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Casson invariant of links of singularities

WALTER NEUMANN AND JONATHAN WAHL

Let Σ be the link of a normal complete intersection surface singularity and let F be the associated Milnor fiber. Thus, Σ is a closed oriented 3-manifold and F is a compact, simply-connected, parallelizable 4-manifold with boundary Σ. Assume Σ is a homology 3-sphere; then the Casson invariant $\lambda(\Sigma)$ has been defined (e.g., [A] or [F–S]) as minus one-half the "number" of non-trivial SU(2)-representations of $\pi_1(\Sigma)$, where "number" is given by an appropriate algebraic count.

CONJECTURE. *The Casson invariant* $\lambda(\Sigma)$ *equals* $\frac{1}{8}$ *sign* (F).

This has been proved by Fintushel and Stern in [F–S] for the Brieskorn sphere $\Sigma(p, q, r)$, which is the link of the singularity at 0 of the hypersurface $x^p + y^q + z^r = 0$ in $\mathbb{C}^3$. Namely, the number of SU(2) representations of $\pi_1(\Sigma(p, q, r))$ is $\frac{1}{4}N(p, q, r)$, where $N(p, q, r)$ is the number of integer lattice points in the interior of the tetrahedron with vertices $(p, 0, 0)$, $(0, q, 0)$, $(0, 0, r)$, (p, q, r) (this has been observed by several people in one form or another; it essentially goes back to Greenberg [G]). On the other hand, $-N(p, q, r)$ is easily equated with Brieskorn's formula for sign (F) ([B]). Thus the core of Fintushel and Stern's result is that in this case the "algebraic count" of SU(2)-representations is the actual count.

In this paper we use their result to

(a) compute the Casson invariant for arbitrary graph manifold homology spheres (Remark 1.14),
(b) confirm the above Conjecture for weighted homogeneous surface singularities (Proposition 1.1),
(c) confirm the Conjecture for links of hypersurface singularities given by an equation of the form $f(x, y) + z^n = 0$ (Proposition 2.5),
(d) confirm the Conjecture for a family of complete intersection singularities in $\mathbb{C}^4$.

It is tempting to make the same conjecture for any smoothing of a Gorenstein surface singularity with homology sphere link (see §3); but incredibly, we know of no examples of such singularities which are not complete intersections!

The calculational methods we use here do not hint at the general reason why the Conjecture should be true, but we give some speculation in §3. Aside from the above evidence, we note that the Casson invariant must be $\frac{1}{8}$ the signature of *some* simply-connected spin 4-manifold which bounds Σ (see §3); it is thus natural to try the Milnor fiber.

The main work in every case is computing sign (F). Along the way we classify homology sphere cyclic branched covers of graph links (Theorem 2.4), and generalize signature formulae of Shinohara (Theorem 2.14) and Hirzebruch (Proposition 1.12) and a result of Mordell (Remark 1.16).

(Added February 1989). After writing this paper we learned that Fukuhara, Matsumoto and Sakamoto have independently proved Proposition 1.1 by essentially the same method, see [F–M–S]. In addition, K. Walker in [W] has extended the definition of the Casson invariant to rational homology spheres (see also Boyer and Lines [B–L], who describe the same extension for homology lens spaces); it would be interesting to know to what extent the above conjecture generalizes.

§1. Seifert fibered homology spheres

Let $a_1, \ldots, a_n$ be pairwise coprime positive integers. The Seifert fibered homology sphere $\Sigma(a_1, \ldots, a_n)$ is the link of the singularity of $f^{-1}(0)$, where $f: \mathbb{C}^n \to \mathbb{C}^{n-2}$ is a map of the form $f(z_1, \ldots, z_n) = (\Sigma_1^n b_{1,j} z_j^{a_j}, \ldots, \Sigma_1^n b_{n-2,j} z_j^{a_j})$ with sufficiently general coefficient matrix (b_{ij}). Every weighted homogeneous surface singularity with homology sphere link is equivalent to one of these ([N2]).

Let $\sigma(a_1, \ldots, a_n)$ denote the signature sign $(f^{-1}(\delta))$ of a nonsingular fiber of f. Let $\lambda(a_1, \ldots, a_n)$ denote the Casson invariant of $\Sigma(a_1, \ldots, a_n)$.

PROPOSITION 1.1. $\lambda(a_1, \ldots, a_n) = \frac{1}{8}\sigma(a_1, \ldots, a_n)$

Proof. This was proved for $n = 3$ by Fintushel and Stern [F–S]. The general case is a trivial induction using the following lemma. □

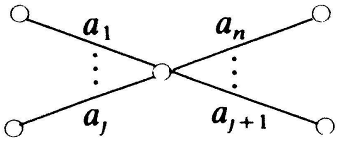
LEMMA 1.2. *For any* $1 < j < n - 1$,

$$\lambda(a_1, \ldots, a_n) = \lambda(a_1, \ldots, a_j, a_{j+1} \cdots a_n) + \lambda(a_1 \cdots a_j, a_{j+1}, \ldots, a_n), \tag{1.3}$$

$$\sigma(a_1, \ldots, a_n) = \sigma(a_1, \ldots, a_j, a_{j+1} \cdots a_n) + \sigma(a_1 \cdots a_j, a_{j+1}, \ldots, a_n). \tag{1.4}$$

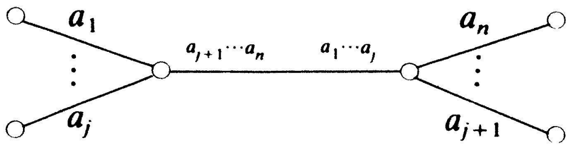
Proof. To prove (1.3) we use the *splice diagrams* of [E–N] and [Si]; they will also be useful later. The homology sphere $\Sigma(a_1, \ldots, a_n)$ is obtainable from

$\Sigma(a_1, \ldots, a_j, a_{j+1} \cdots a_n)$ and $\Sigma(a_1 \cdots a_j, a_{j+1}, \ldots, a_n)$ by splicing along appropriate singular fibers – see [E–N, Lemma 8.4]: specifically, the splice diagram for $\Sigma(a_1, \ldots, a_n)$:



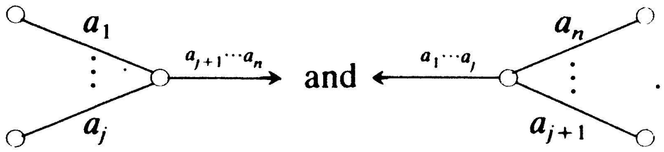
,

is equivalent to



,

which is the splice of



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Therefore, (1.3) is immediate from the additivity of the Casson invariant under splicing, proved independently by Akbulut & McCarthy, Boyer & Nicas, and Fukuhara & Maruyama (according to [B–N]).

