Zeitschrift: Commentarii Mathematici Helvetici
Band: 62 (1987)

Artikel: Power series with integer coefficients in several variables.
Autor: Straube, E.J.
DOI: https://doi.org/10.5169/seals-47364

Comment. Math. Helvetici 62 (1987) 602–615 0010-2571/87/040602-14\$01.50 + 0.20/0

Power series with integer coefficients in several variables*

E. J. STRAUBE

Abstract. A classical theorem of Borel–Pólya, which concerns rationality of an analytic function whose Taylor expansion at a point has integer coefficients, is generalized to several variables.

1. Introduction and main results

A classical theorem of Pólya ([7], [8], see also [3], chapter VII), which generalizes an earlier result of Borel ([1]), may be stated as follows. A function f, analytic in the domain $\mathbb{C}^*\backslash E$, whose Taylor coefficients at ∞ are integers, must be rational if the Čebyšev constant of E is less than 1. Here, E is a compact subset of $\mathbb{C}$ (such that $\mathbb{C}\backslash E$ is connected) and $\mathbb{C}^*$ is the Riemann sphere. It is the purpose of this paper to generalize this result to power series in several variables. As we discuss the main results, we will also point out earlier work dealing with this problem ([5], [6]). Besides their intrinsic interest, results concerning power series with integer coefficients play a role in the theory of arithmetic functions as well (see for example [12]).

We first introduce some notation. For $z=(z_1, \ldots, z_n) \in \mathbb{C}^n$ and $\alpha=(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we set $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, and $|\alpha| := \sum_{j=1}^n \alpha_j$ (so $|\alpha|$ may be negative). We order the set $\mathbb{N}^n$ as follows: $\alpha < \beta$ if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and $(\alpha_1, \ldots, \alpha_n)$ comes before $(\beta_1, \ldots, \beta_n)$ in the lexicographic order.

A polynomial in n (complex) variables will be called *monic*, if has the form

$$P(z) = z^\alpha + \sum_{(0,\ldots,0) \le \beta < \alpha} a_\beta z^\beta, \tag{1}$$

that is, if the leading coefficient is 1.

Let K be a compact subset of $\mathbb{C}^n$. Following [5], [11], we consider a particular Čebyšev constant associated to K, namely

$$\tau^+(K) := \limsup_{j\to\infty} (M_{\alpha^j})^{1/|\alpha^j|} \tag{2}$$

* This research partially supported by an Indiana University Summer Faculty Fellowship.

Here, α^j denotes the j-th element of $\mathbb{N}^n$ (in the order defined above), and for $\gamma \in \mathbb{N}^n$

$$M_\gamma = \inf\left\{\sup_{z\in K}\{|P(z)|\} \Big/ P(z) = z^\gamma + \sum_{(0,\dots,0)\le\beta<\gamma} a_\beta z^\beta\right\} \tag{3}$$

Remarks. (1) It can be seen from simple examples that in contrast to the one-dimensional case, the lim sup in (2) is not a limit. However, if certain restrictions are imposed on the sequence $\{\alpha^j\}$, the limit will exist ([11]).

(2) If $K = K_1 x \cdots x K_n$, where the K_j are compact sets in $\mathbb{C}$, then $\tau^+(K) = \tau^+(K_1 x \cdots x K_n) = \max_{1\le j\le n}\{\tau(K_j)\}$. Here, $\tau(K_j)$ is the classical Čebyšev constant of the compact set K_j in $\mathbb{C}$. If we denote by Π_j the projection of $\mathbb{C}^n$ onto the j-th coordinate axis, we have therefore in particular the estimate

$$\tau^+(K) \le \max_{1\le j\le n}\{\tau(\Pi_j(K))\} \tag{4}$$

(since $K \subset \Pi_1(K)x \cdots x\Pi_n(K)$).

Finally, we make the convention that the homological condition which appears in the theorems below is to be understood in the sense of $(C^\infty -)$ differentiable homology.

THEOREM 1. *Let Ω^* be a domain in $(\mathbb{C}^*)^n$ which contains the point $(\infty, \dots, \infty)$. Let $\Omega := \Omega^* \cap \mathbb{C}^n$. Suppose there is an n-cycle W in Ω with $\tau^+(W) < 1$, and such that for all k, $1 \le k \le n$, W is homologous in $\Omega \setminus \bigcup_{j=1, j\ne k}^{n} \{z_j = 0\}$ to tori $\{|z_j| = R_j \mid 1 \le j \le n\}$ contained in arbitrarily small neighborhoods of $(\infty, \dots, \infty)$. Then any function analytic in Ω, with integer Taylor coefficients at $(\infty, \dots, \infty)$, is a rational function*

$$f = \frac{P}{Q};$$

moreover, the polynomials P and Q can be taken to have integer coefficients, with Q monic.

