2$ *and* $\beta \geqq 2$.

Furthermore suppose that there exists a framed link λ in S^3 such that the linking matrix of λ is congruent to $L + {}^tL$ over $\mathbb{Z}$ and such that λ has at least one  *separated from other components by an embedded 2-sphere. Then there exists a simple fibered 3-knot (S^5, K) whose Seifert matrix is L and such that $K \approx \partial M_\lambda$.*

Remark 3.4. Our method can be used to give another proof of the 4-dimensional embedding theorem of Cappell–Shaneson ([4, Theorems 1 and 2]) in the simply connected case.

We now give an example showing that the map Φ_2 defined in Theorem A is not injective. Let $L = 2L_1 \oplus 2L_2$ where

$$L_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad L_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix},$$

then L satisfies the condition of Theorem 3.2. Thus there exists a simple fibered 3-knot (S^5, K_1) whose Seifert matrix is L and such that $K_1 \approx S^3$. Furthermore define the framed link λ in S^3 as in Figure 1. ($\partial M_\lambda \approx 2\Sigma(2, 3, 5)$ where $\Sigma(2, 3, 5)$ is the dodecahedral space.) Then L and λ satisfy the condition of Theorem 3.3.

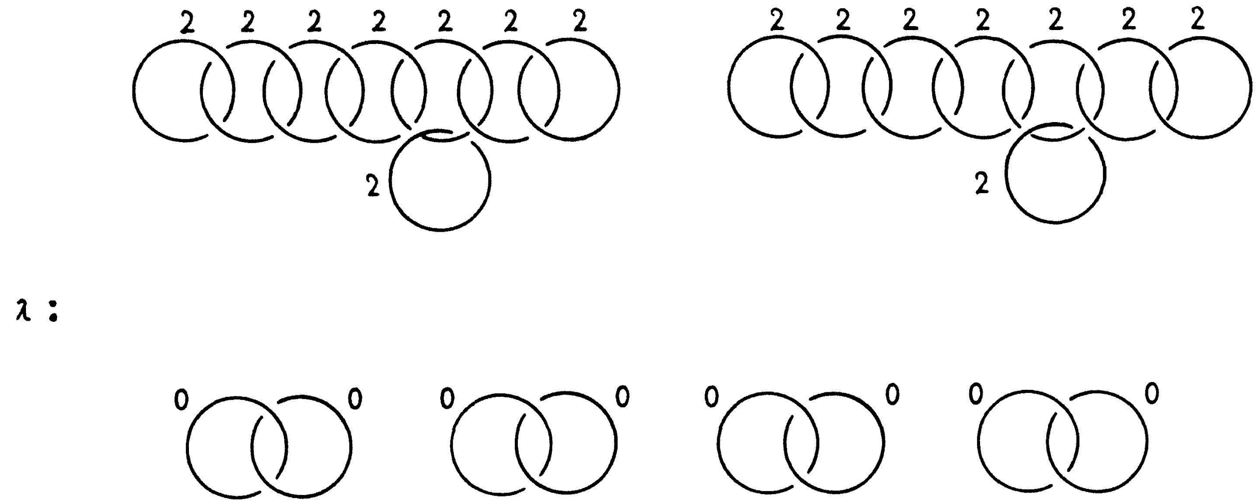


Fig. 1

Thus there exists a simple fibered 3-knot (S^5, K_2) whose Seifert matrix is L and such that $\pi_1(K_2) \neq 1$. The 3-manifolds K_1 and K_2 do not have the same homotopy type; hence the corresponding knots cannot be isotopic. Therefore the map Φ_2 in §2 is not injective.

Remember that in higher dimensions the diffeomorphism type of a simple fibered knot is determined by its Seifert matrix; in fact the isotopy type is determined.

§4. When are two simple fibered 3-knots isotopic?

In this section we consider simple fibered 3-knots which are homology 3-spheres. For these knots diffeomorphism types and Seifert matrices are complete invariants. More precisely we can prove the following theorem.

