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A Riemann–Hurwitz formula for the Selmer group of an elliptic curve with complex multiplication

KAY WINGBERG

In 1981 Iwasawa considered p -adic Galois representations obtained by the action of the Galois group of a finite p -extension on a \mathbf{Q}_p -vector space defined by the ideal class group of the cyclotomic \mathbf{Z}_p -extension of a number field. More precisely his result is as follows [4]: Let p be an odd prime number and let L be a CM -field which is a finite Galois p -extension of the cyclotomic \mathbf{Z}_p -extension k_∞ of a CM -field k with Galois group $G = G(L/k_\infty)$ and ring of integers $\mathcal{O}(L)$. Assuming the Iwasawa- μ -invariant of k_∞ is zero the structure of the minus part of the Pontryagin dual of the flat cohomology group $H^1(\mathcal{O}(L), \mu_{p^\infty})$ as $\mathbf{Q}_p[G]$ -module is given by the isomorphism

$$(H^1(\mathcal{O}(L), \mu_{p^\infty})^-)^* \otimes \mathbf{Q}_p \cong \mathbf{Q}_p^\delta \oplus \mathbf{Q}_p[G]^{\lambda^-(k_\infty) - \delta} \oplus \bigoplus_{\substack{v \nmid p \\ \mu_p \subset k_v^+}} \text{Ind}_{G_v}^G (I(G_v))$$

where δ is equal to 1 if k contains the group μ_p of p -th roots of unity and 0 otherwise; $\lambda^-(k_\infty)$ denotes the λ -invariant of $(H^1(\mathcal{O}(k_\infty), \mu_{p^\infty})^-)^*$, G_v is the decomposition group of G relative to a prime v of k_∞^+ and $I(G_v)$ is the augmentation ideal of $\mathbf{Q}_p[G_v]$. (If $\lambda^-(k_\infty) = 0$ and $\delta = 1$ the isomorphism should be interpreted in the Grothendieck group of finitely generated $\mathbf{Q}_p[G]$ -modules.) Observe that $H^1(\mathcal{O}(L), \mu_{p^\infty})^-$ is just the minus part of the p -component of the Picard group $\text{Pic } \mathcal{O}(L)$.

The corresponding identity between the dimensions on both sides is the Riemann–Hurwitz formula proved by Kida [6].

Our aim is to show an analogous formula for the Selmer group of an elliptic curve E defined over a number field F which has complex multiplication by the ring of integers of an imaginary quadratic field K . In this case p has to be an odd prime number, which splits in K , i.e. $p = \mathfrak{p}\mathfrak{p}^*$, and where E has good (ordinary) reduction. Let F_∞ be the unique \mathbf{Z}_p -extension inside $F(E(\mathfrak{p}))$ and let L be a finite Galois p -extension of F_∞ with Galois group G unramified at all primes above \mathfrak{p}^* . For the Selmer group $H^1(\mathcal{O}(L), E(\mathfrak{p}))$ (we do not distinguish in the notations

between the Néron model and its generic fiber) we will then show the following result,

THEOREM. *Assuming*

$$F(E_p)/K \text{ is abelian} \tag{1.0}$$

then there is a $\mathbf{Q}_p[G]$ -isomorphism

$$H^1(\mathcal{O}(L), E(\mathfrak{p}))^* \otimes \mathbf{Q}_p \cong \mathbf{Q}_p^\varepsilon \oplus \mathbf{Q}_p[G]^{\lambda_p(F_\infty) - \varepsilon} \oplus \bigoplus_{\substack{v \nmid p \\ F_v(E_p) = F_v}} \text{Ind}_{G_v}^G(I(G_v))$$

where ε is equal to 1 if $F = F(E_p)$ and 0 otherwise and $\lambda_p(F_\infty)$ denotes the λ -invariant of $H^1(\mathcal{O}(F_\infty), E(\mathfrak{p}))^*$ (if $\lambda_p(F_\infty) - \varepsilon$ is negative the isomorphism should be considered in the Grothendieck group). In particular, if e_w is the ramification index of L/F_∞ relative to a prime w of L , then

$$\varepsilon - \lambda_p(L) = (\varepsilon - \lambda_p(F_\infty))[L : F_\infty] - \sum_{\substack{w \nmid p \\ L_w(E_p) = L_w}} (e_w - 1).$$

