Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 64 (1989)

Artikel: The level of real projective spaces.

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DOI: https://doi.org/10.5169/seals-48969

Comment. Math. Helvetici 64 (1989) 661–674 0010–2571/89/040661–14$01.50 + 0.20/0

The level of real projective spaces

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1. Introduction

In this paper we determine the level of the real projective space $\mathbf{RP}^{2m-1}$ with the $\mathbf{Z}/2$-action induced by multiplication by the complex number i. By definition (see [DL]), the level of a topological space X with a free $\mathbf{Z}/2$-action is the number

$$s(X) = \min \{n : \text{there exists a } \mathbf{Z}/2\text{-equivariant map } f : X \to S^{n-1}\},$$

where the sphere S^{n-1} is equipped with the antipodal $\mathbf{Z}/2$-action. We abbreviate $s(\mathbf{RP}^{2m-1})$ by $s(m)$.

The previously known results about $s(m)$ seem to be the following, P. E. Conner and E. E. Floyd proved $s(1) = 2$, $s(2) = 3$, $s(3) = 5$ [CF] and A. Pfister and the author obtained the estimates $m + 1 \leq s(m) \leq \frac{1}{2}(3m + 1)$ [PS].

The main result of this paper is the computation of $s(m)$.*

THEOREM. *Let $m \geq 2$. Then*

$$s(m) = \begin{cases} m+1 & \text{if } m = 0, 2 \bmod 8 \\ m+2 & \text{if } m = 1, 3, 4, 5, 7 \bmod 8 \\ m+3 & \text{if } m = 6 \bmod 8 \end{cases}$$

Remark. The invariant $s(m)$ is related to the following purely algebraic invariant

$$r(m) = \min \left\{ n : \begin{array}{l} \text{there exists a complex quadratic form } q : \mathbf{C}^m \to \mathbf{C}^n \\ \text{such that } \operatorname{im}(q) : \mathbf{R}^{2m} \to \mathbf{R}^n \text{ is anisotropic} \end{array} \right\}$$

Here $\operatorname{im}(q)$ denotes the imaginary part of q which is a real quadratic form. It is called anisotropic if $\operatorname{im}(q)^{-1}(0) = 0$. By normalizing and restricting $\operatorname{im}(q)$ it

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* This result was also proved by M. C. Crabb using somewhat different arguments in his preprint "Periodicity in $\mathbf{Z}/4$-equivariant stable homotopy theory".

induces a $\mathbf{Z}/4$-equivariant map $S^{2m-1} \to S^{n-1}$ where $\mathbf{Z}/4$ acts by multiplication by i (resp. -1) on the domain (resp. range). Passing to the quotient we get a $\mathbf{Z}/2$-equivariant map $\mathbf{RP}^{2m-1} \to S^{n-1}$. This shows $r(m) \geq s(m)$. The 8-periodicity of $s(m)$ suggests that there might be a way to use Clifford algebras to construct $\mathbf{Z}/2$-equivariant maps $\mathbf{RP}^{2m-1} \to S^{s(m)-1}$ or even quadratic forms $\mathbf{C}^m \to \mathbf{C}^{s(m)}$ with anisotropic imaginary part.

The proof of the theorem uses the following reformulation of the level of X. Let L be the real line bundle $X \times_{\mathbf{Z}/2} \mathbf{R} \to Y$ over the quotient space $Y = X/\mathbf{Z}/2$. If $f: X \to S^{n-1}$ is a $\mathbf{Z}/2$-equivariant map then by passing to the quotient the equivariant map $id \times f: X \to X \times S^{n-1}$ gives a nowhere vanishing section of nL. Conversely a nowhere vanishing section of nL gives rise to an equivariant map f as above. Hence the level of X can equivalently be characterized as the smallest n such that nL has a nowhere vanishing section. An obstruction for the existence of such a section is the cohomotopy Euler class, which we discuss in section 2.

In section 3 we use K-theory methods to show the non-vanishing of the cohomotopy Euler class of nL for certain n's, where L is the non-trivial line bundle over the $\mathbf{Z}/4$-lens space L^{2m-1}, the quotient space of $\mathbf{RP}^{2m-1}$. This implies a lower bound for $s(m)$. It should be emphasized that these K-theory restrictions are stronger than those imposed by the vanishing of the K-theory Euler class. A study of the K-theory Euler class only leads to the lower bound $s(m) \geq m+1$, the same bound as obtained in [PS].

In section 4 we use the Adams spectral sequence and a vanishing result for its E_2-term to show that the cohomotopy Euler class vanishes in certain cases. That leads to an upper bound for $s(m)$ which agrees with the lower bound derived in section 3 except for $m = 4 \bmod 8$.

Finally in section 5 we prove the inequality $s(m+n) \geq s(m) + s(n)$ and use it to compute $s(m)$ for $m = 4 \bmod 8$.

My thanks go to Bill Dwyer and Larry Taylor for helpful comments.

2. The cohomotopy Euler class

In this section we discuss the cohomotopy Euler class and its properties and recall the definition of the (cohomotopy) Gysin sequence.