THEOREM 4.1. *Let* (S^5, K_i) *be two simple fibered* 3*-knots with Seifert matrices* L_i *respectively* $(i = 1, 2)$. *Suppose that* K_i *are homology* 3*-spheres,* $K_1 \approx K_2$, *and* $L_1 \simeq L_2$. *Then* (S^5, K_1) *is isotopic to* (S^5, K_2) *as a knot.*

Remark 4.2. In higher dimensions, isotopic simple fibered knots have the same fibering structure ([5]). However we do not know whether this is also true for simple fibered 3-knots.

To prove Theorem 4.1, we need the following lemmas.

LEMMA C. *Let* F_i $(i = 1, 2)$ *be compact* 1*-connected oriented spin* 4*-manifolds with boundaries* $\partial F_1 \approx \partial F_2$. *Suppose that* $\partial F_1 \approx \partial F_2$ *is connected, that* $H^1(\partial F_i;$

$\mathbb{Z}/2\mathbb{Z}) = 0$ $(i = 1, 2)$, *and that the signature of* F_1 *is equal to that of* F_2. *Then* $F_1 \# k(S^2 \times S^2) \approx F_2 \# k'(S^2 \times S^2)$ *for some non-negative integers* k *and* k'.

Lemma C was essentially proved by R. Kirby. See [13, Remark 3].

LEMMA 4.3. *Let* F *be a compact* 1-*connected spin* 4-*manifold with boundary* ∂F *a (connected) homology* 3-*sphere. Then* $F \# k(S^2 \times S^2)$ *has a special handlebody decomposition for some non-negative integer* k.

Proof. By [9] there exists a framed link λ in S^3 with $\partial M_\lambda \approx \partial F$ such that all the framings of λ are even. Set $W = F \cup (-M_\lambda)$ identified along $\partial F \approx \partial M_\lambda$. Then W is a closed oriented 1-connected spin 4-manifold, so that the signature of W is a multiple of 16 by Rohlin's theorem. Since $\sigma(W) = \sigma(F) - \sigma(M_\lambda)$ (σ denotes the signature), we can change λ so that $\sigma(F) = \sigma(M_\lambda)$ and that $\partial M_\lambda \approx \partial F$ by adding a number of copies of λ_1 or λ_2 where λ_i are framed links in S^3 with $\partial M_{\lambda_i} \approx S^3$ such that $\sigma(M_{\lambda_1}) = 16$ and $\sigma(M_{\lambda_2}) = -16$ (see [9, Th 3.3], [18, p. 66]). Then by Lemma C, $F \# k(S^2 \times S^2) \approx M_\lambda \# k'(S^2 \times S^2)$ for some k and k'. Clearly $M_\lambda \# k'(S^2 \times S^2)$ has a special handlebody decomposition. This completes the proof of Lemma 4.3.

Using these lemmas, one can prove Theorem 4.1 by the same argument as in [17, pp. 192–194].

Remark 4.4. By Theorem 4.1, simple fibered 3-knots in S^5, diffeomorphic to a given homology 3-sphere, are isotopic if and only if their Seifert matrices are congruent. If the homology 3-sphere is S^3, this result goes back to J. Levine [17].

§5. Proof of the main result

Let $g(x, y)$ be the same polynomial $y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$ as in §4 of [19]. We define the polynomial $f_r(x, y, z)$ by $f_r(x, y, z) = g(x, y) + z^r$, where $r \geqq 2$ is an integer. The purpose of this section is to prove the following theorem.

THEOREM 5.1. *The algebraic* 3-*knot associated with* f_r *is decomposable for* $r \equiv 5 \pmod{78}$. *In fact it is the connected sum of two non-trivial simple fibered* 3-*knots*.

Remark 5.2. By [1, p. 155], each eigenvalue of the monodromy for $g(x, y)$ is a 156-th root of unity. Since the monodromy for $f_r(x, y, z)$ is the tensor product of those for $g(x, y)$ and for z^r, its eigenvalues are products of those for $g(x, y)$ and for z^r. Thus if r is prime to 156, each eigenvalue for f_r is of composite order. By results on cyclotomic polynomials, one obtains $\Delta_f(1)=1$, where $\Delta_f(t)$ is the characteristic polynomial of the monodromy for $f=f_r$. This implies that the algebraic knot associated with f_r is a homology 3-sphere.