Instead of assuming $\mathcal{F} = F(E_p)$ to be an abelian extension of K we will prove the theorem under the more general conditions:

$$\mathcal{F}_\infty \text{ satisfies the weak } p\text{-adic Leopold-conjecture, see [1] p. 124.} \tag{1.1}$$

$$\text{The } \mu\text{-invariant of } H^1(\mathcal{O}(\mathcal{F}_\infty), E(\mathfrak{p}))^* \text{ is zero.} \tag{1.2}$$

These assertions are true if \mathcal{F}/K is abelian, [1] Proposition 15, [2] Theorem 3.4. As a consequence of (1.1) and (1.2) the following is true:

(2) The Pontryagin dual of

$$H^1(\mathcal{O}(F_\infty), E(\mathfrak{p})) \cong H^1(G(\mathcal{F}_{S_p}/\mathcal{F}_\infty), E(\mathfrak{p}))^\Delta$$

is a free \mathbf{Z}_p -module of finite rank $\lambda_p(F_\infty)$, [1] Theorem 12, Proposition 22, where L_S is the maximal p -extensions of a field L unramified outside a set S of primes of L , $S_p = \{v \mid p\}$ and $\Delta = G(\mathcal{F}/F)$. Furthermore, let S be a finite set of primes such that $S \cap S_p = S_p$ then according to [8], Theorem, the assertions (1.2) and (2) imply:

$$G(F_S/F_\infty) \text{ is free pro-} p\text{-group of finite rank.} \tag{3.1}$$

$$\text{For } T \supseteq S \text{ the canonical map} \tag{3.2}$$

$$*_{v \in T \setminus S(F_S)} T_v(F(p)/F_\infty) \xrightarrow{\sim} G(F_T/F_S)$$

from the free pro- p -product of inertia groups into $G(F_T/F_S)$ induced by the maps

$$T_v(F(p)/F_\infty) = T_v(F(p)/F_S) \hookrightarrow G(F(p)/F_S) \twoheadrightarrow G(F_T/F_S)$$

is an isomorphism.

LEMMA 4. *The assertions (3.1) and (3.2) are stable under base change by a finite Galois p -extension unramified at all primes above \mathfrak{p}^* .*

Proof. Let L/F_∞ be a finite Galois p -extension unramified at $S_{\mathfrak{p}^*}$ and let T be a finite set of primes such that $L \subset F_T$ and $T \cap S_p = S_{\mathfrak{p}^*}$. Then by (3.1) the Galois group $G(L_T/L)$ of $L_T = F_T$ over L is free of finite rank. Hence we obtain for the factor group $G(L_{S_{\mathfrak{p}^*}}/L)$

$$\begin{aligned} \text{rank}_\Lambda G(L_{S_{\mathfrak{p}^*}}/L)^{ab} &= 0, \\ \mu(G(L_{S_{\mathfrak{p}^*}}/L)^{ab}) &= \mu(G(L_T/L)^{ab}) = 0. \end{aligned}$$

Again by [8], Theorem, the assertions (3.1) and (3.2) are true for L .

LEMMA 5. *Let G and Δ be finite groups of p -power order and order prime to p , respectively. Let M be a \mathbf{Z}_p -torsion free $\mathbf{Z}_p[G \times \Delta]$ -module with the properties*

- (a) $H^1(G, M) = 0$,
- (b) $H^2(G, M) \cong \mathbf{Z}/(G : 1)\mathbf{Z}$,
- (c) $H^2(G \times \Delta, M) \neq 0$.

Then for every \mathbf{Z}_p -irreducible character χ of Δ , i.e., $\mathbf{Z}_p[\Delta] = \bigoplus_\chi \mathbf{Z}_p[\Delta]^\chi$, there are numbers $m_\chi \geq 0$ and $\mathbf{Z}_p[G]$ -isomorphisms

$$\begin{aligned} M^\chi &\cong \mathbf{Z}_p[G]^{m_\chi} \quad \text{for } \chi \neq \chi_0 := 1, \\ M^{\chi_0} &\cong R_d^{ab} \oplus \mathbf{Z}_p[G]^{m_{\chi_0}}, \end{aligned}$$

where R_d is defined by a minimal presentation $1 \rightarrow R_d \rightarrow F_d \rightarrow G \rightarrow 1$ of the group G by a free pro- p -group F_d of rank d .