Throughout this section let X be a finite CW complex and let α be an n-dimensional vector bundle over X. We choose a metric for α and denote by $S(\alpha)$ (resp. $D(\alpha)$) the sphere bundle (resp. disk bundle) of α. The Thom space $T(\alpha)$ is by definition the quotient space $D(\alpha)/S(\alpha)$. The zero section of α induces a map $i: X \to T(\alpha)$ or, more generally, a map $i: T(\beta) \to T(\alpha \oplus \beta)$ for a vector bundle β over X. If α' is an n'-dimensional inverse bundle of α then a trivialization of $\alpha \oplus \alpha'$ induces a map $t: T(\alpha \oplus \alpha') \to S^{n+n'}$. For n' large the vector bundle α' is unique and we define the cohomotopy Euler class $e(\alpha)$ as the composition $T(\alpha') \to T(\alpha \oplus \alpha') \to S^{n+n'}$ of i and t.

If α has a nowhere vanishing section s then the zero section can be deformed into s and hence i is homotopic to the constant map since we can assume that s is a section of $S(\alpha)$. Thus $e(\alpha)$ is homotopic to the constant map.

At this point it is convenient to use the language of Thom spectra. A general reference for spectra is [S]. With our assumption that X is a finite CW-complex Thom spectra of (virtual) vector bundles over X are easily defined as follows. If α is a n-dimensional vector bundle then its Thom spectrum $M\alpha$ is the n-th desuspension of the suspension spectrum of $T(\alpha)$. Note that with this definition the bottom cell of $M\alpha$ is in dimension 0. The notion of Thom spectrum can be extended to virtual vector bundles. For example $M(-\alpha) = M(\alpha')$, where α' is an inverse to α.

For n' large the set $[T(\alpha'), S^{n+n'}]$ of homotopy classes of maps from $T(\alpha')$ to $S^{n+n'}$ is isomorphic to $\{T(\alpha'), S^{n+n'}\}$, the group of homotopy classes of maps from the suspension spectrum of $T(\alpha')$ to the suspension spectrum of $S^{n+n'}$. Via suspension isomorphism $\{T(\alpha'), S^{n+n'}\}$ can be identified with $\{M(-\alpha), S^n\} = \pi^n(M(-\alpha))$.

Using these identifications the cohomotopy Euler class $e(\alpha)$ is an element of $\pi^n(M(-\alpha))$. We think of $\pi^n(M(-\alpha))$ as a "twisted" cohomotopy group of X and hence we use the notation $\pi^n(X; -\alpha)$. The big advantage of the cohomotopy Euler class is the following.

PROPOSITION 2.1 ([C, Prop. 2.4]). *If α is an n-dimensional vector bundle over a finite* CW*-complex X and* $\dim X < 2(n-1)$ *then α has a nowhere vanishing section if and only if its cohomotopy Euler class vanishes.*

The classical obstruction for finding a non-where vanishing section of an orientable vector bundle α is the usual Euler class of α which is an element of $H^n(X; \mathbf{Z})$ (see e.g. [MS]). If α is a complex vector bundle of dimension k this Euler class is the k-th Chern class $c_k(\alpha) \in H^{2k}(X; \mathbf{Z})$. The usual Euler class and the cohomotopy Euler class are related as follows. Using the notation $H^n(X; -\alpha)$ for $H^n(M\alpha; \mathbf{Z})$ the Hurewicz homomorphism

$$h : \pi^n(X; -\alpha) = \pi^n(M(-\alpha)) \to H^n(M\alpha; \mathbf{Z}) = H^n(X; -\alpha) \tag{2.2}$$

maps $e(\alpha)$ to a (twisted) cohomology class $e_{\mathbf{Z}}(\alpha)$ which we call the cohomology Euler class of α. If α is oriented $e_{\mathbf{Z}}(\alpha)$ corresponds to the usual Euler class under the Thom isomorphism $H^n(X; -\alpha) \cong H(X; \mathbf{Z})$.

Replacing $\mathbf{Z}$-cohomology by $\mathbf{Z}/2$-cohomology there is a corresponding Hurewicz map $h_{\mathbf{Z}/2} : \pi^n(X; -\alpha) \to H^n(X; \mathbf{Z}/2)$ (note that here we don't need α to be oriented) and

$$h_{\mathbf{Z}/2}(e(\alpha)) = w_n(\alpha) \text{ (the } n\text{-th Stiefel Whitney class of } \alpha). \tag{2.3}$$

The Euler class has the following multiplicative property. Assume that α and β are n-dimensional (resp. m-dimensional) vector bundles over X. Then

$$e(\alpha \oplus \beta) = e(\alpha)e(\beta), \tag{2.4}$$

where the product on the right hand side is the cup product for (twisted) cohomotopy

$$\pi^n(X; -\alpha) \otimes \pi^m(X; -\beta) \to \pi^{n+m}(X; -(\alpha \oplus \beta))$$

defined as follows. Let f, g be elements of $\pi^n(X; -\alpha)$ resp. $\pi^m(X; -\beta)$ which are represented by maps of spectra $f: M(\alpha') \to S^n$ resp. $g: M(\beta') \to S^m$, where α' resp. β' are inverse bundles of α resp. β. Then their cup product is given by the composition

$$M(\alpha' \oplus \beta') \xrightarrow{M\Delta} M(\alpha' \times \beta') = M(\alpha') \wedge M(\beta') \xrightarrow{f \wedge g} S^n \wedge S^m = S^{n+m}, \tag{2.5}$$

where $\alpha' \times \beta'$ is the product bundle over $X \times X$ whose Thom spectrum can be identified canonically with the smash product $M(\alpha') \wedge M(\beta')$. The diagonal map $\Delta: X \to X \times X$ is covered by a bundle map $\alpha' \oplus \beta' \to \alpha' \times \beta'$ which induces a map $M\Delta$ between the Thom spectra. The multiplicative property (2.4) follows easily from the definitions of the Euler class and the cup product.