Remark 5.3. The condition $r \equiv 5 \pmod{78}$ does not seem to be essential. Possibly the algebraic knot associated with f_r will be decomposable for any r prime to 156.

Proof of Theorem 5.1. First we calculate the Seifert matrix of the algebraic 1-knot l associated with g. The Puiseux expansion of g at the origin is $y=x^{3/2}+x^{7/4}$. Therefore l is the $\{(3,2),(13,2)\}$-iterated torus knot (for example see [1]). If we let A, B_1 be the Seifert matrices of the $(13,2)$-torus knot and the $(3,2)$-torus knot respectively, then A is the 12×12 matrix given by

$$\begin{pmatrix} -1 & 1 & & & 0 \\ & -1 & 1 & & \\ & & -1 & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & -1 \end{pmatrix} \quad \text{and} \quad B_1=\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

(for example see [25]).

By [28] the Seifert matrix L_1 of l is the 16×16 matrix given by

$$L_1=\begin{pmatrix} B_1 & {}^tB_1 & 0 \\ B_1 & B_1 & 0 \\ 0 & 0 & A \end{pmatrix}.$$

Let

$$B=\begin{pmatrix} B_1 & {}^tB_1 \\ B_1 & B_1 \end{pmatrix},$$

then we have $L_1=B\oplus A$.

Now let L be the Seifert matrix of the algebraic knot (S^5, K_f) associated with $f=f_r$. By [25] we have $L=L_1\otimes C_r=(B\otimes C_r)\oplus(A\otimes C_r)$ where C_r is the $(r-1)\times(r-1)$ matrix given by

$$C_r=\begin{pmatrix} 1 & -1 & & & 0 \\ & 1 & -1 & & \\ & & 1 & \ddots & \\ & & & \ddots & -1 \\ 0 & & & & 1 \end{pmatrix}.$$

Let $D=A\otimes C_r$ and $E=B\otimes C_r$, so that $L=E\oplus D$.

We want to realize D and E as Seifert matrices of simple fibered 3-knots. In order to use Theorems 3.2 and 3.3, we must calculate the signatures of $D + {}^tD$ and $E + {}^tE$. Note that $L + {}^tL$ is unimodular by Remark 5.2, so that $D + {}^tD$ and $E + {}^tE$ are also unimodular. Let $r = 78q + 5$ $(q \geqq 0)$.

(1) *The signature σ_D of $D + {}^tD$.*

By [25] we see that $D = A \otimes C_r$ is the Seifert matrix of the algebraic 3-knot associated with $x^2 + y^{13} + z^r$. By [3], $\sigma_D = \sigma_{D+} - \sigma_{D-}$, where

$$\sigma_{D+} = \text{number of triples } (i, j, k) \text{ of integers, } 0 < i < 2,\ 0 < j < 13,\ 0 < k < r$$
$$\text{such that } 0 < i/2 + j/13 + k/r < 1 \bmod 2$$

and

$$\sigma_{D-} = \text{number of triples such that } -1 < i/2 + j/13 + k/r < 0 \bmod 2.$$

By a direct computation, we see that the number of integers k with $0 < k < r$ and $0 < \frac{1}{2} + j/13 + k/r < 1 \bmod 2$ is $[(|13 - 2j|/26) \cdot r]$ for each j $(j = 1, 2, \ldots, 12)$. Hence,

$$\sigma_{D+} = \sum_{j=1}^{12} \left[\frac{|13 - 2j|}{26} \cdot r \right] = 216q + 8.$$

Thus,

$$\sigma_D = \sigma_{D+} - \sigma_{D-} = \sigma_{D+} - (12(r-1) - \sigma_{D+}) = -504q - 32.$$

Since $D + {}^tD$ is a symmetric unimodular matrix of type II and of rank $12(78q + 4)$, $D + {}^tD \simeq -(63q + 4)E_8 \oplus (216q + 8)U$ by Serre's theorem ([27, p. 93]).