Remarks. (1) "Taylor coefficients at $(\infty, \dots, \infty)$" refers to the coefficients of the expansion of f in powers of $(1/z_1, \dots, 1/z_n)$. Though formally a Laurent expansion, it is a Taylor expansion in terms of the standard local coordinates at $(\infty, \dots, \infty)$.

(2) It should be noted that in the one-dimensional case, the condition in

Theorem 1 is just the one in the classical Borel–Pólya Theorem: in this case the homological condition just says that W must be homologous to circles $\{|z|=R\}$ (for arbitrarily large R) in Ω. The existence of such a W is easily seen to be equivalent to the singular set of the functions having Čebyšev constant (= transfinite diameter ([3])) less than one.

In [5], the problem of finding sufficient conditions on a domain which guarantee that functions which are analytic in the domain and whose Taylor coefficients at a point are integers, must be rational, was formulated for domains in $\mathbb{C}^n$. Via the inversion $(z_1, \ldots, z_n) \mapsto (1/z_1, \ldots, 1/z_n)$, this reduces to the situation considered in Theorem 1 (assuming that the point where the Taylor expansion is considered is $(0, \ldots, 0)$). However, the situation is more special: since the domain is in $\mathbb{C}^n$ (rather than $(\mathbb{C}^*)^n$), its image under the inversion does not intersect the coordinate hyperplanes at $(0, \ldots, 0)$, so that they are automatically excluded. An immediate corollary of Theorem 1 is therefore:

THEOREM 1′. *Let Ω be a domain in $\mathbb{C}^n$ containing $(0, \ldots, 0)$. Suppose there is an n-cycle W with $\tau^+(\{(1/z_1, \ldots, 1/z_n) \mid (z_1, \ldots, z_n) \in W\}) < 1$ and such that W is homologous in $\Omega \setminus \bigcup_{j=1}^n \{z_j = 0\}$ to tori $\{|z_j| = r_j \mid 1 \le j \le n\}$ contained in arbitrarily small neighborhoods of $(0, \ldots, 0)$. Then any function f analytic in Ω, with integer Taylor coefficients at $(0, \ldots, 0)$, is rational*

$$f = \frac{P}{Q};$$

moreover, the polynomials P and Q can be taken to have integer coefficients.

Remark. The cycle W, together with the condition on the Čebyšev constant of the inverted cycle, first appears in Lelong's paper [5]. In that paper, a weaker version of Theorem 1′ was proved for the case of $\mathbb{C}^2$; somewhat restrictive additional conditions were imposed on W (condition c) in Théorème 1 in [5]). But it was indicated that the theorem might be true without these restrictions.

It is instructive to elaborate a little on the conditions on W. First note that both the conditions $\tau^+(W) < 1$ and the homological condition taken by themselves are trivial; it is only their combination that restricts Ω^*. In some sense, Ω^* must be big enough. The requirement that W be homologous in $\Omega \setminus \bigcup_{j=1, j\neq k}^n \{z_j = 0\}$ to big tori (for all k) precisely serves the purpose to make the topology of the domain where homology takes place sufficiently non-trivial (at the level of n-th homology), so that the combination of conditions on W becomes effective. For illustration, consider the following simple

EXAMPLE. Let $\Omega^* := \{z \in (\mathbb{C}^*)^2 / |z_1| > 2\}$. Then $\Omega = \{z \in \mathbb{C}^2 / |z_1| > 2\}$. The tori $\{|z_j| = R_j\}$ are null-homologous in Ω, if $R_1 > 2$. Thus any 2-torus W, centered at some point of Ω, and with small 2-radius, satisfies $\tau^+(W) < 1$ and is homologous to $\{|z_j| = R_j\}$ in $\Omega = \Omega \backslash \{z_1 = 0\}$ (since W is also null-homologous). Thus all conditions are satisfied except the one requiring homology in $\Omega \backslash \{z_2 = 0\}$ (i.e. $k = 2$). This suffices to make the theorem fail: the conclusion of Theorem 1 does not hold for Ω^*, compare Proposition 3 below (take as counterexample a lacunary power series in $1/z_1$).