Proof. Let M^χ be the eigenspace of M with respect to χ , then

$$\begin{aligned} H^1(G, M) &= \bigoplus_\chi H^1(G, M^\chi) = 0, \\ H^2(G, M) &= \bigoplus_\chi H^2(G, M^\chi) \cong \mathbf{Z}/(G : 1)\mathbf{Z} \end{aligned}$$

Because $H^2(G, M^{\chi_0}) \cong H^2(G \times \Delta, M) \neq 0$ we obtain

$$H^2(G, M^\chi) \cong \begin{cases} \mathbf{Z}/(G:1)\mathbf{Z}, & \chi = \chi_0 \\ 0, & \chi \neq \chi_0. \end{cases}$$

This shows that for $\chi \neq \chi_0$ the $\mathbf{Z}_p[G]$ -module M^χ is cohomologically trivial and therefore $\mathbf{Z}_p[G]$ -free, as M is torsion free, [5] Lemma 1.6. According to [5] Korollar 1.8 we obtain the assertion for the eigenspace of the trivial character.

COROLLARY 6. *Let*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 1 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & H & \longrightarrow & \mathcal{F} & \longrightarrow & G \times \Delta & \longrightarrow & 1 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \Delta & = & \Delta & & \end{array}$$

be a commutative and exact diagram of profinite groups, where F is a free pro- p -group of finite rank and Δ is a finite group of order prime to p . Then for every \mathbf{Z}_p -irreducible character χ of Δ there is a $\mathbf{Q}_p[G]$ -isomorphism

$$(H^{ab})^\chi \otimes \mathbf{Q}_p \cong \mathbf{Q}_p[G]^{n_\chi - \delta_\chi} \oplus \mathbf{Q}_p^{\delta_\chi}$$

where

$$\delta_\chi = \begin{cases} 1, & \chi = \chi_0 \\ 0, & \chi \neq \chi_0 \end{cases} \quad \text{and} \quad n_\chi = \text{rank}_{\mathbf{Z}_p}(F^{ab})^\chi$$

Proof. For the $\mathbf{Z}_p[G \times \Delta]$ -module H^{ab} the conditions of Lemma 5 are fulfilled because $\text{scd}_p(F) \leq 2$, [3] Definition 10 and Proposition 11, and

$$H^2(G \times \Delta, H^{ab}) \xrightarrow{\text{res}} H^2(G, H^{ab})^\Delta \xrightarrow{\sim} \hat{H}^0(G, \mathbf{Z}_p)_\Delta \cong \mathbf{Z}/(G:1)\mathbf{Z}.$$

Since

$$H^{ab} \otimes \mathbf{Q}_p \cong \mathbf{Q}_p[G]^{d-1} \oplus \mathbf{Q}_p, \quad d = \text{rank } F,$$

we obtain $\mathbf{Q}_p[G]$ -isomorphisms

$$(H^{ab})^\chi \otimes \mathbf{Q}_p \cong \mathbf{Q}_p[G]^{m_\chi} \oplus \mathbf{Q}_p^{\delta_\chi}$$

for some $m_\chi \geq 0$. These numbers are easily calculated. Since G and Δ commute we get by taking G -coinvariants

$$\begin{aligned} m_\chi + \delta_\chi &= \text{rank}_{\mathbf{Z}_p} ((H^{ab})^\chi)_G \\ &= \text{rank}_{\mathbf{Z}_p} (H_G^{ab})^\chi \\ &= \text{rank}_{\mathbf{Z}_p} (F^{ab})^\chi = n_\chi. \end{aligned}$$