To prove (1.4) we need the following. Put $N = a_1 \cdots a_n$ and $b_\nu = N/a_\nu$.

LEMMA 1.5.

$$(1.6) \quad \sigma(a_1, \ldots, a_n) = -1 + \frac{1}{3N}\left(1 - (n-2)N^2 + \sum_{\nu=1}^{n} b_\nu^2\right) + \sum_{\nu=1}^{n} d(a_\nu; b_\nu).$$

where, for coprime integers a and b with $a > 0$,

$$d(a; b) = \frac{1}{a} \sum_{\substack{\zeta^a = 1 \\ \zeta \neq 1}} \frac{\zeta + 1}{\zeta - 1} \frac{\zeta^b + 1}{\zeta^b - 1}.$$

REMARK 1.7. For $n = 3$ this lemma is in the work of Hirzebruch and Zagier ([H1] and [H–Z]), and also Mordell [Mo] (given that $-\sigma(a_1, a_2, a_3)$ is the number of lattice points in a tetrahedron as described above). $d(a; b)$ is a version of the Dedekind sums discussed there ($d(a; b) = 1/a \operatorname{def}(a; b, 1)$ in Hirzebruch's terminology).

We first show how this lemma implies (1.4).

Applying the lemma to $(a, b, 1)$ and noting that $\sigma(a, b, 1) = d(1; ab) = 0$, we get:

$$0 = -1 + \frac{1 + a^2 + b^2}{3ab} + d(a; b) + d(b; a),$$

so

$$(1.8)\quad d(a;b)+d(b;a)=1-\frac{1+a^2+b^2}{3ab},$$

which is the *Dedekind reciprocity formula* (cf. [H1] and [H–Z]). Now if we apply (1.6) to expand the right side of (1.4) we obtain, with $A=a_1\cdots a_j$ and $B=a_{j+1}\cdots a_n$:

$$\begin{aligned}
&-1+\frac{1}{3N}\left(1-(j-1)N^2+\sum_{\nu=1}^{j} b_\nu^2+A^2\right)+d(B;A)+\sum_{\nu=1}^{j} d(a_\nu;b_\nu)\\
&\quad+-1+\frac{1}{3N}\left(1-(n-j-1)N^2+\sum_{\nu=j+1}^{n} b_\nu^2+B^2\right)+d(A;B)+\sum_{\nu=j+1}^{n} d(a_\nu;b_\nu)\\
&=-1+\frac{1}{3N}\left(1-(n-2)N^2+\sum_{\nu=1}^{n} b_\nu^2\right)+\sum_{\nu=1}^{n} d(a_\nu;b_\nu)\\
&\quad+-1+\frac{1}{3N}(1+A^2+B^2)+d(A;B)+d(B;A).
\end{aligned}$$

By (1.6) and (1.8), this equals the left side of (1.4). □

Proof of Lemma 1.5: We use Hirzebruch's formula ([H2]):

$$\sigma(a_1,\ldots,a_n)=-\sum_{\substack{1\le j<2N\\ j\text{ odd}}}\operatorname{res}_{\pi ij/2N}\left((\tanh Nz)^{n-2}\coth z\prod_{\nu=1}^{n}\coth b_\nu z\right)dz.$$

Making the substitution $w=\exp(2z)$, this becomes

$$\sigma(a_1,\ldots,a_n)=-\sum_{\zeta^N=-1}\operatorname{res}_\zeta\left(\left(\frac{w^N-1}{w^N+1}\right)^{n-2}\frac{w+1}{w-1}\prod_{\nu=1}^{n}\left(\frac{w^{b_\nu}+1}{w^{b_\nu}-1}\right)\right)\frac{dw}{2w}.$$

Note that

$$\omega:=\left(\left(\frac{w^N-1}{w^N+1}\right)^{n-2}\frac{w+1}{w-1}\prod_{\nu=1}^{n}\left(\frac{w^{b_\nu}+1}{w^{b_\nu}-1}\right)\right)\frac{dw}{2w}$$

has poles only at 0, ∞, and at certain $2N$-th roots of unity. Thus, by the residue theorem,

$$\sigma(a_1,\ldots,a_n)=\operatorname{res}_0\omega+\operatorname{res}_\infty\omega+\sum_{\zeta^N=1}\operatorname{res}_\zeta\omega=-\tfrac12-\tfrac12+\sum_{\zeta^N=1}\operatorname{res}_\zeta\omega.$$

Now the only poles at N-th roots of unity are a triple pole at 1 and simple poles at each a_ν-th root of unity other than 1. Thus

$$\sigma(a_1, \dots, a_n) = -1 + \operatorname{res}_1 \omega + \sum_{\nu=1}^{n} \sum_{\substack{\zeta^{a_\nu}=1 \\ \zeta\neq 1}} \operatorname{res}_\zeta \omega. \tag{1.9}$$

But

$$\begin{aligned}
\sum_{\substack{\zeta^{a_\nu}=1 \\ \zeta\neq 1}} \operatorname{res}_\zeta \omega &= \sum_{\substack{\zeta^{a_\nu}=1 \\ \zeta\neq 1}} \operatorname{res}_\zeta \left(\left(\frac{w^N-1}{w^N+1}\right)^{n-2} \frac{w+1}{w-1} \prod_{\nu=1}^{n} \left(\frac{w^{b_\nu}+1}{w^{b_\nu}-1}\right)\right) \frac{dw}{2w} \\
&= \sum_{\substack{\zeta^{a_\nu}=1 \\ \zeta\neq 1}} \operatorname{res}_1 \left(\left(\frac{w^N-1}{w^N+1}\right)^{n-2} \frac{\zeta w+1}{\zeta w-1} \frac{(\zeta w)^{b_\nu}+1}{(\zeta w)^{b_\nu}-1} \prod_{\mu\neq\nu} \left(\frac{w^{b_\mu}+1}{w^{b_\mu}-1}\right)\right) \frac{dw}{2w} \\
&= \sum_{\substack{\zeta^{a_\nu}=1 \\ \zeta\neq 1}} \frac{1}{2}\left(\frac{N}{2}\right)^{n-2} \frac{\zeta+1}{\zeta-1} \frac{\zeta^{b_\nu}+1}{\zeta^{b_\nu}-1} \prod_{\mu\neq\nu} \frac{2}{b_\mu} \\
&= \frac{1}{a_\nu} \sum_{\substack{\zeta^{a_\nu}=1 \\ \zeta\neq 1}} \frac{\zeta+1}{\zeta-1} \frac{\zeta^{b_\nu}+1}{\zeta^{b_\nu}-1} = d(a_\nu; b_\nu).
\end{aligned} \tag{1.10}$$

