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Multiplicative stability for the cohomology of finite Chevalley groups

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To Beno Eckmann on the occasion of his 70th birthday

Stability theorems for the cohomology of linear groups are quite familiar, useful in the study of the algebraic K-theory of a ring R. Such a theorem asserts that for n sufficiently large with respect to i, $GL_n(R) \to GL_{n+1}(R)$ induces an isomorphism $H^i(GL_{n+1}(R), A) \to H^i(GL_n(R), A)$ for some coefficient module A. Generalizations of such theorems to other classical families of groups have also been considered. In these situations, isomorphisms only occur in low degrees and no assertion is made for arbitrarily high cohomological degree.

In this paper, we investigate the very special case in which the ring R is a finite field $\mathbf{F}_{p^d}$ and in which the coefficient module is the ring $\mathbf{Z}/l$ for some prime $l \neq p$. The stabilization process we consider is with respect to d (i.e., change of rings from $\mathbf{F}_{p^d}$ to $\mathbf{F}_{p^{de}}$) and the result we obtain concerns stability of the cohomology algebras $H^*(GL_n(\mathbf{F}_{p^d}), \mathbf{Z}/l)$. Corollary 5 provides such a multiplicative stability result not only for GL_n but more generally for any (connected) reductive algebraic group G defined and split over the integers.

In considering multiplicative stability, we also verify a close relationship between $H^*(G(\bar{\mathbf{F}}_p), \mathbf{Z}/l)$ and $H^*(G(\mathbf{F}_{p^d}), \mathbf{Z}/l)$ for d sufficiently large, where $\bar{\mathbf{F}}_p$ is an algebraic closure of $\mathbf{F}_p$. This enables us to extend to the non-compact group $G(\bar{\mathbf{F}}_p)$ D. Quillen's theorem relating the $\mathbf{Z}/l$-cohomology of the classifying space of a compact group to that of its elementary abelian l-subgroups ([4; 7.2]). An amusing consequence is a derivation of this Quillen theorem for the case of a compact connected group from the case of finite groups.

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To begin, we establish the following conventions. We fix distinct primes p and l. We consider an arbitrary (connected) reductive algebraic group $G_{\mathbf{Z}}$ defined and split over the ring $\mathbf{Z}$ of rational integers. For any ring R, we denote by $G(R)$ the discrete group of R-rational points of $G_{\mathbf{Z}}$ and we denote by $G\mathbf{C}$ the complex Lie group with underlying discrete group $G(\mathbf{C})$. (The generality of our context is reflected in the observation that any compact connected Lie group is a compact form of $G\mathbf{C}$ for some such reductive group $G_{\mathbf{Z}}$). Except in statements of results, we abbreviate the finite group $G(\mathbf{F}_{p^d})$ of $\mathbf{F}_{p^d}$-rational points of G by $G(p^d)$, the discrete group $G(\bar{\mathbf{F}}_p)$ by $G(p^\infty)$, and the cohomology functor $H^*(\,,\mathbf{Z}/l)$ by $H^*(\ )$. We leave implicit the name of the restriction map $H^*(\Gamma)\to H^*(\pi)$ associated to an inclusion $\pi < \Gamma$ of discrete groups.

THEOREM 1. *There exists a positive integer e such that for any non-trivial pth power p^d*

a) $H^*(G(\bar{\mathbf{F}}_p), \mathbf{Z}/l)\to H^*(G(\mathbf{F}_{p^{de}}), \mathbf{Z}/l)$ *is injective.*

b) $im\{H^*(G(\bar{\mathbf{F}}_p), \mathbf{Z}/l)\to H^*(G(\mathbf{F}_{p^d}), \mathbf{Z}/l)\}\to im\{H^*(G(\mathbf{F}_{p^{de}}), \mathbf{Z}/l)\to H^*(G(\mathbf{F}_{p^d}), \mathbf{Z}/l)\}$ *is bijective.*