When verifying the condition $\tau^+(W) < 1$, one can sometimes verify more, namely $\max_{1 \le j \le n} \{\tau(\Pi_j(W))\} < 1$. In this case, a stronger conclusion is available:

THEOREM 2. *Assumptions as in Theorem* 1, *but with the condition* $\tau^+(W) < 1$ *replaced by* $\max_{1 \le j \le n} \{\tau(\Pi_j(W))\} < 1$. *Then f is rational of the form*

$$f(z_1, \ldots, z_n) = \frac{P(z_1, \ldots, z_n)}{Q_1(z_1) \cdots Q_n(z_n)}; \tag{5}$$

the Q_j $(1 \le j \le n)$ *are monic polynomials of* one *variable only and* Q_j *as well as P have integer coefficients.*

Remarks. (1) Theorem 2 was proved in [6] in the case where f is analytic in $(\mathbb{C}^* \backslash K_1) x \cdots x (\mathbb{C}^* \backslash K_n)$, and $\tau(K_j) < 1$, $1 \le j \le n$. This is a special case of our result: if $\tau(K_j) < 1$, then $\tau(K_j \cup \{0\}) < 1$. Thus there are 1-cycles W_j, with $\tau(W_j) < 1$ and such that W_j is homologous in $\mathbb{C} \backslash (K_j \cup \{0\})$ to circles $\{|z| = R\}$ for arbitrarily large R. The assumptions of Theorem 2 are satisfied with $W := W_1 x \cdots x W_n$, in view of (4).

(2) As for Theorem 1, there is a version (i.e. Theorem 2′) of Theorem 2 for domains in $\mathbb{C}^n$, which is analogous to Theorem 1′. As in Theorem 2 the conclusion is that the denominator is a product of polynomials in one variable.

By making suitable coordinate changes, or by scrutinizing the proofs (especially for Theorem 2), one may obtain theorems when unions of hyperplanes passing through certain other points than $(0, \ldots, 0)$ are removed in the formulation of the homological condition on W. More generally, it might be interesting to know what sets could serve the same purpose. We do not pursue this here. Rather, we would now like to point out some generalizations of the previous theorems along the classical lines. Just as in the classical case ([7] p. 27, [8]), the assumption that the Taylor coefficients of f are integers may be relaxed. Let θ be a solution of $z^2 + pz + q = 0$, p and $q \in \mathbb{Z}$, $p^2 - 4q < 0$, and denote by

$\mathbb{Z}(\theta)$ the subring of $\mathbb{C}$ generated by $\mathbb{Z}$ and θ. Then it suffices to assume that the coefficients are in $\mathbb{Z}(\theta)$. The polynomials involved will then also have coefficients in $\mathbb{Z}(\theta)$. This generalization follows by inspection of the proofs below (sections 2 and 3). Martineau ([6]) has pointed out that the classical proofs actually yield a stronger result than is commonly stated: instead of one function f, one may consider a normal family $\{f_\sigma\}$. Then there exist P_σ and Q, all with $\mathbb{Z}(\theta)$ coefficients, such that $f_\sigma = P_\sigma/Q$. This generalization also holds for Theorems 1(1′), and 2(2′): in both cases the denominator has the indicated form, but may be taken independent of σ. Note that since all f_σ are analytic near $(\infty, \ldots, \infty)$, the degree of P_σ is less than that of Q (or $Q_1 \cdots Q_n$); this gives in particular a bound on the degree of P_σ which is independent of σ. Again, the proof is by inspection of the proofs below; if (f_σ) is a normal family, all the estimates in the proofs will be uniform in σ.

In the case of one variable, Pólya ([7]) has shown that the conditions in Theorem 1 are not only sufficient, but also necessary, at least for simply connected (in $\mathbb{C}^*$) Ω. Let $\Omega^* = \Omega_1^* x \cdots x \Omega_n^*$, $\infty \in \Omega_j^* \subset \mathbb{C}^*$, Ω_j^* simply connected, for $1 \le j \le n$. Then, by Pólya's result, for the conclusion of Theorem 1 to hold, it is necessary that $\tau(\mathbb{C}^* \backslash \Omega_j^*) < 1$, $1 \le j \le n$ (otherwise consider a function of the variable z_j only to get a contradiction). Since then also $\tau((\mathbb{C}^* \backslash \Omega_j^*) \cup \{0\}) < 1$, there are cycles W_j with $\tau(W_j) < 1$, W_j homologous in $\Omega_j^* \backslash \{\infty, 0\}$ to circles $\{|z| = R\}$, for arbitrarily large R. Thus Ω^* must indeed contain a cycle W as in Theorem 1: $W := W_1 x \cdots x W_n$. Hence in this case the conditions in Theorem 1 are also necessary. Another class of domains where Theorem 1 is sharp is provided by the following proposition, proved in [5] (for $\mathbb{C}^2$, but the arguments carry over to $\mathbb{C}^n$).

PROPOSITION 3 ([5]). *Let Ω be a Reinhardt domain of holomorphy containing $(\infty, \ldots, \infty)$ in $(\mathbb{C}^*)^n$. If Ω contains the n-torus $\{z \in \mathbb{C}^n \backslash\, |z_j| = 1, 1 \le j \le n\}$, then every function analytic in Ω, with integer Taylor coefficients at $(\infty, \ldots, \infty)$ is a polynomial in $(1/z_1, \ldots, 1/z_n)$. If Ω does not contain this n-torus, there exist functions analytic in Ω, with integer coefficients at $(\infty, \ldots, \infty)$, which are not rational.*

Remarks. (1) In the question of necessity it is reasonable to assume that the domain is a domain of holomorphy.