(Here the second line is the substitution $w \mapsto \zeta w$ and the next line follows by removing all factors $(w-1)$ and then evaluating at $w=1$.) To compute the residue of ω at 1 we need the third term in the Laurent expansion of ω in terms of $(w-1)$, since 1 is a triple pole. We can compute this by taking the first three terms of the expansion for each factor of ω and multiplying. We omit the details of this elementary computation, which gives the answer

$$\operatorname{res}_1 \omega = \frac{1}{3N}\left(1 - (n-2)N^2 + \sum_{\nu=1}^{n} b_\nu^2\right). \tag{1.11}$$

Inserting (1.10) and (1.11) into (1.9) proves the lemma. □

One can perform the analogous computation to Lemma 1.5 also when the a_i are not coprime. It is a digression and the computational details are similar, so we just describe the final result. Let now $N = \operatorname{lcm}(a_1, \dots, a_n)$, and denote $b_\nu = N/a_\nu$, $t_\nu = N/\operatorname{lcm}(a_1, \dots, \hat{a}_\nu, \dots, a_n)$, and $s_\nu = a_1 \cdots \hat{a}_\nu \cdots a_n / \operatorname{lcm}(a_1, \dots, \hat{a}_\nu, \dots, a_n)$. Then

PROPOSITION 1.12. *The signature of the* 2 *complex dimensional smooth Brieskorn complete intersection with exponents* $a_1, \dots, a_n$ *is*

$$\sigma(a_1, \dots, a_n) = -1 + \frac{\prod_{i=1}^{n} a_i}{3N^2}\left(1 - (n-2)N^2 + \sum_{\nu=1}^{n} b_\nu^2\right) + \sum_{\nu=1}^{n} s_\nu\, d(t_\nu; b_\nu). \tag{1.13}$$

□

This formula involves several topological ingredients: by [N–R], $\Sigma(a_1, \ldots, a_n)$ is Seifert fibered with exactly s_ν multiple fibers of multiplicity t_ν for each ν with $t_\nu > 1$, these are the only multiple fibers, and the Seifert invariant of each t_ν-multiple fiber is (t_ν, β_ν) with $b_\nu\beta_\nu \equiv 1$ (*modulo* t_ν); moreover, the euler number of this Seifert fibering is $(\prod_{i=1}^n a_i)/N^2$. (With the information that the euler characteristic of the base surface is $-(n-2)(\prod_{i=1}^n a_i)/N + \sum_{\nu=1}^n s_\nu$, this determines the topology of $\Sigma(a_1, \ldots, a_n)$ completely.) However, easy examples show that $\sigma(a_1, \ldots, a_n)$ is not always a topological invariant of $\Sigma(a_1, \ldots, a_n)$.

REMARKS 1.14. As described in [E–N] and [Si], any graph manifold homology sphere can be obtained by iteratively splicing Seifert fibered homology spheres along fibers. Since λ is additive under splicing, the formula of Lemma 1.5 for $\sigma(a_1, \ldots, a_n)$, together with Proposition 1.1, give an efficient computation of the Casson invariant for any graph manifold homology sphere.

1.15. Don Zagier pointed out that Lemma 1.5 also enables quick proofs or generalizations of some of the results of [H–Z] about $d(a; b)$ and $\sigma(a, b, c)$ (there called $t(a, b, c)$). For instance, we have already deduced Dedekind reciprocity. By applying reciprocity to one term of (1.3) we get

$$\begin{aligned}\sigma(a_1, \ldots, a_n) &= \frac{1}{3N}\left(-(n-2)N^2 + \sum_{\nu=1}^{n-1} b_\nu^2 - a_n^2\right) + \sum_{\nu=1}^{n-1} d(a_\nu; b_\nu) - d(b_n; a_n) \\ &= -\frac{1}{3}\left((n-2)N - \sum_{\nu=1}^{n-1} \frac{b_\nu}{a_\nu} + \frac{a_n}{b_n}\right) + \sum_{\nu=1}^{n-1} d(a_\nu; b_\nu) - d(b_n; a_n),\end{aligned}$$

so

$$\sigma(a_1, \ldots, a_{n-1}, a_n + b_n) = \sigma(a_1, \ldots, a_n) - \frac{1}{3}\left((n-2)b_n^2 - \sum_{\nu=1}^{n-1} \frac{b_n^2}{a_\nu^2} + 1\right),$$

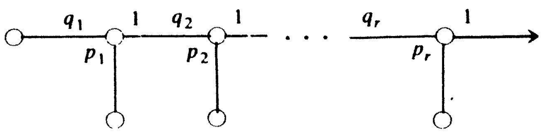
showing that $\sigma(a_1, \ldots, a_{n-1}, a)$ is a linear-plus-periodic function of a. This periodicity of signatures has been discussed by several people (see [H–Z] for references). Interpreted as a periodicity for the Casson invariant, it is just an instance of Casson's surgery formula, since $\Sigma(a_1, \ldots, a_{n-1}, a_n + b_n)$ is obtained by (-1)-Dehn surgery on the degree a_n fiber in the Seifert fibering of $\Sigma(a_1, \ldots, a_n)$.

1.16. For $n = 3$ formula (1.13) generalizes Mordell's result [Mo] (cf. Remark 1.7), since $\sigma(a_1, a_2, a_3)$ is a lattice point count by [B], even if the a_i are not coprime.

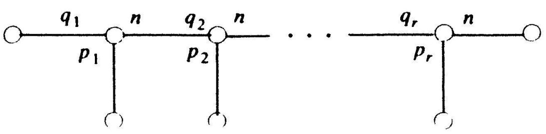
§2. Some hypersurfaces

Let $g(x, y, z) = f(x, y) + z^n$ define an analytic map $g: (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0 and let Σ be the link of the singularity at 0 of $V = g^{-1}(0)$.

PROPOSITION 2.1. *Σ is a homology sphere if and only if the plane curve $f(x, y) = 0$ has only one branch at 0 and n is prime to each entry of each essential Pulseux pair for this plane curve. In fact, if the splice diagram for the plane curve singularity is*

(2.2)  ,

then the condition is that n be prime to each p_i and q_i, and the splice diagram for Σ is then

(2.3) 

The "Newton–Puiseux pairs" (p_i, q_i) are related to the standard Puiseux pairs (n_i, m_i) by the recursion $p_i = n_i$, $q_1 = m_1$, $q_i = m_i - m_{i-1}n_i$, for $i > 1$ ([E–N p. 49]). It follows that n is prime to each p_i and q_i if and only if it is prime to each m_i and n_i, so it does not matter which version of the Puiseux pairs is referred to in the first sentence of the Proposition.

The above notation means that the link $\mathscr{L} = (S^3, L)$ of the singularity at 0 of $f(x, y) = 0$ is the knot $\mathcal{O}(p_1, q_1; p_2, q_2; \ldots; p_r, q_r)$ obtained by iterative (p_i, q_i) cabling starting from the unknot $\mathcal{O}$.