Proof. For any positive integer f, we consider the spectral sequence

$$E_2^{i,j}(p^f) = H^i(BG\mathbf{C}, H^j(G\mathbf{C}))\Rightarrow H^{i+j}(G(p^f))$$

associated to the fibration sequence $\mathrm{fib}\,(D_{p^f})\to BG(p^f)\to(\mathbf{Z}/l)_\infty BG\mathbf{C}$, where $H^*(G\mathbf{C})$ is the $\mathbf{Z}/l$-cohomology of $G\mathbf{C}$ viewed as a topological manifold ([2; 1.2]). For any $k\geq 1$, the inclusion $G(p^f)\to G(p^{fk})$ extends to a map of fibration sequences

$$\begin{array}{ccccc} \mathrm{fib}\,(D_{p^f}) & \longrightarrow & BG(p^f) & \longrightarrow & (\mathbf{Z}/l)_\infty G\mathbf{C} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{fib}\,(D_{p^{fk}}) & \longrightarrow & BG(p^{fk}) & \longrightarrow & (\mathbf{Z}/l)_\infty BG\mathbf{C} \end{array} \qquad (1.1)$$

and thus to a map of spectral sequences $\{E_r^{*,*}(p^{fk})\}\to\{E_r^{*,*}(p^f)\}$. As in [2; 1.3, 1.4], there exists $k\geq 1$ such that for any $f\geq 1$ the map $E_2^{i,j}(p^{fk})\to E_2^{i,j}(p^f)$ is the 0-map for all $i\geq 0$, $j>0$. We shall verify that $e=k^{D+1}$ satisfies a) and b), where $D=\max\{t: H^t(G\mathbf{C})\neq 0\}$.

For notational simplicity, set $q_m=(p^d)^{k^m}$, so that $q_0=p^d$ and $q_{D+1}=p^{de}$. Using [2; 1.4], the edge homomorphism $E_2^{*,0}(q_m)\to E^*(q_m)$ can be identified with the restriction map $H^*(G(p^\infty))\to H^*(G(q^m))$. Since $E_{D+3}^{*,0}(p^f)\cong E_\infty^{*,0}(p^f)$ for any $f\geq 1$ for dimension reasons, a) is equivalent to the assertion that $E_2^{*,0}(p^{de})\to$

$E_{D+3}^{*,0}(p^{de})$ is an isomorphism. We proceed to verify by induction with respect to $r \geq 2$ that if $m \geq r-2$ then $E_2^{*,0}(q_m) \tilde{\to} E_r^{*,0}(q_m)$. The case $r=2$ is trivial. Consider the commutative diagram

$$\begin{array}{ccccc} E_r^{*-r,r-1}(q_{m+1}) & \xrightarrow{d_r} & E_r^{*,0}(q_{m+1}) & \longleftarrow & E_2^{*,0}(q_{m+1}) \\ \downarrow & & \downarrow & & \downarrow \\ E_r^{*-r,r-1}(q_m) & \xrightarrow{d_r} & E_r^{*,0}(q_m) & \longleftarrow & E_2^{*,0}(q_m) \end{array} \tag{1.2}$$

By definition of k and $q_{m+1} = q_m^k$, the left vertical map of (1.2) is the 0-map whereas the right vertical map is an isomorphism, because the right vertical map of (1.1) is a homotopy equivalence. For $m \geq r-2$, the right horizontal maps of (1.2) are isomorphisms by induction. Consequently, we conclude that the middle vertical map of (1.2) is an isomorphism, so that $d_r: E_r^{*-r,r-1}(q_{m+1}) \to E_r^{*,0}(q_{m+1})$ is the 0-map. This implies that $E_2^{*,0}(q_{m+1}) \tilde{\to} E_{r+1}^{*,0}(q_{m+1})$ for any $m \geq r-2$ as required to conclude a).

To prove b), we observe that $im\{H^n(G(p^\infty)) \to H^n(G(p^d))\}$ consists of those cohomology classes in $H^n(G(p^d))$ of filtration degree n (i.e., of maximal filtration degree). Since the definition of k implies that $H^n(G(p^{m+1})) \to H^n(G(p^m))$ increases the filtration degree of any cohomology class of filtration degree less than n, we conclude that $im\{H^n(G(p^{de})) \to H^n(G(p^d))\}$ also consists of those cohomology classes of filtration degree n.

By applying Theorem 1 in the case in which $d = e$, we immediately obtain the following corollary. We should emphasize that the splitting obtained does not respect the higher order Bockstein transformation on $H^*(G(p^f), \mathbf{Z}/l)$.

COROLLARY 2. *Whenever e^2 divides f, $H^*(G(\bar{\mathbf{F}}_p), \mathbf{Z}/l) \to H^*(G(\mathbf{F}_{p^f}), \mathbf{Z}/l)$ is a split inclusion of graded $\mathbf{Z}/l$-algebras, where e is as in Theorem* 1.

Following Quillen, we consider the category $\mathscr{A}(H)$ whose objects are elementary abelian l-subgroups of a group H and whose maps are group homomorphisms given by the restriction of some inner automorphism of H.