Another tool we need is the Gysin sequence. Let α be an n-dimensional vector bundle over X. Then by definition of the Thom space there is a cofibration

$$S(\alpha) \xrightarrow{p} X \xrightarrow{i} T(\alpha) = \Sigma^n M\alpha, \tag{2.6}$$

where p is the projection map and i denotes the inclusion of the zero section. It induces long exact sequences

$$\to \pi^{i-n}(X; \alpha) \xrightarrow{i^*} \pi^i X \xrightarrow{p^*} \pi^i S(\alpha) \xrightarrow{\partial} \pi^{i-n+1}(X; \alpha) \to \text{and} \tag{2.7}$$

$$\to H^{i-n}(X; \alpha) \xrightarrow{i^*} H^i(X; \mathbf{Z}) \xrightarrow{p^*} H^i(S(\alpha); \mathbf{Z}) \xrightarrow{\partial} H^{i-n+1}(X; \alpha) \to, \tag{2.8}$$

which we refer to as the cohomotopy (resp. cohomology) Gysin sequence for $S(\alpha)$. If α is orientable we can replace the twisted cohomology group $H^{i-n}(X;\alpha)=H^{i-n}(M\alpha;\mathbf{Z})$ by $H^{i-n}(X;\mathbf{Z})$ using the Thom isomorphism and this gives the usual Gysin sequence (see e.g. [MS]). More generally, if β is a vector bundle over X then there is a cofibration

$$T(p^*\beta)\xrightarrow{p}T(\beta)\xrightarrow{i}T(\alpha\oplus\beta) \tag{2.9}$$

inducing long exact sequences

$$\to\pi^{i-n}(X;\alpha\oplus\beta)\xrightarrow{i^*}\pi^i(X;\beta)\xrightarrow{p^*}\pi^i(S(\alpha);p^*\beta)\xrightarrow{\partial}\pi^{i-n+1}(X;\alpha\oplus\beta) \tag{2.10}$$

and

$$\to H^{i-n}(X;\alpha\oplus\beta)\xrightarrow{i^*}H^i(X;\beta)\xrightarrow{p^*}H^i(S(\alpha);p^*\beta)\xrightarrow{\partial}H^{i-n+1}(X;\alpha\oplus\beta), \tag{2.11}$$

which we call the cohomotopy (resp. cohomology) Gysin sequence for $S(\alpha)$ with coefficients in β. It follows from the definition of the cohomotopy Euler class that the map i^* in these sequences is the multiplication by the cohomotopy (resp. cohomology) Euler class.

3. A lower bound for $s(m)$

The goal of this section is the proof of the following.

PROPOSITION 3.1. *Let L be the non-trivial real line bundle over the $\mathbf{Z}/4$-lens space L^{2m-1} with $m\geq 2$. If $m=2k-2$ and $k=0 \bmod 4$ or $m=2k-1$ then the cohomotopy Euler class of $2kL$ is non-trivial.*

This implies that $2kL$ does not have a nowhere vanishing section or, equivalently, there is no $\mathbf{Z}/2$-equivariant map $\mathbf{RP}^{2m-1}\to S^{2k-1}$. Hence we obtain the following estimate on $s(m)$.

COROLLARY 3.2. *Let $m\geq 2$. Then*

$$s(m)\geq\begin{cases} m+1 & \text{if } m=0,2,4 \bmod 8\\ m+2 & \text{if } m=1,3,5,7 \bmod 8.\\ m+3 & \text{if } m=6 \bmod 8\end{cases}$$

Proof of Proposition 3.1. We observe that L^{2m-1} can be identified with the sphere bundle of H^4, the fourth tensor power of the Hopf bundle H over the complex projective space $\mathbf{CP}^{m-1}$. Moreover the pull back of H^2 under the projection map $p: L^{2m-1} = S(H^4) \to \mathbf{CP}^{m-1}$ is $2L$.

This can be seen as follows. The Hopf bundle H can be written as the vector bundle associated to the standard 1-dimensional complex representation of S^1 given by multiplication by $z \in S^1$. Thus H^2 corresponds to the representation given by multiplication by z^2 and $p^*(H^2)$ corresponds to its restriction to the subgroup $\mathbf{Z}/4$ of S^1 generated by $i \in S^1$. This representation of $\mathbf{Z}/4$ is the sum of two copies of the non-trivial 1-dimensional real representation of $\mathbf{Z}/4$ whose assocated vector bundle is L.