(2) *The signature σ_L of $L + {}^tL$.*

We compute the signature σ_L of $L + {}^tL$ using the formula of A. Durfee [7].

Let $V = f^{-1}(0) \cap D^6_\varepsilon$ for $\varepsilon > 0$ sufficiently small. There exists a minimal good resolution $\pi : \tilde{V} \to V$ of V at the origin $0 \in V$. (As a general reference for this see [14].) Let $E = \pi^{-1}(0)$ be the exceptional locus, which is described by the dual graph as in Figure 2 $(q \geqq 2)$. Each vertex represents a non-singular rational curve, the number attached is the Chern class of the normal bundle to that curve, and two vertices are joined if the corresponding curves intersect. Thus the number of vertices $s = 4q + 16$ and $h = \operatorname{rank} H_1(E) = 0$. Next let $K \in H_2(\tilde{V}; \mathbb{Q})$ be the canonical class (for the definition see [7]). Using the adjunction formula [7, Lemma 1.4], we see that the self-intersection number K^2 of K is $-676q - 24$. Since the Milnor number μ of V at the origin is $16(78q + 4)$, we obtain $\sigma_L = -\frac{1}{3} \cdot (2\mu + K^2 + s + 2h) = -608q - 40$ by the formula of [7]. By similar arguments we see that this equality is also valid for $q = 0, 1$.

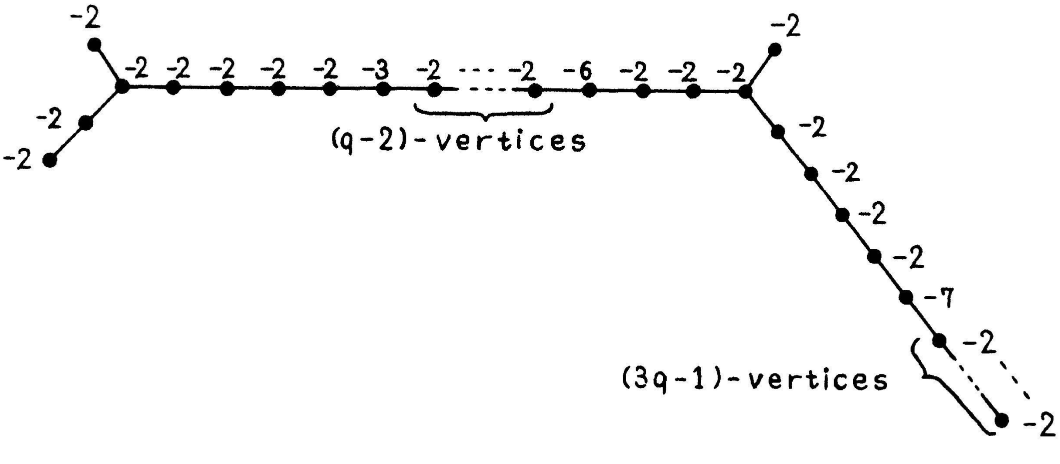


Fig. 2

(3) *The signature* σ_E *of* $E + {}'E$.

Since $L = E \oplus D$, $\sigma_L = \sigma_E + \sigma_D$. Thus $\sigma_E = \sigma_L - \sigma_D = -104q - 8$. By Serre's theorem $E + {}'E \simeq -(13q+1)E_8 \oplus (104q+4)U$.

Next we find a framed link λ_1 in S^3 such that every framing of λ_1 is even and $\partial M_{\lambda_1} \approx K_f$.

LEMMA 5.4. *There exists a framed link* λ_1 *in* S^3 *which has the following properties.*

(1) $\partial M_{\lambda_1} \approx K_f$.

(2) *Let* Q *be the linking matrix of* λ_1. *Then,*

$$Q \simeq \begin{cases} -E_8 \oplus (4q+7)U \text{ if } q \text{ is even } (q \geqq 2) \\ E_8 \oplus (4q+15)U \text{ if } q \text{ is odd } (q \geqq 3). \end{cases}$$

Proof. We consider the case of q even. The case of q odd can be handled similarly.