LEMMA 3. *There exists a positive integer s such that whenever t is a positive multiple of s the natural function $\mathscr{A}(F(\mathbf{F}_{p^t})) \to \mathscr{A}(G(\bar{\mathbf{F}}_p))$ is an equivalence of categories.*

Proof. Since any elementary abelian l-subgroup of $G(p^\infty)$ normalizes a maximal torus and since the maximal tori of $G(p^\infty)$ are conjugate, any

$E \in \mathscr{A}_\infty = \mathscr{A}(G(p^\infty))$ is isomorphic (i.e., conjugate) to a subgroup of N_∞, the normalizer in $G(p^\infty)$ of the $\bar{\mathbf{F}}_p$-rational points of a maximal torus $T \subset G$ defined and split over $\mathbf{F}_p$. Let q' be a pth power with the properties that $l \mid q'-1$ and that every l-torsion element (i.e., element whose lth power is 1) of $W = N_\infty / T(p^\infty)$ which lifts to an l-torsion element of N_∞ also lifts to $N_{q'}$, the normalizer of $T(q')$ in $G(q')$. Then $N_{q'} \subset N_\infty$ contains all l-torsion elements of N_∞, so that $\mathscr{A}_\infty$ necessarily has only finitely many isomorphism classes of objects. Since $\mathscr{A}(G(p^f)) \to \mathscr{A}_\infty$ is a faithful inclusion for any $f \geq 1$, and since there are only finitely many maps between any two objects of $\mathscr{A}_\infty$, we conclude that $\mathscr{A}(G(p^s)) \to \varinjlim_q \mathscr{A}(G(q)) = \mathscr{A}_\infty$ is an equivalence of categories where p^s is a sufficiently large power of q'. Clearly, $\mathscr{A}(p^t) \to \mathscr{A}_\infty$ is therefore also an isomorphism for t any positive multiple of s.

In seeking a multiplicative stability theorem, care must be taken, as can be seen in the following extremely simple example of the multiplicative group G_m. In this case, $H^*(G_m(p^\infty)) \cong H^*(\mathbf{Q}_l/\mathbf{Z}_l)$ and $H^*(G_m(p^d)) \cong H^*(\mathbf{Z}/l^{f(d)})$, where $f(d)$ is the largest power of l dividing the order of the (cyclic) group of units of $\mathbf{F}_{p^d}$. For any $n \geq 0$, $H^{2n+1}(G_m(p^\infty)) = 0$ whereas $H^{2n+1}(G_m(p^d)) \cong \mathbf{Z}/l$ if $f(d) \geq 1$. Moreover, $H^{2n+1}(G_m(p^{de})) \to H^{2n+1}(G_m(p^d))$ is the 0-map whenever $f(de) > f(d)$.

If $p = 2$ (respectively, $p > 2$), we let $H^*(\ )_{\text{red}}$ denote the quotient of $H^*(\ )$ (resp., $H^{ev}(\ )$) by the subfunctor of nilpotent elements.

THEOREM 4. *There exists a* pth *power* q *such that for any positive integer* r, $H^*(G(\bar{\mathbf{F}}_p), \mathbf{Z}/l) \to H^*(G(\mathbf{F}_{q^r}), \mathbf{Z}/l)$ *induces an isomorphism*

$$H^*(G(\bar{\mathbf{F}}_p), \mathbf{Z}/l)_{\text{red}} \xrightarrow{\sim} H^*(G(\mathbf{F}_{q^r}), \mathbf{Z}/l)_{\text{red}}.$$

Proof. Let e and s be chosen as in Theorem 1 and Lemma 3 respectively, and set $q' = p^{se}$, $q = p^{se^2}$. We consider the following commutative diagram of $\mathbf{Z}/l$-algebras determined by Theorem 1

$$\begin{array}{ccc} H^*(G(p^\infty)) & \longrightarrow & H^*(G(q^r)) \\ \downarrow & & \downarrow \\ H^*(G(q'^r)) & \longleftarrow & H^*(G(p^\infty)) \end{array} \tag{4.1}$$

in which the composite $H^*(G(p^\infty)) \to H^*(G(q^r)) \to H^*(G(p^\infty))$ is the identity map and the composite $H^*(G(q^r)) \to H^*(G(p^\infty)) \to H^*(G(q'^r))$ is the restriction map. Because $\mathscr{A}(G(q'^r)) \to \mathscr{A}(G(q^r))$ is an equivalence of categories by Lemma

3, [4; 7.2] (see also (6.1) below) implies that $H^*(G(q^r))_{\text{red}} \to H^*(G(q^{r'}))_{\text{red}}$ is injective. The theorem now follows by inspecting the commutative square obtained by applying $(\ )_{\text{red}}$ to (4.1).