The naturality of the Euler class then implies $p^*(e(kH^2)) = e(2kL)$. To analyze $p^*(e(kH^2))$ we use the Gysin sequence for the sphere bundle $S(H^4)$. Writing down the Gysin sequences for cohomotopy (resp. cohomology) with coefficients in $-kH^2$ (see (2.10) resp. (2.11)) and identifying the twisted cohomology groups with untwisted ones using the Thom isomorphism we get the following commutative diagram

$$\begin{array}{ccccccc}
\pi^{2k-2}(\mathbf{CP}^{m-1}; H^4\text{-}kH^2) & \xrightarrow{i^*} & \pi^{2k}(\mathbf{CP}^{m-1}; \text{-}kH^2) & \xrightarrow{p^*} & \pi^{2k}(L^{2m-1}; \text{-}2kL) & \longrightarrow \\
\downarrow h & & \downarrow h & & \downarrow h & \\
\longrightarrow H^{2k-2}(\mathbf{CP}^{m-1}; \mathbf{Z}) & \xrightarrow{i^*} & H^{2k}(\mathbf{CP}^{m-1}; \mathbf{Z}) & \xrightarrow{p^*} & H^{2k}(L^{2m-1}; \mathbf{Z}) & \longrightarrow
\end{array}$$

Here the vertical map h is the Hurewicz map. It maps the cohomotopy Euler class of kH^2 to the cohomology Euler class $e_{\mathbf{Z}}(kH^2)$.

Recall that the cohomology of $\mathbf{CP}^{m-1}$ is a truncated polynomial ring $H^*(\mathbf{CP}^{m-1}; \mathbf{Z}) \cong \mathbf{Z}[x]/(x^m)$ whose generator $x \in H^2(\mathbf{CP}^{m-1}; \mathbf{Z})$ is the first Chern class of the Hopf bundle. Hence $e_{\mathbf{Z}}(H^2) = c_1(H^2) = 2x$ and $e_{\mathbf{Z}}(kH^2) = (e_{\mathbf{Z}}(H^2))^k = 2^k x^k$. The induced map i^* in cohomology is multiplication by $e_{\mathbf{Z}}(H^4) = c_1(H^4) = 4x$.

To prove proposition 3.1 assume $e(2kL) = 0$. Then the cohomotopy exact sequence implies that $e(kH^2)$ is of the form $i^*(y)$ for some $y \in \pi^{2k-2}(\mathbf{CP}^{m-1}; H^4 - kH^2)$. The commutativity of the diagram implies $i^*(h(y)) = h(i^*(y)) = h(e(kH^2)) = e_{\mathbf{Z}}(kH^2) = 2^k x^k$ and hence $h(y) = 2^{k-2} x^{k-1}$. But this contradicts the following proposition.

PROPOSITION 3.3. *Let $m \geq 2$. If $m = 2k - 2$ and $k = 0 \bmod 4$ or $m = 2k - 1$ then the index of the Hurewicz homomorphism $h: \pi^{2k-2}(\mathbf{CP}^{m-1}; H^4 - kH^2) \to H^{2k-2}(\mathbf{CP}^{m-1}; \mathbf{Z}) \cong \mathbf{Z}$ is multiple of 2^{k-1}.*

To prove this proposition we first characterize the index of h as the "codegree" of some vector bundle and then use the K-theory methods of [CK] of obtain estimates for this codegree. If α is an orientable (virtual) vector bundle over a space X then $cd(\alpha)$, the codegree of α, is defined as the index of the Hurewicz map $\pi^0 M \in \to H^0(M\alpha; \mathbf{Z}) \cong \mathbf{Z}$.

LEMMA 3.4. *If α is some (virtual) vector bundle over $\mathbf{CP}^{m-1}$ then the index of the Hurewicz map $h: \pi^{2r}(\mathbf{CP}^{m-1}; \alpha) \to H^{2r}(\mathbf{CP}^{m-1}; \mathbf{Z})$ is the codegree of $\alpha + rH$ over $\mathbf{CP}^{m-r-1}$.*

Proof. Consider the cofibration

$$\mathbf{CP}^{r-1} \to \mathbf{CP}^{m-1} \xrightarrow{pr} \mathbf{CP}^{m-1}/\mathbf{CP}^{r-1}.$$

It is well known that the cofiber $\mathbf{CP}^{m-1}/\mathbf{CP}^{r-1}$ can be identified with the Thom space of the vector bundle rH over $\mathbf{CP}^{m-r-1}$. Moreover there is a corresponding cofibration with "coefficients in α" which induces the following long exact sequence of cohomotopy groups.

$$\pi^{2r-1}(\mathbf{CP}^{r-1}; \alpha) \to \pi^0(\mathbf{CP}^{m-r-1}; \alpha + rH) \xrightarrow{pr^*} \pi^{2r}(\mathbf{CP}^{m-1}; \alpha)$$
$$\to \pi^{2r}(\mathbf{CP}^{r-1}; \alpha)$$

The groups $\pi^{2r-1}(\mathbf{CP}^{r-1}; \alpha)$ and $\pi^{2r}(\mathbf{CP}^{r-1}; \alpha)$ vanish for dimensional reasons and hence pr^* is an isomorphism. The same argument shows that pr induces an isomorphism in cohomology, too. Hence the index of the Hurewicz map

$$h: \pi^{2r}(\mathbf{CP}^{m-1}; \alpha) \to H^{2r}(\mathbf{CP}^{m-1}; \mathbf{Z})$$

is equal to the index of

$$h: \pi^0(\mathbf{CP}^{m-r-1}; \alpha + rH) \to H^0(\mathbf{CP}^{m-r-1}; \mathbf{Z}),$$

which is the codegree of $\alpha + rH$. Q.E.D.