Let λ_1' be the framed link in S^3 induced by the resolution diagrm (Figure 2). The characteristic sublink of λ_1' (for the definition see [9, p. 240]) is described in Figure 3.

Using the framed link calculus of Kirby [13], we can change λ_1', without changing $M_{\lambda_1'}$, so that its characteristic sublink κ consists of one unknotted component with framing $-4q-8$. Then by [9, Th 4.2], we can find a framed link λ_1 with $\partial M_{\lambda_1} \approx \partial M_{\lambda_1'}$, $\operatorname{rank} H_2(M_{\lambda_1}) = \operatorname{rank} H_2(M_{\lambda_1'}) + ||-4q-8|-1| - 1 = 8q + 22$, and $\sigma(M_{\lambda_1}) = \sigma(M_{\lambda_1'}) - (-4q-8)$. (Note that the characteristic sublink κ is an unknotted circle.) Since λ_1' is induced by the resolution diagram of a normal surface singularity, its linking matrix is negative definite ([21]). Hence, $\sigma(M_{\lambda_1}) =$

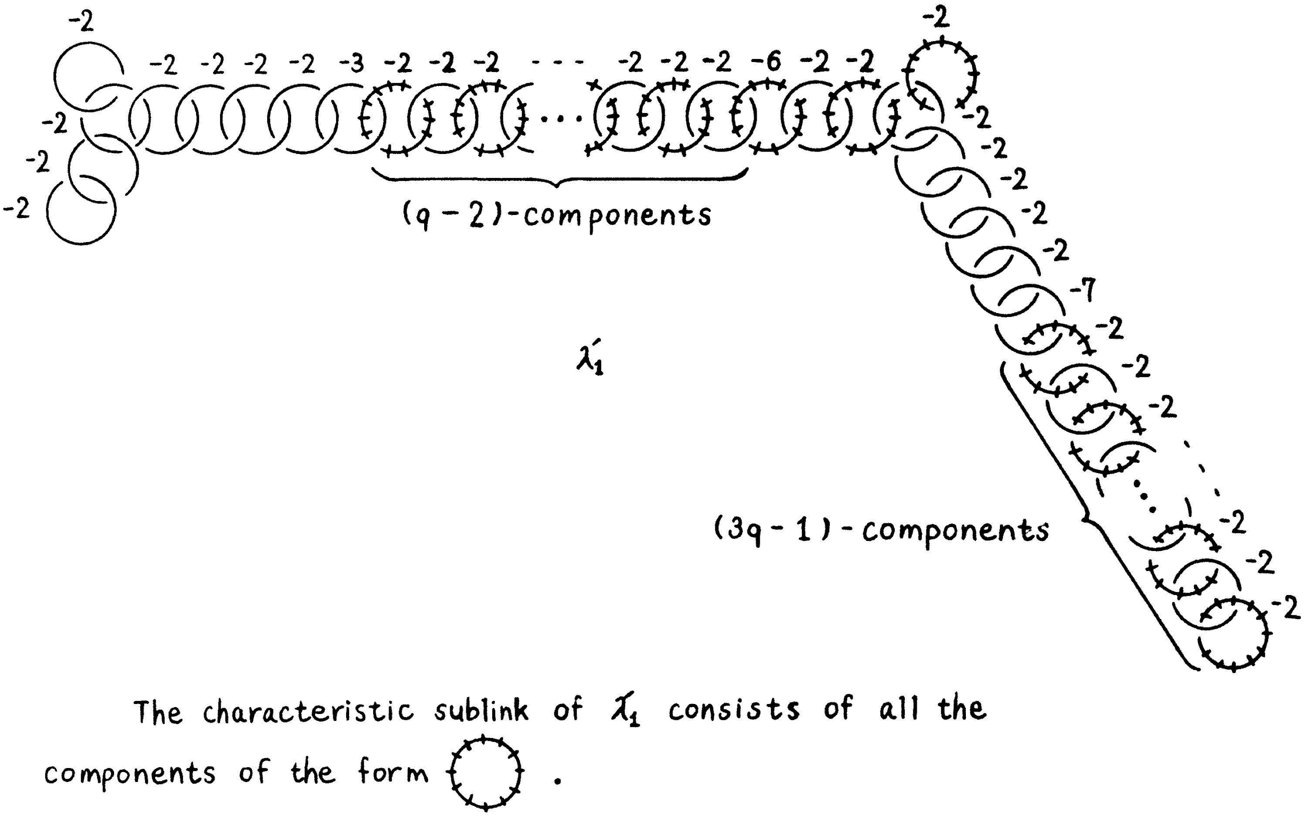


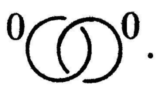
Fig. 3

−8. Thus the linking matrix Q of λ_1 is congruent to $-E_8 \oplus (4q+7)U$ by Serre's theorem. This completes the proof of Lemma 5.4.

Next we realize D and E as Seifert matrices.

Case I. q is even ($q \geqq 2$).

By (1) $D + {}^tD \simeq -(63q+4)E_8 \oplus (216q+8)U$. Thus D satisfies the hypothesis of Theorem 3.2. Therefore there exists a simple fibered 3-knot (S^5, K_D) such that its Seifert matrix is D and $K_D \approx S^3$.

Next we consider E. By (3) $E + {}^tE \simeq -(13q+1)E_8 \oplus (104q+4)U$. On the other hand, the framed link λ_1 of Lemma 5.4 has linking matrix congruent to $-E_8 \oplus (4q+7)U$. To change this matrix to $-(13q+1)E_8 \oplus (104q+4)U$, we must add $-13qE_8 \oplus (100q-3)U$. Since $100q - 3 > \frac{3}{2} \cdot 13q$, this can be done geometrically by adding to λ_1 a number of copies of λ_2 and λ_3 where λ_2 is Kaplan's framed link (see [9, Th 3.3], [18, p. 66]) which has 22 components and linking matrix congruent to $-2E_8 \oplus 3U$ and λ_3 is 