Proof of 3.1. Let (S^3, L) be the link for $f(x, y) = 0$. The link Σ for $f(x, y) + z^n = 0$ is the n-fold cyclic branched cover of S^3 along L. By P. A. Smith theory, for any prime p dividing n, the $\mathbb{Z}/p$-action contained in the $\mathbb{Z}/n$ covering transformation group on the homology sphere Σ must have connected fixed point set. But the fixed point set is L, so L is connected and the plane curve has just one branch at 0.

Now suppose (S^3, L) is the knot $\mathcal{O}(p_1, q_1; p_2, q_2; \ldots; p_r, q_r)$ with splice diagram Γ of (2.2) above. Let Σ be the n-fold cyclic cover of S^3 along L. We want to see that Σ is a homology sphere if and only if n is prime to the p_i and q_i. We first consider the case $r = 1$. Then Γ is the splice diagram for the (p_1, q_1) torus knot, which is the link of the singularity $x^{p_1} + y^{q_1} = 0$. Hence Σ is the link of the surface singularity $x^{p_1} + y^{q_1} + z^n = 0$, that is, Σ is the Brieskorn manifold $\Sigma(p_1, q_1, n)$. This is a homology sphere if and only if p_1, q_1, and n are pairwise coprime ([B]; more generally, the 3-dimensional link $\Sigma(a_1, \ldots, a_n)$ of a Brieskorn complete intersection is a homology sphere if and only if the a_i are pairwise coprime – this follows from [N–R, Theorem 2.1]).

Now for general r let $k \leq r$ and consider the link $(S_k^3, L_0 \cup L_1)$ with splice diagram

$$\begin{tikzcd} L_0 & \cdot \arrow[l, "q_k"'] \arrow[r, "1"] \arrow[d, dash, "p_k"'] & L_1 . \\ & {} & \end{tikzcd}$$

The ambient sphere S^3 of the link (S^3, L) is obtained from the ambient sphere S_k^3 by replacing tubular neighborhoods of the link components L_0 and L_1 by suitable homology solid tori N_0 and N_1 (N_1 is a genuine solid torus, but this is irrelevant). Any homology solid torus N has a degree 1 map to a true solid torus, by collapsing the complement of a collar neighborhood of ∂N to a circle. We can thus construct a degree 1 map $f: S^3 \to S_k^3$ which collapses N_0 and N_1 to circles and is a homeomorphism from $S^3 - (N_0 \cup N_1)$ to $S_k^3 - (L_0 \cup L_1)$.

Now suppose n is prime to p_i for $i = k+1, \cdots, r$. Then n is prime to the linking number $p_{k+1} \cdots p_r$ of a meridian of L_1 with L, so the n-fold cyclic cover of the exterior of L in S^3 induces the n-fold cyclic cover on the exterior of L_1 in S_k^2. Thus, taking n-fold cyclic branched covers of $f: S^3 \to S_k^3$, we get a map $\bar{f}: \Sigma \to \Sigma(p_k, q_k, n)$ which is a homeomorphism over the complement of two circles (the degree q_k and degree n fibers in the Seifert fibering of $\Sigma(p_k, q_k, n)$). Since $\bar{f}$ has degree 1, it is surjective on homology. Thus, if Σ is a homology sphere then $\Sigma(p_k, q_k, n)$ is a homology sphere, and hence n is prime to p_k and q_k. By induction, if Σ is a homology sphere then n is prime to each p_i and q_i.

On the other hand, if n is prime to each p_i and q_i, then the above argument identifies Σ inductively as the result of splicing together homology spheres $\Sigma(p_i, q_i, n)$ according to the splice diagram (2.3), so Σ is a homology sphere. □

Remark. The same argument applies more generally. Suppose (S, L) is a graph knot in a homology sphere given by a splice diagram Γ. Every internal edge of Γ has two weights and every external edge has one. Call a weight on an edge of Γ "near" or "far" according as it is on the end of the edge nearest to or furthest from the arrowhead of Γ (thus the weight on an external edge is near, except for the edge with the arrowhead).

THEOREM 2.4. *The n-fold cyclic cover of S branched along L is a homology sphere Σ if and only if n is prime to all near weights in Γ, and Σ is then given by the splice diagram obtained from Γ by multiplying each far weight by n.* □

This theorem can be applied iteratively to complete intersection singularities given by systems of equations of the form $f_i(z_1, \ldots, z_{i+1}) + z_{i+2}^{n_i} = 0$, $i = 1, \ldots, k$, and allows us to compute the Casson invariant when the link is a homology sphere. Unfortunately, except when $k = 1$ we have been able to compute the signature of the Milnor fiber in only very few cases; these cases confirm the Conjecture of the Introduction.

PROPOSITION 2.5. *Let* $g(x, y, z) = f(x, y) + z^n$ *define an analytic map* $g:(\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ *with an isolated singularity at* 0 *and let the link* Σ *at* 0 *of* $V = g^{-1}(0)$ *be a homology sphere with splice diagram* (2.3) *above. Then the signature of the Milnor fiber* F *for* g *and the Casson invariant* $\lambda(\Sigma)$ *are*:

$$\operatorname{sign}(F) = \sum_{\nu=1}^{r} \sigma(p_\nu, q_\nu, n); \qquad \lambda(\Sigma) = \tfrac{1}{8}\operatorname{sign}(F).$$

Proof. The formula for $\lambda(\Sigma)$ follows from the formula for $\operatorname{sign}(F)$, the additivity of λ under splicing, and Proposition 1.1, so we must just prove the formula for $\operatorname{sign}(F)$.

We must remind the reader of a general construction. If $\mathcal{L} = (S^3, L)$ is a knot, then the *n-cyclic suspension* $L \otimes [n]$ is defined as a special case of the knot product of [K–N]; it is a knot $\mathcal{L} \otimes [n] = (S^5, \Sigma)$ in the 5-sphere and Σ is the n-fold cyclic cover of S^3 branched along L. Moreover, if $\mathcal{L}$ is the link of a plane curve singularity given at $0 \in \mathbb{C}^2$ by $f(x, y) = 0$ say, then $\mathcal{L} \otimes [n]$ is the link of the singularity at $0 \in \mathbb{C}^3$ of $f(x, y) + z^n = 0$. Thus Proposition 2.5 follows by induction from the more general:

PROPOSITION 2.6. *Let* $\mathcal{L}$ *be a knot in* S^3 *and* $\mathcal{L}(p, q)$ *the* (p, q)*-cable on* $\mathcal{L}$. *Suppose* $\gcd(n, p) = d$ ($d = 1$ *in the application to Proposition* 2.5). *Then*

(2.7) $\operatorname{sign}(\mathcal{L}(p, q) \otimes [n]) = d \operatorname{sign}(\mathcal{L} \otimes [n/d]) + \sigma(p, q, n)$,

where $\operatorname{sign}(\mathcal{K})$ *means signature of a Seifert surface of the knot* $\mathcal{K}$ *in* S^5.