We estimate the codegree of $H^4 - kH^2 + (k-1)H$ using the K-theory method of [CK]. It is based on the fact that the Hurewicz map factors through K-theory. More precisely the Hurewicz map $h: \pi^0 M\alpha \to H^0(M\alpha; \mathbf{Z})$ composed with the inclusion $i: H^0(M\alpha; \mathbf{Z}) \to H^*(M\alpha; \mathbf{Q})$ is the composition of the K-theory Hurewicz map $h_K: \pi^0 M\alpha \to K^0 M\alpha$ and the Chern character $ch: K^0 M\alpha \to H^*(M\alpha; \mathbf{Q})$.

The codegree of α is by definition the index of $\operatorname{im}(h)$ in $H^0(M\alpha; \mathbf{Z})$ or, alternatively, the index of $\operatorname{im}(i \circ h)$ in $\operatorname{im}(i)$. It is hence a multiple of the index of $\operatorname{im}(i) \cap \operatorname{im}(ch)$ in $\operatorname{im}(i)$ which is called the K-theory codegree of α and denoted by $cd^K(\alpha)$.

For computations the following characterization of $cd^K(\alpha)$ is useful.

LEMMA 3.5 ([CK], Prop. 3.2). *Let α be a complex vector bundle over a finite* CW *complex X with torsion free homology. Then*

$$cd^K(\alpha) = \min\{m \in N \mid m \cdot ch^{-1}\operatorname{Todd}(-\alpha) \in K^0X \otimes \mathbf{Q} \text{ is integral}\}$$

Here $\operatorname{Todd}(\alpha) \in H^*(X; \mathbf{Q})$ *is the* Todd *genus of α. It is multiplicative, i.e.*

$$\operatorname{Todd}(\alpha + \beta) = \operatorname{Todd}(\alpha) \cdot \operatorname{Todd}(\beta),$$

and if L is a complex line bundle then

$$\operatorname{Todd}(L) = (\exp(c_1(L)) - 1)/c_1(L).$$

LEMMA 3.6 ([CK], p. 16). *Let L be a complex line bundle. Then* $ch^{-1}\operatorname{Todd}(-L) = \log(\lambda + 1)/\lambda \in K^0X \otimes \mathbf{Q}$, *where* $\lambda = L - 1 \in K^0X$ *and* $\log(\lambda + 1)$ *is the standard power series of the natural logarithm.*

Proof. $ch(\log(\lambda + 1)/\lambda) = \log(ch(\lambda + 1)/ch(\lambda)) = \log(ch(L)/(ch(L) - 1)) = c_1(L)/(\exp(c_1(L)) - 1) = \operatorname{Todd}(L)^{-1} = \operatorname{Todd}(-L)$. Q.E.D.

LEMMA 3.7. *The K-theory codegree of $H^4 - kH^2 + (k-1)H$ over $\mathbf{CP}^{k-1}$ is a multiple of 2^{k-1}.*

Proof. Recall that $K^0\mathbf{CP}^{k-1}$ is the truncated polynomial ring $\mathbf{Z}[\eta]/(\eta^k)$ where $\eta = H - 1$. To compute the highest power of 2 in the denominator of $ch^{-1}\operatorname{Todd}(-(H^4 - kH^2 + (k-1)H))$ it is convenient to rewrite everything in terms of the new variable $y = \eta/2$. A look at the power series

$$\left(\frac{\log(\eta + 1)}{\eta}\right) = 1 - \frac{\eta}{2} + \frac{\eta^2}{3} - \frac{\eta^3}{4} + \cdots$$

shows that it represents an element in $\mathbf{Z}_{(2)}[y]$, where $\mathbf{Z}_{(2)}$ denotes the integers localized at 2, i.e. all rational numbers whose denominator is prime to 2. Moreover computing modulo the ideal $2\mathbf{Z}_{(2)}[y]$ we have $\log(\eta + 1)/\eta = 1 - y$. More generally, if λ is an element of $\mathbf{Z}[\eta]$ with vanishing constant term then

$$\left(\frac{\log(\lambda + 1)}{\lambda}\right) = 1 - \frac{\lambda}{2} + \frac{\lambda^2}{3} - \frac{\lambda^3}{4} + \cdots = 1 - \frac{\lambda}{2} \bmod 2\mathbf{Z}_{(2)}[y].$$

In particular we get

$$ch^{-1}\,\mathrm{Todd}\,(-H^4)=\frac{\log(\eta+1)^4}{(\eta+1)^4-1}=1-\frac{4\eta+6\eta^2+4\eta^3+\eta^4}{2}=1 \bmod 2\mathbf{Z}_{(2)}[y]$$

and

$$ch^{-1}\,\mathrm{Todd}\,(-H^2)=\frac{\log((\eta+1)^2)}{(\eta+1)^2-1}=1-\frac{2\eta+\eta^2}{2}=1 \bmod 2\mathbf{Z}_{(2)}[y].$$

Using the multiplicativity of the Todd genus and the fact that the Chern character is a ring homomorphism we obtain

$$ch^{-1}\,\mathrm{Todd}\,(-H^4-kH^2+(k-1)H))=(1-y)^{k-1} \bmod 2\mathbf{Z}_{(2)}[y].$$

Expressing $(1-y)^{k-1}$ as a power series in η we see that $m=2^{k-1}$ is the smallest power of 2 such that $m(1-y)^{k-1}\in\mathbf{Z}_{(4)}[\eta]/(\eta^k)$. Since $2^{k-2}(2\mathbf{Z}_{(2)}[y])$ is contained in $\mathbf{Z}_{(2)}[\eta]/(\eta^k)$ the same conclusion holds for $ch^{-1}\,\mathrm{Todd}\,(-(H^4-kH^2+(k-1)H))$. It follows from (3.5) that the codegree of $H^4-kH^2+(k-1)H$ is a multiple of 2^{k-1}. Q.E.D.