(2) Note that the conclusion that f is a polynomial in $(1/z_1, \ldots, 1/z_n)$ is compatible with the conclusion of Theorem 2, which in this case applies.

(3) Actually, the non-rational functions constructed in [5] in the case where Ω does not contain the n-torus of polyradius 1 have a stronger property: they cannot be continued beyond the Reinhardt domain of convergence of the series

expansion about $(\infty, \ldots, \infty)$. This follows from a generalization of Ostrowski's gap theorem to several variables, due to Siciak, see [10], in particular the corollary on p. 573.

We conclude this introduction with some remarks about the proof of Theorems 1 and 2. The classical proofs of the Borel–Pólya theorem ([1], [7], [8], [3]) as well as the proof in [5] are all based on a characterization of rational functions by the vanishing of certain Hankel determinants formed from the Taylor coefficients of a germ of the function (for details see [4], §7.5 and [9], part 7, §2). The condition $\tau^+(W) < 1$ is exploited to show that these determinants must be small; since they must also be integers, they must vanish, whence the result. Our proof of Theorems 1 and 2 proceeds along quite different lines. It is based on the observation that $\tau^+(W) < 1$ implies the existence of monic polynomials which are not only small on W (this trivially follows from the definition of $\tau^+(W)$), but which have "integer" coefficients: coefficients in $\mathbb{Z}(i) = \{k + im \mid k, m \in \mathbb{Z}\}$. For the one variable case, the existence of these polynomials was observed in [2], and this one-variable result was shown to be useful in the present context in [6]. We will use the special polynomials in an inductive procedure: at the k-th step, the function is multiplied by a special polynomial, chosen so that the product contains only terms z^α in its expansion at $(\infty, \ldots, \infty)$ with at least k of the α_j non-negative. At the n-th step, we arrive at a polynomial. For $n = 1$, this gives a new, direct proof of Pólya's classical result.

The remainder of the paper is organized as follows: section 2 contains the result concerning small polynomials with "integer" coefficients on sets K with $\tau^+(K) < 1$. Section 3 contains the proof of Theorems 1 and 2.

2. On the condition $\tau^+(K) < 1$

Denote by $\mathbb{Z}(i)$ the subring of $\mathbb{C}$ generated by $\mathbb{Z}$ and i (as in section 1). K will be a compact subset of $\mathbb{C}^n$ throughout this section.

PROPOSITION 4. *Assume $\tau^+(K) < 1$. For each j $(1 \le j \le n)$ there exists a monic polynomial of the special form*

$$B_j(z) = z_j^h + \sum_{\beta < (0, \ldots, \underset{\substack{\nearrow \\ j\text{-th position}}}{h}, \ldots, 0)} a_\beta z^\beta \tag{1}$$

with coefficients in $\mathbb{Z}(i)$, *such that*

$$\|B_j\|_K := \sup_{z\in K} \{|B_j(z)|\} < 1. \tag{2}$$

Proof. The proof consists of an adaption of the arguments in §2 of [2]. Choose ρ such that $\tau^+(K) < \rho < 1$. Then there is $\alpha^0 \in \mathbb{N}^n$ such that for all $\gamma > \alpha^0$ (in the order defined in section 1) there exists a monic polynomial P_γ with leading power z^γ and with

$$\|P_\gamma\|_K < \rho^{|\gamma|} \tag{3}$$

For $\alpha, \beta \in \mathbb{N}^n$, $\beta < \alpha$, consider now linear combinations of the polynomials P_γ of the form

$$S_{\alpha,\beta} = P_\alpha + \sum_{\beta\le\gamma<\alpha} \lambda_\gamma P_\gamma \tag{4}$$

Starting with the biggest γ between β and α, one can choose special coefficients $\lambda_\gamma^{\alpha,\beta}$ recursively in such a way that

$$|\lambda_\gamma^{\alpha,\beta}| \le 1 \tag{5}$$

and such that in $S_{\alpha,\beta}$ the coefficients of the powers of z^γ for $\beta \le \gamma \le \alpha$ are all in $\mathbb{Z}(i)$ (since every complex number has a distance less than one from the lattice formed by the elements of $\mathbb{Z}(i)$). Therefore, with this choice of the coefficients in (4), we have