. The resulting framed link λ and the unimodular matrix E satisfy the hypothesis of Theorem 3.3. Thus there exists a simple fibered 3-knot (S^5, K_E) such that its Seifert matrix is E and $K_E \approx \partial M_\lambda \approx \partial M_{\lambda_1} \approx K_f$.

Case II. q is odd ($q \geqq 3$).

By a similar argument, we can show that there exist simple fibered 3-knots (S^5, K_D), (S^5, K_E) whose Seifert matrices are D and E respectively and such that $K_D \approx K_f$ and $K_E \approx S^3$.

Thus the connected sum $(S^5, K_D) \# (S^5, K_E)$ is a simple fibered 3-knot whose Seifert matrix is $D \oplus E (\cong L)$ and such that $K_D \# K_E \approx K_f$. Therefore by Theorem 4.1, the algebraic knot (S^5, K_f) is isotopic to $(S^5, K_D) \# (S^5, K_E)$.

For $q = 0$, 1, similar arguments can be used. This completes the proof of Theorem 5.1.

Remark 5.5. Due to a theorem of N. A'Campo, an algebraic knot cannot be a connected sum of two non-trivial algebraic knots ([2]). In the above example, when q is even, the knot (S^5, K_E) is not algebraic because the trace of its monodromy is zero. (For a non-trivial algebraic knot, the trace of its monodromy is ± 1. See [2].) The knot (S^5, K_D) is not algebraic, either, because it is non-trivial and $K_D \approx S^3$ (see [21]). When q is odd, we do not know whether the knot (S^5, K_D) is algebraic or not.

Remark 5.6. Let (S^5, K) be an algebraic 3-knot. Then by W. Neumann, K is always irreducible as a 3-manifold (see [23], [8]). Hence, if an algebraic 3-knot is a non-trivial connected sum, one of the summands must be a spherical knot.