Together the lemmata 3.4 and 3.7 provide the proof of proposition 3.3 except if $k=0 \bmod 4$. In that case we have to show that the codegree of $H^4-kH^2+(k-1)H$ over $\mathbf{CP}^{k-2}$ is a multiple of 2^{k-1}. This sharper estimate can be obtained by considering the KO-theory codegree which is defined analogous to the K-theory codegree by replacing the Chern character $ch: K^0M\alpha\to H^*(M\alpha;\mathbf{Q})$ by the Pontrjagin character $ph: KO^0M\alpha\to H^*(M\alpha;\mathbf{Q})$ which is the composition of the complexification map $KO^0M\alpha\to K^0M\alpha$ and the Chern character. The same arguments as before show that the codegree is a multiple of the KO-theory codegree which in turn is a multiple of the K-theory codegree. Hence the proof of proposition 3.3 is completed with the proof of the following lemma.

LEMMA 3.8. *Let $k=0 \bmod 4$. Then the KO-theory codegree of $H^4-kH^2+(k-1)H$ over $\mathbf{CP}^{k-2}$ is a multiple of 2^{k-1}.*

Proof. Consider the cofibration $\mathbf{CP}^{k-2}\to\mathbf{CP}^{k-1}\to\mathbf{CP}^{k-1}/\mathbf{CP}^{k-2}=S^{2k-2}$ and its induced long exact sequence in KO-theory

$$\to KO^{-1}S^{2k-2}\to KO^0\mathbf{CP}^{k-1}\to KO^0\mathbf{CP}^{k-2}\to KO^0S^{2k-2}\to.$$

It follows that $KO^0\mathbf{CP}^{k-1}\to KO^0\mathbf{CP}^{k-2}$ is an isomorphism since the other two terms vanish by Bott periodicity. Hence the KO-codegree of $H^4 - kH^2 + (k-1)H$ as a bundle over $\mathbf{CP}^{k-2}$ is the same as its codegree as a bundle over $\mathbf{CP}^{k-1}$ which is a multiple of 2^{k-1} by (3.7). Q.E.D.

4. An upper bound for $s(m)$

The main result of this section is the following.

PROPOSITION 4.1. *Assume* $m = 2k$ *and* $k = 0, 1 \bmod 4$ *or* $m = 2k - 1$. *Then the cohomotopy Euler class of* $(2k+1)L$ *over* L^{2m-1} *vanishes.*

By proposition 2.1 this implies that $(2k+1)L$ has a nowhere vanishing section or, equivalently, that there is a $\mathbf{Z}/2$-equivariant map $\mathbf{RP}^{2m-1} \to S^{2k}$. Hence we obtain the following upper estimate for $s(m)$.

COROLLARY 4.2.

$$s(m) \leq \begin{cases} m+1 & \text{if } m = 0, 2 \bmod 8 \\ m+2 & \text{if } m = 1, 3, 5, 7 \bmod 8 \\ m+3 & \text{if } m = 4, 6 \bmod 8 \end{cases}$$

Proposition 4.1 is proved using the Adams spectral sequence, notably a "vanishing line" for its E_2-term (see 4.4). We begin by describing the properties of the Adams spectral sequence which are relevant to us. General references are the books of Adams [A] and Switzer [S].

Let X, Y be finite spectra and let p be a fixed prime. We say that a map $X \to Y$ has $\mathbf{Z}/p$-Adams filtration $\geq s$ if it can be written as a composition

$$X \to Z_1 \to \cdots \to Z_{s-1} \to Y$$

of s maps which are all trivial in Z/p-cohomology. This defines a filtration on the abelian group $[X, Y]$ of homotopy classes of maps $X \to Y$ or, more generally, on $[X, Y]_n = [\Sigma^n X, Y]$. We denote by $F_s[X, Y]_n$ the subgroup of elements of filtration $\geq s$ in $[X, Y]_n$. Note that in the case where X (resp. Y) is the sphere spectrum S^0 this defines a filtration of the homotopy (resp. cohomotopy) groups of spectra.

This filtration is compatible with the smash product, i.e. if $f \in F_s[X, Y]_n$ and $f' \in F_{s'}[X', Y']_{n'}$ then $f \wedge f' \in F_{s+s'}[X \wedge X', Y \wedge Y']_{n+n'}$. This follows directly from the definition since if f factors as $X \to Z_1 \to \cdots \to Z_{s-1} \to Y$ and f' factors as $X' \to Z'_1 \to \cdots \to Z'_{s-1} \to Y'$ then there is the following factorization for $f \wedge f'$.

$$X \wedge X' \to Z_1 \wedge X' \to \cdots \to Z_{s-1} \wedge X' \to Y \wedge X' \to Y \wedge Z'_1$$
$$\to \cdots \to Y \wedge Z'_{s'-1} \to Y \wedge Y'$$

The compatibility of the Adams filtration with the smash product implies its compatibility with the cup product (see 2.5), which we state as a lemma for further reference.