Proof of the Theorem. Let $L | F_\infty$ be a finite Galois p -extension unramified at all primes above \mathfrak{p}^* and contained in F_S , S a finite set with $S \cap S_p = S_p$. Let $\mathcal{L} = L(E_p)$, $\Delta = G(\mathcal{L}/L)$ and $G = G(L/F_\infty)$. Then we obtain a commutative and exact diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & G(\mathcal{F}_S/\mathcal{L}) & \longrightarrow & G(\mathcal{F}_S/\mathcal{F}_\infty) & \longrightarrow & G & \longrightarrow & 1 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G(\mathcal{F}_S/\mathcal{L}) & \longrightarrow & G(\mathcal{F}_S/F_\infty) & \longrightarrow & G \times \Delta & \longrightarrow & 1 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \Delta & = & \Delta & & \end{array}$$

From Corollary 6 it follows that

$$\begin{aligned} (H^1(G(\mathcal{F}_S/\mathcal{L}), E(\mathfrak{p}))^\Delta)^* \otimes \mathbf{Q}_p &= (H^1(G(\mathcal{F}_S/\mathcal{L}), \mathbf{Q}_p/\mathbf{Z}_p)^\chi(1))^* \otimes \mathbf{Q}_p \\ &\cong \mathbf{Q}_p[G]^{\lambda_p(F_\infty) + \#\{v \in S \setminus S_p \mid F_v(E_p) = F_v\} - \varepsilon} \oplus \mathbf{Q}_p^\varepsilon, \end{aligned}$$

where (1) denotes the twist with $E(\mathfrak{p})$ and χ is the character given by the action of Δ on E_p . Here we use (3.1) and (3.2) which give

$$\begin{aligned} \text{corank}_{\mathbf{Z}_p} H^1(G(\mathcal{F}_S/\mathcal{F}_\infty), \mathbf{Q}_p/\mathbf{Z}_p)^\chi &= \text{corank}_{\mathbf{Z}_p} H^1(G(\mathcal{F}_{S_p}/\mathcal{F}_\infty), \mathbf{Q}_p/\mathbf{Z}_p)^\chi \\ &\quad + \sum_{v \in S \setminus S_p} \text{corank}_{\mathbf{Z}_p} H^1(T_v(\mathcal{F}(p)/\mathcal{F}_\infty), \mathbf{Q}_p/\mathbf{Z}_p)^\chi \\ &= \lambda_p(F_\infty) + \#\{v \in S \setminus S_p \mid F_v(E_p) = F_v\} \end{aligned}$$

Again with (3.1) and (3.2) and Lemma 4 we obtain

$$\begin{aligned} (H^1(G(\mathcal{F}_S/\mathcal{L}), E(\mathfrak{p}))^\Delta)^* \otimes \mathbf{Q}_p &\cong (H^1(G(\mathcal{L}_{S_p}/\mathcal{L}), E(\mathfrak{p}))^\Delta)^* \otimes \mathbf{Q}_p \\ &\quad \oplus \bigoplus_{v \in S \setminus S_p(L)} (H^1(T_v(\mathcal{F}(p)/\mathcal{L}), \mathbf{Q}_p/\mathbf{Z}_p)^\chi(1))^* \otimes \mathbf{Q}_p \\ &\cong (H^1(G(\mathcal{L}_{S_p}/\mathcal{L}), E(\mathfrak{p}))^\Delta)^* \otimes \mathbf{Q}_p \\ &\quad \oplus \bigoplus_{\substack{w \in S \setminus S_p(L) \\ L_w(E_p) = L_w}} \mathbf{Q}_p \end{aligned}$$

Thus we get an isomorphism

$$\begin{aligned} (H^1(\mathcal{O}(L), E(p)))^* \otimes \mathbf{Q}_p \oplus \mathbf{Q}_p^{\#\{w \in S \setminus S_p(L) \mid L_w(E_p) = L_w\}} \\ \cong \mathbf{Q}_p^\varepsilon \oplus \mathbf{Q}_p[G]^{\lambda_p(F_x) - \varepsilon} \oplus \bigoplus_{\substack{v \in S \setminus S_p(F_x) \\ F_v(E_p) = F_v}} \text{Ind}_{G_v}^G \mathbf{Q}_p[G_v] \end{aligned}$$

which proves the theorem.

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