$$S_{\alpha,\beta} = T_{\alpha,\beta} + R_{\alpha,\beta}, \tag{6}$$

where the coefficients of $T_{\alpha,\beta}$ are in $\mathbb{Z}(i)$ and $R_{\alpha,\beta}$ contains only powers z^γ with $\gamma < \beta$; moreover, the coefficients of $R_{\alpha,\beta}$ are all less than one in modulus. If now $\beta > \alpha^0$, $S_{\alpha,\beta}$ will satisfy the estimate

$$\begin{aligned}\|S_{\alpha,\beta}\|_K &\le \|P_\alpha\|_K + \sum_{\beta\le\gamma<\alpha} |\lambda_\gamma^{\alpha,\beta}|\, \|P_\gamma\|_K \\ &\le \sum_{\beta\le\gamma\le\alpha} \rho^{|\gamma|} \le \sum_{s=|\beta|}^{\infty} b_s\rho^s =: c_\beta \end{aligned} \tag{7}$$

Here, b_s is the number of $\gamma \in \mathbb{N}^n$ with $|\gamma| = s$. Note that the last estimate is independent of α; also, since

$$\sum_{s=0}^{\infty} b_s \rho^s = \frac{1}{(1-\rho)^n} < \infty, \tag{8}$$

c_β can be made arbitrarily small, provided only $|\beta|$ is big enough.

Choose now an integer β_1 such that $\beta := (\beta_1, 0, \ldots, 0) > \alpha^0$ and such that $c_\beta < 1/3$. Let $\alpha^s := (\beta_1 + s, 0, \ldots, 0)$ and consider the sequence

$$S_{\alpha^s,\beta} = T_{\alpha^s,\beta} + R_{\alpha^s,\beta} \tag{9}$$

Since the coefficients in $R_{\alpha^s,\beta}$ are always less than one in modulus and since the "degree" of $R_{\alpha^s,\beta}$ does not exceed β, there is a subsequence such that all coefficients converge. In particular, this subsequence of $R_{\alpha^s,\beta}$ converges uniformly on K. Thus there exist s_1 and s_2, $s_1 > s_2$, such that

$$\|R_{\alpha^{s_1},\beta} - R_{\alpha^{s_2},\beta}\|_K < 1/3. \tag{10}$$

Combining this with (9), (7) and the fact that $c_\beta < 1/3$, we obtain

$$\begin{aligned}\|T_{\alpha^{s_1},\beta} - T_{\alpha^{s_2},\beta}\|_K &\le \|S_{\alpha^{s_1},\beta}\|_K + \|S_{\alpha^{s_2},\beta}\|_K \\ &\quad + \|R_{\alpha^{s_1},\beta} - R_{\alpha^{s_2},\beta}\|_K \\ &< 1/3 + 1/3 + 1/3 = 1\end{aligned} \tag{11}$$

Thus $B_1(z) := T_{\alpha^{s_1},\beta}(z) - T_{\alpha^{s_2},\beta}(z)$ satisfies (2). By construction, B_1 has coefficients in $\mathbb{Z}(i)$ and is of the special form (1) (for $j = 1$). For $j = 2, \ldots, n$, the B_j are obtained similarly, and the proof of Proposition 4 is complete.

3. Proof of Theorems 1 and 2

We first prove Theorem 1. Let W be the n-cycle given by the assumption: then $\tau^+(w) < 1$. Denote by B_j, $1 \le j \le n$, the polynomials associated to W by Proposition 4. Set

$$\mu := \max_{1 \le j \le n} \{\|B_j\|_W^{1/h_j}\} < 1 \tag{1}$$

Here, h_j is the degree of B_j. For $\alpha \in \mathbb{N}^n$ such that $(0, \ldots, 0) \leq \alpha < (h_1, \ldots, h_n)$ and non-negative integers $m_1, \ldots, m_n$, we have

$$\begin{aligned} \|B_1^{m_1} \cdots B_n^{m_n} z^\alpha\|_W &\leq \mu^{m_1h_1+\cdots+m_nh_n}\Big(\max_j \{\|z_j\|_W\}\Big)^{|\alpha|} \\ &\leq C\mu^{m_1h_1+\cdots+m_nh_n} \end{aligned} \tag{2}$$

for some C independent of α and $m_1, \ldots, m_n$ (as long as α satisfies the restriction stated above). Since $\mu < 1$, (2) implies that

$$\left|\frac{1}{(2\Pi i)^n}\int_W f(z)B_1^{m_1}(z) \cdots B_n^{m_n}(z)z^\alpha \, dz_1 \wedge \cdots \wedge dz_n\right| < 1, \tag{3}$$