Proof. First some notation. If A is an $r \times r$ matrix over $\mathbb{C}$, let $A^{(p)}$ denote the $rp \times rp$ matrix (A^* means transpose):

$$A^{(p)} = \begin{bmatrix} A & A^* & \cdots & A^* \\ A & A & \cdots & A^* \\ \vdots & \vdots & \ddots & \vdots \\ A & A & \cdots & A \end{bmatrix}.$$

For $p > 0$ let Λ_p be the $(p-1) \times (p-1)$ matrix

$$\Lambda_p = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

and let $\Lambda_{p,q} = -\Lambda_p \otimes \Lambda_q$.

Let A be a Seifert form of $\mathscr{L}$. Then it is well known (see e.g. [E–N sect. 15]) that $\mathscr{L}(p, q)$ has Seifert form $A^{(p)} \oplus \Lambda_{p,q}$. Moreover, by [K–N], $\mathscr{L} \otimes [n]$ has Seifert form $A \otimes \Lambda_n$. In particular, $\Lambda_{p,q}$ is the Seifert form of the (p, q) torus knot and $\Lambda_{p,q} \otimes \Lambda_n$ is the Seifert form of the link of the Brieskorn singularity $x^p + y^q + z^n = 0$. Since the intersection form on the fiber of a fibered knot in S^5 is the symmetrized Seifert form, the equation (2.7) to be proved is:

$$\operatorname{sign}^+((A^{(p)} \oplus \Lambda_{p,q}) \otimes \Lambda_n) = d \operatorname{sign}^+(A \otimes \Lambda_{n'}) + \sigma(p, q, n),$$

where $\operatorname{sign}^+(A)$ means $\operatorname{sign}(A + A^*)$ and $n' = n/d$. Since $\sigma(p, q, n) = \operatorname{sign}(\Lambda_{p,q} \otimes \Lambda_n)$, this simplifies to:

$$\text{(2.8)} \quad \operatorname{sign}^+(A^{(p)} \otimes \Lambda_n) = d \operatorname{sign}^+(A \otimes \Lambda_{n'}).$$

This equation should have a more elementary proof than what follows, but we have not found one (except when $n = 2$). Our proof will use the decomposition of the above signatures into "equivariant signatures" (for a fibered knot this is the decomposition according to eigenvalues of the monodromy). There are many different versions of the equivariant signatures in the literature and results equivalent to our formula (2.10) below have been proved geometrically by Litherland [L] and using the Blanchfield pairing by Kearton [K]. However, it is more transparant (and slightly more general) to give a direct algebraic approach here than to extract what we need from their results.

We shall work over $\mathbb{C}$ and consider our matrices to represent sesquilinear forms over $\mathbb{C}$. Thus for a general sesquilinear form, $\operatorname{sign}^+(A)$ means $\operatorname{sign}(A + A^*)$, where A^* now means conjugate transpose. If A is a Seifert form of a knot in S^3, then $S = A - A^*$ is the intersection form of the fiber and is thus also non-singular. We shall only consider forms A which satisfy:

$$\text{(2.9)} \quad S = A - A^* \text{ is non-singular.}$$

Note that by a simple change of basis, $A^{(p)}$ can be put in the form

$$A^{(p)} \sim \begin{bmatrix} A & -S & 0 & \dots & 0 \\ 0 & S & -S & \dots & 0 \\ 0 & 0 & S & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & S \end{bmatrix},$$

and is then easily seen to satisfy (2.9) also. Define the *algebraic bordism group* $C_-(\mathbb{C})$ as the group generated by isomorphism classes of sesquilinear forms A as above, with orthogonal sum as addition, modulo the relations that a form A on a space V represents 0 if V contains a subspace W with $2\dim(W)=\dim(V)$ and $W={}^{\perp}W\cap W^{\perp}$ (since the form A may not be symmetric, W has two orthogonal complements with respect to A, a left complement ${}^{\perp}W$ and a right complement $W^{\perp}$).

We recall briefly the computation of $C_-(\mathbb{C})$. It is easy to see that up to algebraic bordism A can be assumed non-singular, and then the sesquilinear space (V, A) determines and is determined by the isometric structure (V, iS, H) where $H=A^{-1}A^*$ is an isometry for the hermitian form iS and H has no eigenvalue 1 (if A is the Seifert form of a fibered knot then H is the algebraic monodromy of the knot). It follows easily that $C_-(\mathbb{C})$ equals the Witt group of such hermitian isometric structures, and by Milnor [M], this is just the free group on the classes of 1-dimensional structures $(\mathbb{C}, (1), e^{i2\theta})$ with $0<\theta<\pi$. The sesquilinear space corresponding to this $(\mathbb{C}, (1), e^{i2\theta})$ is easily seen to be equivalent to $(\mathbb{C}, (e^{i\theta}))$.

Note that both sides of equation (2.8) are invariants of the algebraic bordism class of A. We must thus just verify (2.8) for the generators of $C_-(\mathbb{C})$. Thus let $A=(e^{\pi it})$ with $0<t<1$. Then

$$A^{(p)}=\begin{bmatrix} e^{\pi it} & e^{-\pi it} & \dots & e^{-\pi it} \\ e^{\pi it} & e^{\pi it} & \dots & e^{-\pi it} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\pi it} & e^{\pi it} & \dots & e^{\pi it} \end{bmatrix}.$$

If $\beta=e^{2\pi it/p}$, $\zeta=e^{2\pi i/p}$, and $B=(b_{jk})$ is the matrix $b_{jk}=(\zeta^{k-1}\beta)^{j-1}$, then one verifies that $B^*A^{(p)}B=\operatorname{diag}(r_1e^{\pi it/p}, r_2e^{\pi i(t+1)/p}, \dots, r_pe^{\pi i(t+(p-1))/p})$, where the r_j are positive reals. (The columns of B are the eigenvectors for the corresponding monodromy matrix.) Thus

$$(2.10)\quad A^{(p)}\sim \bigoplus_{j=0,\dots,p-1}(\exp \pi i(t+j)/p)$$

Similarly one shows that

$$\Lambda_n\sim \bigoplus_{k=1,\dots,n-1}\left(\exp \pi i\left(\frac{k}{n}-\frac{1}{2}\right)\right).$$

Thus

$$(2.11)\quad A^{(p)}\otimes\Lambda_n\sim \bigoplus_{\substack{j=0,\dots,p-1\\ k=1,\dots,n-1}}\left(\exp \pi i\left(\frac{t}{p}+\frac{j}{p}+\frac{k}{n}-\frac{1}{2}\right)\right)$$