LEMMA 4.3. *If α and α' are vector bundles over a space X and f, f' are elements of $\pi^n(X; \alpha)$ (res. $\pi^{n'}(X; \alpha')$) of Adams filtration $\geq s$ (resp. $\geq s'$) then their cup product has filtration $\geq s + s'$.*

Associated to the Adams filtration on $[X, Y]_n$ there is a corresponding spectral sequence $E_r^{s,t}(X, Y)$, the Adams spectral sequence. It converges to the p-primary part of $[X, Y]_n$, i.e.

$$E_\infty^{s,t}(X, Y) \cong F_s[X, Y]_{t-s}/F_{s+1}[X, Y]_{t-s},$$

where $F_s[X, Y]_{t-s}$ denotes the elements of filtration s in $[X, Y]_{t-s}$. Moreover the intersection of all $F_s[X, Y]_{t-s}$ consists of the torsion elements of $[X, Y]_{t-s}$ whose order is prime to p. Its E_2-term is

$$E_2^{s,t}(X, Y) = \mathrm{Ext}_A^{s,t}(H^*Y, H^*X),$$

where H^*X (resp. H^*Y) denotes the cohomology of X (resp. Y) with coefficients in $\mathbf{Z}/p$, which is a module over the mod p Steenrod algebra A. The differentials have the form

$$d_r: E_r^{s,t}(X, Y) \to E_r^{s+r,t+r-1}(X, Y).$$

For $p = 2$ let A_0 be the subalgebra of A which is generated by $Sq^1 \in A$. This is an exterior algebra since $Sq^1Sq^1 = 0$. J. F. Adams proved the following homological vanishing theorem.

PROPOSITION 4.4 ([A], Thm. 3, p. 62]). *Let M be a graded A-module which is free over A_0 and $(l-1)$-connected, i.e. trivial in domensions $<l$. Then* $\mathrm{Ext}_A^{s,t}(M, \mathbf{Z}/2)$ *is zero if $t - s < l + F(s)$, where $F(s)$ is the numerical function defined by $F(4r) = 8r$, $F(4r+1) = 8r+1$, $F(4r+2) = 8r+2$ and $F(4r+3) = 8r+4$.*

COROLLARY 4.5. *Let X be a finite spectrum whose $\mathbf{Z}/p$-cohomology vanishes for p odd and whose $\mathbf{Z}/2$-cohomology is free as an A_0-module and trivial above dimension d. Let $\alpha \in \pi^n X$ be an element of Adams filtration s. Then $\alpha = 0$ provided $d - n < F(s)$.*

Proof of the corollary. Consider the Adams spectral sequence $E_r^{s,t}(X, S^0)$ converging to $[X, S^0]_{-n} = \pi^n X$. For p odd all terms are zero and hence the cohomotopy groups of X are torsion groups whose orders are powers of 2.

From now on let $p = 2$. $E_2^{s,t}(X, S^0)$ is equal to $\mathrm{Ext}_A^{s,t}(\mathbf{Z}/2, H^*X) = \mathrm{Ext}_A^{s,t}(DH^*X, \mathbf{Z}/2)$, where DH^*X is the dual of the graded A-module H^*X which is defined as follows. If M is a graded A-module and M_i denotes the elements of degree i in M then $(DM_i) = \mathrm{Hom}\,(M_{-i}, \mathbf{Z}/2)$. The left A-module structure on M induces a right A-module structure on $DM = \mathrm{Hom}\,(M, \mathbf{Z}/2)$ which is then converted into a left A-module structure using the canonical anti-automorphism χ of the Steenrod algebra.

Our assumption that H^*X vanishes in dimensions bigger than d implies that DH^*X is $(-d-1)$-connected. Moreover, DH^*X is free as A_0-module since H^*X is A_0-free and $\chi(Sq^1) = Sq^1$. It follows from proposition 4.4 that $E_2^{s,t}(X, S^0)$ and hence $E_\infty^{s,t}(X, S^0)$ vanishes for $t - s + d < F(s)$. This means that the filtration quotient $F_s\pi^n X / F_{s+1}\pi^n X = E_\infty^{s,t}(X, S^0)$ is zero for $d - n = d + t - s < F(s)$, which implies that the element $\alpha \in \pi^n X$ is in the intersection of all filtration groups and hence a torsion element of odd order. Thus $\alpha = 0$. Q.E.D.