The reader wishing to replace $\mathbf{Z}/l$ in the above theorem by $\mathbf{Z}/p$ should consider the example of GL_{2n} for some $n \geq 1$. Then $H^i(GL_{2n}(p^\infty), \mathbf{Z}/p) = 0$ for $i > 0$, whereas the Krull dimension of $H^*(GL_{2n}(p^d), \mathbf{Z}/p)$ equals n^2d.

As an immediate corollary of Theorem 4, we conclude the following "multiplicative stability" for $\{H^*(G(p^d)); d \geq 1\}$.

COROLLARY 5. *There exists a* pth *power* q *such that for any positive integer* r *the restriction map*

$$H^*(G(\mathbf{F}_{q^r}), \mathbf{Z}/l)_{\text{red}} \to H^*(G(\mathbf{F}_q), \mathbf{Z}/l)_{\text{red}}$$

is an isomorphism.

We recall Quillen's terminology of a "uniform F-isomorphism" $h: R \to S$ between graded, anti-commutative $\mathbf{Z}/l$-algebras: h must be a homomorphism for which there exists a positive integer N such that

i) if $h(r) = 0$ for some homogeneous element $r \in R$, then $r^N = 0$ and

ii) for each homogeneous element $s \in S$, $s^{l^N} \in h(R)$.

In [4; 7.2], Quillen proved

$$H^*(BK, \mathbf{Z}/l) \to \varprojlim_{E \in \mathcal{A}(K)} H^*(E, \mathbf{Z}/l) \text{ is a uniform } F\text{-isomorphism,} \tag{6.1}$$

any compact Lie group K. The following extension of (6.1) to the non-compact group $G(p^\infty)$ is an immediate consequence of (6.1) applied to the finite groups $G(p^d)$ together with Theorem 1 and Lemma 3.

COROLLARY 6. *The canonical map*

$$H^*(G(\bar{\mathbf{F}}_p), \mathbf{Z}/l) \to \varprojlim_{E \in \mathcal{A}(G(\bar{\mathbf{F}}_p))} H^*(E, \mathbf{Z}/l)$$

is a uniform F-isomorphism.

We conclude by demonstrating how Corollary 6 can be used to prove (6.1) for compact, connected Lie groups. Since relatively elementary algebraic proofs of (6.1) for finite groups are now available (e.g., [1; 2.26]), this demonstration might prove of interest to algebraists unfamiliar with equivariant cohomology theories.

COROLLARY 7. *Let K be a compact, connected Lie group and l a prime. Then the canonical map*

$$H^*(BK, \mathbf{Z}/l) \to \varprojlim_{E \in \mathscr{A}(K)} H^*(E, \mathbf{Z}/l)$$

is a uniform F-isomorphism.

Proof. Let $G\mathbf{C}$ be the complex form of K. As is well-known (see, for example, [3]), $H^*(BK) \to \varprojlim_{E \in \mathscr{A}(K)} H^*(E)$ is isomorphic to $H^*(G\mathbf{C}) \to \varprojlim_{E \in \mathscr{A}(G\mathbf{C})} H^*(E)$. Let p be any prime different from l and consider (as in [3; 3.1]) the following commutative square

$$\begin{array}{ccc} H^*(BG\mathbf{C})) & \longrightarrow & \varprojlim_{E \in \mathscr{A}(G\mathbf{C}))} H^*(E) \\ \downarrow & & \downarrow \\ H^*(G(p^\infty)) & \longrightarrow & \varprojlim_{E' \in \mathscr{A}(G(p^\infty))} H^*(E') \end{array} \tag{7.1}$$

By [2; 1.4], the left vertical map of (7.1) is an isomorphism, whereas Corollary 6 asserts that the lower horizontal map is a uniform F-isomorphism. By [3; 1.4, 1.7], "lifting to characteristic 0" determines a faithful, essentially surjective functor $\mathscr{A}(G(p^\infty)) \to \mathscr{A}(G\mathbf{C})$ which induces the right vertical map. As argued in [3; 2.1], this right vertical map must be injective. We now conclude by inspection of (7.1) that the upper horizontal map must also be a uniform F-isomorphism.

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