Now $\operatorname{sign}^+(e^{i\theta})$ equals the sign of the real part of $e^{i\theta}$, which is $+1$, -1, or 0 according as $\theta/\pi + \frac{1}{2} \pmod 2$ is between 0 and 1, between 1 and 2, or integral. Thus

$$(2.12)\quad \operatorname{sign}^+(A^{(p)} \otimes \Lambda_n) = \sum_{\substack{j=0,\dots,p-1\\ k=1,\dots,n-1}} \left(\left(\frac{t}{p}+\frac{j}{p}+\frac{k}{n}\right)\right),$$

with

$$((a)) := \begin{cases} 1 & \text{if } 0<a<1, \\ 0 & \text{if } a=1, \\ -1 & \text{if } 1<a<2. \end{cases}$$

As a function of t, (2.12) defines a step function with discontinuities only at values of t where j and k exist with $\frac{t}{p}+\frac{j}{p}+\frac{k}{n}$ in $\mathbb{Z}$. Putting $p' = p/d$, this means

$$tn' \equiv -jn' - kp' \left(\operatorname{mod} \frac{pn}{d}\right),$$

which has solutions only when

$$t = \frac{1}{n'}, \frac{2}{n'}, \dots, \frac{n'-1}{n'}$$

and has exactly d solutions with $0 \le j < p$ and $0 < k < n$ for each such t. Thus the step function decreases by $2d$ at each discontinuity

$$t = \frac{1}{n'}, \frac{2}{n'}, \dots, \frac{n'-1}{n'}.$$

On the other hand, it just changes sign under the transformation $t \mapsto 1-t$ (use the change of indices $k \mapsto n-k$, $j \mapsto p-1-j$). It follows that the function is

$$(2.13)\quad \operatorname{sign}^+(A^{(p)} \otimes \Lambda_n) = d f_{n'}(t),$$

where

$$f_{n'}(t)=\begin{cases} n'-1-2\lfloor n't\rfloor & \text{if } n't\notin\mathbb{Z} \\ n'-1-2k+1 & \text{if } t=\dfrac{k}{n'}. \end{cases}$$

In particular, replacing p and n by 1 and n' in (2.13) we get

$$\operatorname{sign}^+(A\otimes\Lambda_{n'})=f_{n'}(t),$$

and with (2.13) this completes the proof of (2.8). □

The above proof works with no essential change for arbitrary companions, giving:

THEOREM 2.14. *Let k be a knot in the standard solid torus T in S^3 and let $\mathscr{L}(k)$ denote the companion to $\mathscr{L}=(S^3, L)$ constructed by replacing a tubular neighborhood of L by the solid torus containing k (this is the same as splicing $\mathscr{L}$ to the link $(S^3, K\cup k)$ along K, where K is the core circle of the complementary solid torus $S^3-\mathrm{int}\ T$). Let p be the winding number of k in T, that is $p=\operatorname{link}(k, K)$, and $d=\gcd(p, n)$. Then*

$$\operatorname{sign}(\mathscr{L}(k)\otimes[n])=d\operatorname{sign}\left(\mathscr{L}\otimes\left[\frac{n}{d}\right]\right)+\operatorname{sign}((S^3, k)\otimes[n]).$$

Note that $\mathscr{L}\otimes[1]$ is the unknot, which has signature 0. For $n=2$, $\operatorname{sign}(\mathscr{L}\otimes[2])$ is the standard knot signature, and Theorem 2.14 becomes a theorem of Shinohara [Sh], also proved by Kearton [K].

§3. Speculation and questions for Gorenstein singularities

Let (X, o) be a germ of a normal complex surface singularity, with X contractible, so $\partial X=\Sigma$ is the link. Choose a "good resolution" $(\tilde{X}, E)\to(X, o)$, that is one for which the exceptional curve $E=\cup E_i$ is a union of smooth curves intersecting transversally. $\tilde{X}$ retracts onto E. The resolution dual graph of E determines the graph manifold Σ, and Σ determines the dual graph for the unique minimal good resolution [N]. Σ is a homology sphere if and only if all E_i are rational, the graph has no loops, and $\det(E_i\cdot E_j)=\pm1$. A resolution graph allows one to define the rational number $c_1^2(\tilde{X})+c_2(\tilde{X})$ ([L–W], §4), which is both a resolution invariant and a topological invariant of Σ. The geometric genus $p_g = \dim H^1(\mathcal{O}_{\tilde{X}})$ is an analytic invariant, not determined by the graph alone. X is called *Gorenstein* if there exists a nowhere vanishing holomorphic 2-form on $X - \{o\}$; complete intersection singularities are Gorenstein.

A *smoothing* of (X, o) is a deformation of X whose typical fiber F (the *Milnor fiber*) is smooth. F is a compact real 4-manifold with boundary Σ (see [L–W] for discussion and references). If X is Gorenstein, then F has first betti number 0, is parallelizable, and has *Milnor number* $\mu = \dim H_2(F)$ given by

$$1 + \mu = 12p_g(X) + (c_1^2(\tilde{X}) + c_2(\tilde{X})).$$

If all E_i are rational and the graph has no loops then the intersection pairing on $H_2(F)$ is non-degenerate with $\mu_+ = 2p_g(X)$. Thus the signature of the Milnor fiber satisfies

$$(3.1)\quad -\text{sign}\,(F) = \mu - 4p_g = 8p_g + (c_1^2(\tilde{X}) + c_2(\tilde{X}) - 1).$$

(For the Milnor fiber associated to the Brieskorn sphere $\Sigma(p, q, r)$, the first equality can be shown to yield Brieskorn's formula for the signature by using a formula for p_g as a number of lattice points.) Therefore, given a link Σ, one may have many different Gorenstein (X, o) with link Σ, but if X is smoothable then the invariants sign (F), μ, and $p_g(X)$ determine each other.

Assume from now on that Σ is a homology sphere. A complete intersection singularity has essentially only one smoothing. Moreover, F is a simple-connected (by the Lefschetz theorems) spin manifold whose boundary is Σ. Such manifolds can be used to define interesting invariants of Σ such as the Rokhlin invariant (the mod 2 reduction of the signature of F divided by 8). By surgery, one can always construct a simply-connected spin 4-manifold with boundary Σ, whose signature divided by 8 is exactly the Casson invariant of Σ. The Conjecture of the Introduction is that for a homology sphere link of a complete intersection singularity, the Milnor fiber is such a 4-manifold.