provided that $m_1h_1 + \cdots + m_nh_n \geq N_0$ for suitably large $N_0 \in \mathbb{N}$. On the other hand, in view of the closedness of the form $fB_1^{m_1} \cdots B_n^{m_n} z^\alpha \, dz_1 \wedge \cdots \wedge dz_n$ (analyticity of the integrand), we may integrate over a suitable torus T^n in a neighborhood of $(\infty, \ldots, \infty)$ which is homologous to W. Since f has integer Taylor coefficients at $(\infty, \ldots, \infty)$, and the B_j have coefficients in $\mathbb{Z}(i)$, this integral assumes only values in $\mathbb{Z}(i)$. Hence, in order to satisfy (3), it must vanish:

$$\begin{gathered} \int_{T_n} f(z)B_1^{m_1}(z) \cdots B_n^{m_n}(z)z^\alpha \, dz_1, \ldots, dz_n = 0 \\ (0, \ldots, 0) \leq \alpha < (h_1, \ldots, h_n) \\ m_1h_1 + \cdots + m_nh_n \geq N_0 \end{gathered} \tag{4}$$

We will successively multiply f by polynomials formed from the B_j, until we arrive at a polynomial. First consider $fB_1^{N_0}$. It has an expansion of the form

$$f(z)B_1^{N_0}(z) = \sum_{\substack{\beta \in z^n \\ |\beta| \leq N_0 h_1}} b_\beta z^\beta \tag{5}$$

with $b_\beta \in \mathbb{Z}(i)$. We show that

$$b_{-\beta} = 0, \text{ if } \beta_j > 0 \text{ for } 1 \leq j \leq n \tag{6}$$

$(-\beta = (-\beta_1, \ldots, -\beta_n))$. The proof of (6) is by induction on β (in the order defined in section 1). We have

$$b_{-\beta} = \frac{1}{(2\Pi i)^n} \int_{T^n} f(z) B_1^{N_0}(z) z_1^{\beta_1 - 1} \cdots z_n^{\beta_n - 1} \, dz_1 \wedge \cdots \wedge dz_n \tag{7}$$

So for $\beta = (1, \ldots, 1)$, (4) yields the desired conclusion. Assume then that (6) holds up to some β, and call the next index $\gamma = (\gamma_1, \ldots, \gamma_n)$. The components of γ can be written as

$$\gamma_j - 1 = m_j h_j + \alpha_j \qquad 0 \leq \alpha_j < h_j, \qquad 0 \leq m_j, \tag{8}$$

with m_j and $\alpha_j \in \mathbb{N}$. By (4), the induction hypothesis and (7):

$$\begin{aligned} 0 &= \frac{1}{(2\Pi i)^n} \int_{T^n} f(z) B_1^{N_0}(z) B_1^{m_1}(z) \cdots B_n^{m_n}(z) z_1^{\alpha_1} \cdots z_n^{\alpha_n} \, dz_1 \wedge \cdots \wedge dz_n \\ &= \frac{1}{(2\Pi i)^n} \int_{T^n} f(z) B_1^{N_0}(z) z_1^{\gamma_1 - 1} \cdots z_n^{\gamma_n - 1} \, dz_1 \wedge \cdots \wedge dz_n = b_{-\gamma}. \end{aligned} \tag{9}$$

This concludes the induction and thus the proof of (6). In view of (6), and after appropriately collecting terms in (5), we obtain

$$f(z) B_1^{N_0}(z) = \sum_{\substack{\Delta \subseteq \{1, \ldots, n\} \\ \Delta \neq \phi}} \; \sum_{\substack{\gamma \in IN^{|\Delta|} \\ |\gamma| \leq N_0 h_1}} a_\gamma^\Delta(z_{\Delta'}) z_\Delta^\gamma, \tag{10}$$

Here, the outer summation is over all non-empty subsets Δ of $\{1, \ldots, n\}$, $|\Delta|$ is the cardinality of Δ, $\Delta' = \{1, \ldots, n\} \setminus \Delta$; if $\Delta = \{l_1, \ldots, l_{|\Delta|}\}$ with $l_1 < l_2 < \cdots < l_{|\Delta|}$, then $z_\Delta^\gamma = z_{l_1}^{\gamma_1} \cdots z_{l_{|\Delta|}}^{\gamma_{|\Delta|}}$; finally $z_{\Delta'}$ stands for the "remaining" variables, and terms are grouped in such a way in (10) that $a_\gamma^\Delta(z_{\Delta'})$ is a sum of strictly negative powers of the variables $z_{\Delta'}$. Thus the inner sum in (10) contains the terms where precisely the z_j with $j \in \Delta$ have non-negative exponent. (6) is expressed by the fact that the outer summation is only over the non-empty subsets of $\{1, \ldots, n\}$.

We assume now inductively that there is a monic polynomial $P_k(z)$ with coefficients in $\mathbb{Z}(i)$, such that fP_k has a Laurent expansion of the form

$$f(z) P_k(z) = \sum_{\substack{\Delta \subseteq \{1, \ldots, n\} \\ |\Delta| \geq k}} \; \sum_{\substack{\gamma \in IN^{|\Delta|} \\ |\gamma| \leq \deg P_k}} a_\gamma^\Delta(z_{\Delta'}) z_\Delta^\gamma; \tag{11}$$

that is, the outer summation is only over subsets of cardinality at least k.