§6. Every 3-manifold is the binding of an open book decomposition of S^5

In this section we shall prove the following theorem.

THEOREM 6.1. *Let K be a closed orientable (connected) 3-manifold. Then there exists an embedding $\psi: K \rightarrow S^5$ such that $(S^5, \psi(K))$ is a simple fibered 3-knot. (Moreover the fiber of this fibered knot has a special handlebody decomposition.)*

This theorem is motivated by the problem of characterizing diffeomorphism types of algebraic 3-knots (see [6]). In view of the above theorem, we cannot restrict such diffeomorphism types only by the fact that algebraic knots are fibered.

Proof of Theorem 6.1. For K as above, there exists a framed link λ in S^3 with $\partial M_\lambda \approx K$ such that all the framings of λ are even (see [9]). Let Q be the linking matrix of λ. We need the following lemma.

LEMMA 6.2. *There exist a non-negative integer k and an integral unimodular matrix L such that $Q \oplus kU \simeq L + {}^tL$.*

Proof. Let n be the size of Q. Then

$$Q \oplus 2nU \simeq \begin{pmatrix} Q & X_1 & X_1 & X_2 & X_2 & \cdots & X_n & X_n \\ Y_1 & U & & & & & & \\ Y_1 & & U & & & \text{\Large 0} & & \\ Y_2 & & & U & & & & \\ Y_2 & & & & U & & & \\ \vdots & & & & & \ddots & & \\ Y_n & & \text{\Large 0} & & & & U & \\ Y_n & & & & & & & U \end{pmatrix}$$

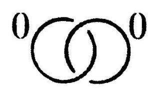
where X_i is an $n \times 2$ matrix given by

$$X_i = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \leftarrow i\text{-th row}$$

and $Y_i = {}^tX_i$ for $i = 1, 2, \ldots, n$. Set

$$L = \begin{pmatrix} R & X_1 & O & X_2 & O & \cdots & X_n & O \\ O & \multicolumn{2}{c}{S} & & & & & \\ Y_1 & & & & & \text{\Large 0} & & \\ O & & & \multicolumn{2}{c}{S} & & & \\ Y_2 & & & & & & & \\ \vdots & & & & & \ddots & & \\ O & & \text{\Large 0} & & & & \multicolumn{2}{c}{S} \\ Y_n & & & & & & & \end{pmatrix} \quad \text{where} \quad S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and R is an $n \times n$ matrix with $R + {}^tR = Q$. Then $L + {}^tL \simeq Q \oplus 2nU$ and L is easily seen to be unimodular. This completes the proof of Lemma 6.2.

Adding a number of copies of  to λ, we may assume that $Q = L + {}^tL$ for some unimodular matrix L and $\partial M_\lambda \approx K$. Then using the argument of M. Kervaire ([12, II.6]), we can embed M_λ in S^5 so that its Seifert matrix is L.

Let T be a tubular neighborhood of ∂M_λ in S^5 and let V be the manifold which is given by $S^5 - \operatorname{Int} T$ cut along M_λ. Since L is unimodular, V is an h-cobordism with boundary. By work of F. Quinn [24] and T. Lawson [15], for some non-negative integer k, there exists a diffeomorphism between $V \#_c k(S^2 \times S^2 \times I)$ and $(M_\lambda \# k(S^2 \times S^2)) \times I$ which extends the product structure on the boundary, where $\#_c$ denotes connected sum along the cobordism (see also [18, Th 6.22]).

In fact there exists a simple fibered 3-knot (S^5, K_0) whose fiber is diffeomorphic to $(S^2 \times S^2 \# S^2 \times S^2)^\circ$ and such that $K_0 \approx S^3$ by Theorem 3.2. (For example its Seifert matrix is

$$\left(\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}.\right)$$

Thus for some k', $(S^5, \partial M_\lambda) \# k'(S^5, K_0)$ is a simple fibered 3-knot. Since $K_0 \approx S^3$, $\partial M_\lambda \# k'K_0 \approx \partial M_\lambda \approx K$. This completes the proof.

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