A general Gorenstein (X, o) may not be smoothable, or, if it is, it might have many different smoothings. These phenomena are "explained" in many cases by [L–W, 4.16]. A *smoothing component* of (X, o) is an irreducible component of the base space of the semi-universal deformation over which the smoothing takes place. According to [L–W], there is a map from the set of smoothing components of (X, o) into a finite set $\mathcal{S}(X)$, computed directly from (X, o), which is frequently bijective. One can check easily that when $\partial X = \Sigma$ is a homology sphere, then $\mathcal{S}(X)$ has at most one element. It also follows from [L–W] that in this case a Milnor fiber F must have vanishing first integral homology; we do not know if it need be simply-connected. The only general classes we know of Gorenstein singularities with unique smoothing and simple-connected Milnor fiber are complete intersections and (possibly) Gorenstein singularities in $\mathbb{C}^5$. These considerations, the Conjecture of the Introduction, and a dearth of examples make natural the following

QUESTION 3.2. *Let (X, o) be a Gorenstein surface singularity whose link Σ is a homology sphere.*

(a) *Is (X, o) a complete intersection?*

(b) *Is $p_g(X)$ uniquely determined by Σ?*

(c) *Do all such (X, o) fit into one equisingular (simultaneous resolution) family?*

Note that the Conjecture of the Introduction would imply (b), at least for complete intersections.

PROPOSITION 3.3. *Let (X, o) be a weighted homogeneous singularity whose link Σ is a homology sphere. Then*

(i) *(X, o) is isomorphic to a Brieskorn complete intersection with link $\Sigma(a_1, \ldots, a_n)$.*

(ii) *If (Y, o) is a Gorenstein singularity with the same link and same p_g, then (Y, o) is an equisingular deformation of (X, o), hence a complete intersection with diffeomorphic Milnor fiber.*

Proof. Statement (i) is found in [N2]; it is this surprising fact which motivates Question 3.2. For (ii) consider the filtration of the local ring of Y by order of vanishing along the central curve of the minimal good resolution. Then the associated graded ring is the graded ring of (X, o), and the degeneration of Y to the spectrum of this graded ring is, by the condition of p_g, equisingular. □

Why might some form of the Conjecture be true? According to C. Taubes (cf. [A]), there is a gauge-theoretic definition of the Casson invariant which makes it similar to some of Donaldson's invariants for differentiable structures on 4-manifolds. From the Donaldson point of view and via the Taubes grafting construction, a smooth M frequently occurs as the boundary (or part of it) of a certain compact moduli space $\mathcal{M}$ (of self-dual or anti-self-dual connections). In algebraic geometry, it sometimes turns out that $\mathcal{M}$ is also a moduli space of certain vector bundles. We ask if F has some natural C^∞ interpretation with respect to some metric on the link Σ.

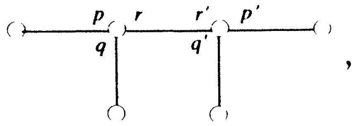
QUESTION 3.4. *Is there an appropriate space of self-dual or anti-self-dual connections modulo gauge equivalence, on a manifold built simply from* Σ *(e.g.* $\Sigma \times \mathbb{R}$*), which gives rise to the Milnor fiber* F? *Does the Milnor fiber parametrize vector bundles of a certain type*?

We point out that for the Brieskorn complete intersection singularities, the Milnor fiber F admits a natural compactification as a smooth projective algebraic variety X, where $D = X - F$ is a normal crossings divisor. Perhaps one should consider bundles on $X - D$ with certain conditions along D.

QUESTION 3.5 (Atiyah). *Is there a Milnor fiber description of the Floer homology of the link*? *For* $\Sigma(p, q, r)$, *is it related to the action of complex conjugation on the homology of the Milnor fiber of* $x^p + y^q + z^r = 1$?

§4. Another class of examples

The homology spheres which are links of surface singularities are classified in [E–N], but we do not know in general which of them are links of complete intersection (or just Gorenstein) singularities. The simplest case is the Seifert fibered case, which has been discussed (they *are* links of complete intersection singularities). The next simplest case is as follows: if p, q, r are relatively prime integers ≥ 1, as are p', q', r', then the homology sphere with splice diagram

(4.1) 

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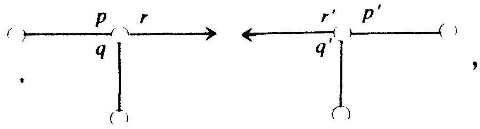
obtained by splicing



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is the link of a singularity if and only if $rr' > pp'qq'$. Only in a few cases do we know if this singularity can be chosen Gorenstein or complete intersection. We denote this homology sphere $\Sigma(p, q, r; p', q', r')$.

EXAMPLE 4.2. (cf. Proposition 2.1): If $r = q'$ then this homology sphere is the link of a hypersurface singularity $z^r = f(x, y)$, where $f(x, y) = 0$ is an irreducible plane curve singularity with two Newton-Puiseux pairs (p, q) and (p', r') (they automatically satisfy the Puiseux inequality $r' > pp'q$.)

By addivity of the Casson invariant under splicing one has

(4.3) $\lambda(\Sigma(p, q, r; p', q', r')) = \lambda(p, q, r) + \lambda(p', q', r')$,

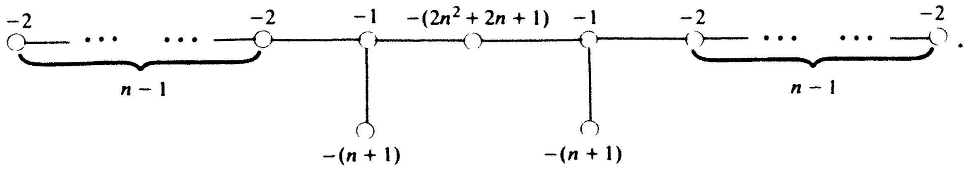
and the last two integers are computed as in section 1 via [F–S]. Even if one can find a Gorenstein singularity with this link, it is in general very difficult to compute p_g, hence the signature of the Milnor fiber F.

PROPOSITION 4.4. *For each integer* $n \geq 1$, *consider the complete intersection singularity* $(X_n, o) = (X, o)$ *in* $\mathbb{C}^4$ *defined by*

$$x^n = u^{n+1} + v^n y$$
$$y^n = v^{n+1} + u^n x.$$

Then

(i) (X, o) *has an isolated singularity at* o *with minimal good resolution dual graph*



(ii) *The link of* (X, o) *is a homology sphere of type*

$$\Sigma = \Sigma(n, n+1, n^2+n+1; n, n+1, n^2+n+1).$$

(iii) *The geometric genus of* X *is*

$$p_g = \tfrac{1}{12} n(n-1)^2(7n+4).$$

(iv) *The Conjecture is true for these examples*; *in fact*:

$$\tfrac{1}{8}\operatorname{sign}(F) = \lambda(\Sigma) = -\tfrac{1}{12}(n-1)n(n+1)(n+2).$$

Proof. A long direct argument shows the singularity is isolated. To resolve, we can project to the (u, v)-plane and blow up the discriminant curve; equivalently, we blow up $\mathbb{C}^4$ along the plane $u = v = 0$, take the proper transform, and normalize.