Let $k<n$, and let $\Lambda=\{n-k+1,\ldots,n\}$. Note that W and T^n (T^n as in the theorem, suitably close to $(\infty,\ldots,\infty)$) are homologous in $\Omega\backslash\bigcup_{j=2}^n\{z_j=0\}$. In that domain the function $f(z)P_k(z)/z_{n-k+1}\cdots z_n$ is analytic. Also, the coefficients of the expansion at $(\infty,\ldots,\infty)$ are in $\mathbb{Z}(i)$. Therefore, the same arguments that lead to (4) yield, when applied to this function, that there exists an integer N_1, such that

$$\begin{gathered}\int_{T^n}\frac{f(z)P_k(z)}{z_{n-k+1}\cdots z_n}B_1^{m_1}(z)\cdots B_{n-k}^{m_{n-k}}(z)z_1^{\beta_1}\cdots z_{n-k}^{\beta_{n-k}}\,dz_1\wedge\cdots\wedge dz_n=0\\ 0\le\beta\le(h_1,\ldots,h_{n-k})\\ m_1h_1+\cdots+m_{n-k}h_{n-k}\ge N_1\end{gathered}\tag{12}$$

Now we plug (11) into (12) and observe that no proper subsets of Λ appear in the summation (since no sets of cardinality less than k appear). Therefore, any contribution coming from a $\Delta\neq\Lambda$ is annihilated by integration with respect to dz_j for a suitable $j\in\Delta\cap\{1,\ldots,n-k\}$. For $\Delta=\Lambda$, all contributions coming from $\gamma\neq(0,\ldots,0)$ vanish, since then at least one of the $z_{n-k+1},\ldots,z_n$ would have non-negative exponent. If for the remaining contribution (i.e. $\Delta=\Lambda$, $\gamma=(0,\ldots,0)$) we perform the integration with respect to $dz_{n-k+1}\wedge\cdots\wedge dz_n$, we find similarly

$$\begin{gathered}\int_{T^{n-k}}a^{\Lambda}_{(0,\ldots,0)}(z_1,\ldots,z_{n-k})B_1^{m_1}(z_1,\ldots,z_{n-k},0,\ldots,0)\cdots\\ \times B_{n-k}^{m_{n-k}}(z_1,\ldots,z_{n-k},0,\ldots,0)z_1^{\beta_1}\cdots z_{n-k}^{\beta_{n-k}}\,dz_1\wedge\cdots\wedge dz_{n-k}=0\\ 0\le\beta\le(h_1,\ldots,h_{n-k})\\ m_1h_1+\cdots+m_{n-k}h_{n-k}\ge N_1\end{gathered}\tag{13}$$

In the same way (4) was used to show (6), (13) implies that

$$\begin{aligned}&a^{\Lambda}_{(0,\ldots,0)}(z_1,\ldots,z_{n-k})B_1^{N_1}(z_1,\ldots,z_{n-k},0,\ldots,0)\\ &\quad=\sum_{\substack{\beta\in\mathbb{Z}^{n-k}\\ \Sigma\beta_j\le N_1h_1\\ \text{not all }\beta_j<0}}C_\beta z_1^{\beta_1}\cdots z_{n-k}^{\beta_{n-k}}\end{aligned}\tag{14}$$

The important point in (14) is that not all $\beta_j<0$. To obtain the analogous conclusion for a^{Λ}_{γ} with $\gamma\neq(0,\ldots,0)$, we replace fP_k by

$(\partial^{|\gamma|}/\partial z_{n-k+1}^{\gamma_1} \cdots \partial z_n^{\gamma_k})(fP_k)$ and apply the preceding discussion. Finally, by symmetry, the conclusion holds for all Δ of cardinality k:

$$a_\gamma^\Delta(z_{\Delta'})B_{j_\Delta}^{N_{\Delta,\gamma}}(0, z_{\Delta'}) = \sum_{\substack{\text{not all} \\ \beta_j<0}} C_\beta^{\Delta,\gamma} z_{\Delta'}^\beta, \qquad |\Delta| = k; \tag{15}$$

j_Δ is an element of Δ'. Let

$$P_{k+1}(z) := P_k(z) \prod_{|\Delta|_\gamma = k} B_{j_\Delta}^{N_{\Delta,\gamma}}(0, z_{\Delta'}) \tag{16}$$