After these preparations we now prove proposition 4.1. The idea is to use corollary 4.5 to prove the vanishing of the cohomotopy Euler class $e((2k+1)L) \in \pi^n M(-(2k+1)L)$. We first show that $M(-(2k+1)L)$ satisfies the assumptions of (4.5), i.e. that

i) $H^*(M(-(2k+1)L); \mathbf{Z}/2)$ is free as A_0-module

ii) $H^*(M(-(2k+1)L); \mathbf{Z}/p) = 0$ for p odd

Ad i) The $\mathbf{Z}/2$-cohomology ring of L^{2m-1} is $\mathbf{Z}[x]/(x^m) \otimes E(y)$, where x is a 2-dimensional cohomology class, $y = w_1(L)$ is the first Stiefel Whitney class of L and $E(y)$ is the exterior algebra generated by y. As abelian group the $\mathbf{Z}/2$-cohomology of the Thom spectrum $M(-(2k+1)L)$ is isomorphic to the $\mathbf{Z}/2$-cohomology of L^{2m-1} via Thom isomorphism. It is given by multiplication with the Thom class $U \in H^0(M(-(2k+1)L); \mathbf{Z}/2)$. The computation $Sq^1 U = w_1(-(2k+1)L)U = yU$, $Sq^1(x^s U) = x^s y U$ for $s < m$ shows that the $\mathbf{Z}/2$-cohomology of the Thom spectrum is a free A_0-module.

Ad ii) Note that $-(2k+1)L$ is non-orientable since its first Stiefel–Whitney class is non-trivial and hence there is no Thom isomorphism for $\mathbf{Z}/p$-cohomology. Instead we use the Gysin sequence for $S(L)$ with coefficients in $-(2k+2)L$ (see

(2.11))

$$\to H^{i-1}(L^{2m-1}; -(2k+1)L) \to H^i(L^{2m-1}; -(2k+2)L)$$
$$\xrightarrow{p^*} H^i(S(L); -(2k+2)p = L) \to .$$

Here $H^i(\ )$ is the cohomology with $\mathbf{Z}/p$-coefficients. The bundle $-(2k+2)L$ is orientable and hence p^* can be identified with the map induced by p in (untwisted) $\mathbf{Z}/p$-cohomology whch is an isomorphism since L^{2m-1} and $S(L) = \mathbf{RP}^{2m-1}$ have the $\mathbf{Z}/p$-cohomology of a point. Thus $H^*(M(-(2k+1)L); \mathbf{Z}/p) = H^*(L^{2m-1}; -(2k+1)L)$ vanishes.

Next we estimate the Adams filtration of the cohomotopy Euler class of $(2k+1)L$ using the general properties of the Euler class stated in section 2. Note that $w_2(2L) = w_1(L)^2 = y^2 = 0$. This implies that $e(2L)$ has at least Adams filtration 1, since $w_2(2L)$ is the image of $e(2L)$ under the Hurewicz map. Hence $e(2kL) = e(2L)^k$ has at least filtration k by (2.4) and (4.3).

Finally we apply (4.5) to the Euler class $e((2k+1)L) \in \pi^{2k+1}M(-(2k+1)L)$. In this case $d = 2m-1$ (the dimension of $M(-(2k+1)L)$), $n = 2k+1$ and $s = k$ (the filtration of $(2k+1)L$). Thus the inequality $d - n < F(s)$ reduces to $2k-2 < F(k)$ (in the case $m = 2k$, $k = 0, 1 \bmod 4$) respectively to $2k-4 < F(k)$ (in the case $m = 2k-1$). Inspection of the numerical function $F(k)$ (see 4.4) shows that these inequalities hold. Corollary (4.5) then implies $e((2k+1)L) = 0$. Q.E.D.

5. Determination of $s(m)$

An inspection of the lower and upper estimates for $s(m)$ obtained in the last two sections show that they agree except for $m = 4 \bmod 8$ where we have the inequalities $m+1 \le s(m) \le m+3$.

PROPOSITION 5.1. $s(m) = m+2$ *for* $m = 4 \bmod 8$.

The main ingredients of the proof are the knowledges of $s(m)$ for other values of m and the following lemma.

LEMMA 5.2. $s(m+n) \le s(m) + s(n)$

Proof of the lemma. Let $f: \mathbf{RP}^{2m-1} \to S^{s(m)-1}$ and $g: \mathbf{RP}^{2n-1} \to S^{s(n)-1}$ be $\mathbf{Z}/2$-equivariant maps. Denote by $\tilde{f}: S^{2m-1} \to S^{s(m)-1}$ resp. $\tilde{g}: S^{2n-1} \to S^{s(n)-1}$ the composition of f resp. g with the projection map from the sphere to projective space. These maps are $\mathbf{Z}/4$-equivariant with respect to the $\mathbf{Z}/4$-action given by multiplication by $i \in \mathbf{C}$ on the domain and multiplication by -1 on the range. Then also their join

$$\tilde{f} * \tilde{g} : S^{2(m+n)-1} = S^{2m-1} * S^{2n-1} \to S^{s(m)-1} * S^{s(n)-1} = S^{s(m)+s(n)-1}$$

is a $\mathbf{Z}/4$-equivariant map. Passing to the quotient we obtain a $\mathbf{Z}/2$-equivariant map $\mathbf{RP}^{2(m+n)-1} \to S^{s(m)+s(n)-1}$ showing that $s(m+n) \le s(m) + s(n)$. Q.E.D.

Proof of the proposition. Let $m = 4 \bmod 8$. Then using the lemma and our computations of $s(m)$ we obtain the inequalities $s(m) \le s(m-2) + s(2) = (m-1) + 3 = m + 2$ and $m + 5 = s(m+2) \le s(m) + s(2) = s(m) + 3$. Thus $s(m) = m + 2$. Q.E.D.

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Received April 19, 1988