On the first patch we adjoin $v/u = w$, yielding equations

$$x^n = u^{n+1} + w^n u^n y$$
$$y^n = w^{n+1} u^{n+1} + u^n x.$$

Normalizing, we let $x' = x/u$, $y' = y/u$, obtaining

$$x'^n = u(1 + w^n y')$$
$$y'^n = y(x' + w^{n+1}).$$

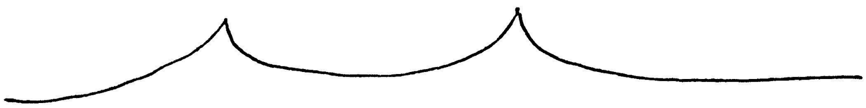
Since $1 + w^n y'$ is non-zero at every point above $0 \in \mathbb{C}^4$, we may invert it in a neighborhood and write $u = ax'^n$, where a is a unit; this gives

$$y'^n = ax'^n(x' + w^{n+1}).$$

Letting $z' = y'/x'$ and normalizing again, we obtain as proper transform of X the surface

$$z'^n = a(x' + w^{n+1}).$$

This is a non-singular surface, so (X, o) has been resolved on this patch. The exceptional fiber is given by $x' = 0$, hence is an irreducible curve with one singular point, analytically equivalent to $z^n = w^{n+1}$. Reversing the roles of u and v gives the second patch, so by symmetry we have a resolution $(\tilde{X}, E) \to (X, o)$, where E is an irreducible curve with two singular points as above.



By pulling back a function like $f = x + u$ from $\mathbb{C}^4$ to $\tilde{X}$, checking its zero-locus $(f)_0$, and noting that $(f)_0 \cdot E = 0$, we deduce that $E \cdot E = -1$. Resolving further the two singular points of E gives the minimal good resolution of (i). Statement (ii) then follows by the dictionary between splice diagrams and resolution diagrams [E–N].

Using the minimal good resolution (or the minimal resolution) one now readily calculates that

(4.5) $c_1^2(\tilde{X}) + c_2(\tilde{X}) = 2 - (2n^2 - 2n - 1)^2.$

The calculation of $p_g(x)$ will take some work. Consider the 1-parameter deformation $(\mathscr{X}, \sigma) \to T$ of (X, o) with fiber (X_t, o) given by

$$x^n = u^{n+1} + v^n y + tg(u, v)$$
$$y^n = v^{n+1} + u^n x + th(u, v),$$

where g, h are homogeneous polynomials of degree $n+1$ in u and v. We claim that for general g and h and small $|t|$, this family admits a simultaneous resolution, inducing a deformation of $(\tilde{X}, E)$ so that the general fiber has one smooth exceptional curve. Resolve as above by blowing up the $u=v=0$ plane in a family, obtaining equations

$$x'^n = u(1 + w^n y' + tg(1, w)) = uA$$
$$y'^n = y(x' + w^{n+1} + th(1, w)) = uB.$$

It is easy to see that for general g, h, there are only finitely many t for which both A and B vanish for some point with $x' = y' = 0$. If A is non-zero, as above one can localize appropriately, and write $u = \alpha x'^n$ for some unit α; normalizing gives the smooth surface

$$z'^n = \alpha(x' + w^{n+1} + th(1, w)).$$

Further, for general h, the exceptional curve $x' = 0$ is smooth for small non-zero $|t|$. This is the desired simultaneous resolution. It follows that $p_g(X) = p_g(X_t)$ for all t.

We next assert that (X_t, o) for $t \neq 0$ occurs among the non-negative weight deformations of the weighted homogeneous singularity (Y, o):

$$z_1^n = g(u, v)$$
$$z_2^n = h(u, v).$$

For, letting $z_1 = t^{-1/n}x$, $z_2 = t^{-1/n}y$, we may write X_t as

$$z_1^n = g(u, v) + t^{-1}u^{n+1} + t^{-1+1/n}v^n z_2$$
$$z_2^n = h(u, v) + t^{-1}v^{n+1} + t^{-1+1/n}u^n z_1.$$

The weights of the variables z_1, z_2, u, v in the equations for (Y, o) are $n+1$, $n+1$, n, n, respectively, and each equation has weight $n(n+1)$, which is less than or equal to the weights of the terms u^{n+1}, $v^n z_2$, v^{n+1}, and $u^n z_1$, added in the equations for X_t. In particular, $p_g(X_t) = p_g(Y)$. The latter may be computed for a weighted homogeneous complete intersection as follows. Let the variables have weights w_i $(i = 1, \ldots, 4)$ and the equations have weights $d_j (j = 1, 2)$. Let $s = \sum w_i - \sum d_j$ ($-s$ is the weight of a nowhere-zero holomorphic 2-form on $Y - \{o\}$). Let the graded ring of (Y, o) be $A = \oplus A_i$. Then

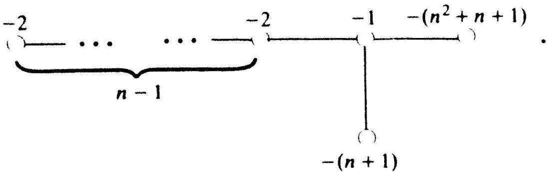
$$p_g(Y) = \sum_{0 \leq i \leq s} \dim A_i.$$

A monomial basis for A is given by $\{z_1^\alpha z_2^\beta u^\gamma v^\delta \mid \alpha, \beta \le n-1\}$; counting the number of monomials of weight $\le s$ is a lengthy but straightforward exercise and gives the formula for $p_g(Y)$, and hence $p_g(X)$, asserted in (iii). (One can also compute $p_g(Y)$ using the Greuel-Hamm formula for μ.)

Combining this expression for $p_g(X)$ and formula (4.5) for $c_1^2(\tilde{X}) + c_2(\tilde{X})$ with the signature formula (3.1) gives the signature expression in (iv) of the theorem. It remains to compute the Casson invariant of Σ. By (4.3) it is $2\lambda(n, n+1, n^2+n+1)$. One may compute $\lambda(n, n+1, n^2+n+1)$ by §1. Alternatively, it is $\frac{1}{8}$ the signature of the Milnor fiber of the Brieskorn hypersurface singularity

$$x^n + y^{n+1} + z^{n^2+n+1} = 0.$$

A resolution dual diagram is



This gives $c_1^2 + c_2 - 1 = -n(n-1)(n^2-n-2)$. Of course, $\mu = (n-1)n^2(n+1)$. Thus, by (3.1), the signature of the Milnor fiber is $-\frac{1}{3}(n-1)n(n+1)(n+2)$. Two times $\frac{1}{8}$ of this is then the Casson invariant of our original link, completing the proof. □

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