Then P_{k+1} is monic, has coefficients in $\mathbb{Z}(i)$, and, as follows from (11) and (15), fP_{k+1} has an expansion like the one in (11), but with only sets of cardinality at least $k+1$ appearing. This completes the inductive step. We therefore find a monic polynomial P_n, with coefficients in $\mathbb{Z}(i)$, such that

$$f(z)P_n(z) = \sum_{\substack{|\gamma| \leq \deg(P_n) \\ \text{all } \gamma_j \geq 0}} a_\gamma z^\gamma =: P(z) \tag{17}$$

The right side of (17) is thus also a polynomial (also with $\mathbb{Z}(i)$ coefficients). Observe now that f has integer (i.e. *real*) Taylor coefficients at ∞. It follows from (17) that

$$f = \frac{\tilde{P}}{\tilde{P}_n}, \tag{18}$$

where $\tilde{P}$ and $\tilde{P}_n$ are the polynomials obtained from P and P_n respectively by taking the real parts of the coefficients. Thus f has the desired form, and the proof of Theorem 1 is complete.

Essentially the same proof works for Theorem 2. We only have to observe that from the stronger assumption

$$\max_{1 \leq j \leq n} \{\tau(\Pi_j(W))\} < 1 \tag{19}$$

it follows that the polynomials B_j from proposition 4 can be chosen to be polynomials of one variable only:

$$B_j(z) = z_j^h + \sum_{0 \leq s < h} a_s z_j^s \tag{20}$$

Then, with the induction hypothesis (11) modified to the effect that P_k is a product of polynomials in one variable, the above induction yields

$$f(z)Q_1(z_1)\cdots Q_n(z_n)=P(z), \tag{21}$$

where the Q_j are monic, and all Q_j as well as P have coefficients in $\mathbb{Z}(i)$. Denote by $\tilde{Q}_j$ the polynomial obtained from Q_j by conjugating the coefficients. Then $Q_j\tilde{Q}_j$ has integer coefficients, and (21) yields

$$f(z)Q_1(z_1)\tilde{Q}_1(z_1)\cdots Q_n(z_n)\tilde{Q}_n(z_n)=P(z)\tilde{Q}_1(z_1)\cdots\tilde{Q}_n(z_n) \tag{22}$$

This establishes Theorem 2, because the coefficients of the polynomial on the right side of (22) are automatically integers (since f and $Q_j\tilde{Q}_j$ have integer coefficients).

Acknowledgement

The author thanks E. Bedford and P. Lelong for pointing out the problem studied in this paper.

REFERENCES

[1] BOREL, É., *Sur une application d'un théorème de M. Hadamard,* Bull. Sc. Math., 2me série, *18* (1894), 22–25.
[2] FEKETE, M. and SZEGÖ, G., *On algebraic equations with integral coefficients whose roots belong to a given point set,* Math. Z. *63* (1955), 158–172.
[3] GOLUZIN, G., *Geometric Theory of Functions of a Complex Variable,* AMS Transl. of Math. Mono. *26*, 1969.
[4] HENRICI, P., *Applied and Computational Complex Analysis,* Vol. I, Pure and Appl. Math., John Wiley & Sons, 1974.
[5] LELONG, P., *Sur les séries de Taylor $F(x, y)$ ayant des coefficients entiers,* Publ. Mathematicae *1*, Nr. 4 (1950), 209–221.
[6] MARTINEAU, A., *Extension en n-variables d'un théorème de Pólya–Carlson concernant les séries de puissances à coefficients entiers*", C.R. Acad. Sc. Paris, Sér. A, *273*, (1971), 1127–1129.
[7] PÓLYA, G., *Sur les séries entières à coefficients entiers*", Proc. London Math. Soc., series 2, *21* (1922), 22–38.
[8] ——, *Über gewisse notwendige Determinantenkriterien für die Fortsetzbarkeit einer Potenzreihe,* Math. Ann. *99* (1928), 687–706.
[9] —— and SZEGÖ, G., *Problems and Theorems in Analysis,* Vol. II, Grundl. d. Math. Wissenschaften, Band 216, Springer 1976.
[10] SICIAK, J., *A generalization of Ostrowski's theorem on lacunary power series,* Bull. Acad. Polon. Sc., série sc. math., astr. et phys., *24*, Nr. 8 (1976), 569–574.

[11] ZAHARJUTA, V., *Transfinite diameter, Čebyšev constants, and capacity for compacta in* $\mathbb{C}^n$, Math. USSR Sbornik *25*, Nr. 3 (1975), 350–364.

[12] AVANISSIAN, V. and GAY, R., *Sur une transformation des fonctionelles analytiques et ses applications aux fonctions entières de plusieurs variables,* Bull. Soc. Math. France *103* (1975), 341–384.

Indiana University,
Bloomington IN 47405/*USA*

Current address
Dept of Mathematics
Texas A&M University
College Station, TX 77843/USA

Received December 27, 1985