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On the nodal lines of second eigenfunctions of the fixed membrane problem

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Abstract. A well-known conjecture about the second eigenfunction of a bounded domain in $\mathbb{R}^2$ states that the nodal line has to intersect the boundary in exactly two points. We give sufficient conditions on the domain for this assertion to hold. For special doubly symmetric domains we also prove that λ_2 is simple and that the nodal line of the second eigenfunction lies on one of the axes.

1. Introduction

Consider the Dirichlet eigenvalue problem for the Laplacian on a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary of class $C^{2,\alpha}$:

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The set of eigenvalues can be arranged in a nondecreasing sequence of positive numbers tending to infinity $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots$.

The corresponding eigenfunctions $\{u_i\}_{i=1}^{\infty}$ are in $C^{2,\alpha}(\bar{\Omega})$ (see [3], Theorem 6.15) and analytic in the interior of Ω. If u is an eigenfunction $N(u) := \{x \in \Omega : u(x) = 0\}$ is called the *nodal set* of u; the connected components of $\Omega \backslash N(u)$ are called *nodal domains*. The Courant nodal domain theorem states that the i-th eigenfunction can possess at most i nodal domains. As a consequence of Courant's theorem, u_1 has exactly one and

u_2 has exactly two nodal domains. (1.2)

Cheng proved in [1] that, for any eigenfunction u, the nodal set $N(u)$ consists of a finite number of C^1-immersed arcs $\phi : (0, 1) \to \Omega$ or circles $\psi : S^1 \to \Omega$. When these arcs or circles intersect or self-intersect, they form an equiangular system. As a consequence of (1.2), we have:

If Ω is simply connected, $N(u_2)$ consists of one embedded arc or one embedded circle only. (1.3)

A conjecture on the configuration of $N(u_2)$ states (see, e.g. [5] or [7]) that the latter case in (1.3) cannot occur, and more precisely:

$$\text{If } \Omega \text{ is simply connected, then } \overline{N(u_2)} \text{ intersects } \partial\Omega \text{ in exactly two points.} \quad (1.4)$$

Hitherto, the conjecture has been proved only for some special classes of domains, all possessing an axial or rotational symmetry. L. Payne showed (1.4) if Ω is symmetric about the axis $\{x_2 = 0\}$ and convex in x_2. C. S. Lin [4] proved (1.4) provided Ω is convex and invariant under rotation by an angle $2\pi/m$ for some $m \geq 2$.

In the paper we are concerned with proving (1.4) under a whole continuum of possible conditions on the domain Ω. For that purpose we introduce the notion of convexity with respect to a point.

DEFINITION 1.1. Let $G \subset \mathbb{R}^2$ be a domain, $p \in \mathbb{R}^2$ a point. We call G convex with respect to p if for every circle C centered in p the intersection $C \cap G$ is either empty or connected.

We then show in Theorem 2.3: If Ω is symmetric about the axis $\{x_2 = 0\}$ and convex with respect to a point $p = a \cdot e_1$ on this axis, $p \notin \Omega$, then (1.4) holds. Payne's condition is then the limit case of our condition for $a \to \infty$ or $a \to -\infty$.

Closely related to the shape of the nodal line of second eigenfunction is the multiplicity m_2 of λ_2. It is known that $m_2 \leq 3$ for simply connected Ω (cf. [1]) and that (1.4) implies $m_2 \leq 2$ (cf. [4]). Also C. L. Shen, for the case of doubly symmetric plane domains and under the conditions

(i) $\bar{\Omega} = \{(x_1, x_2) : -a \leq x_1 \leq a,\ -f(|x_1|) \leq x_2 \leq f(|x_1|\}$,
(ii) $f \in C([0, a]), f > 0$ on $[0, a)$, $f(a) = 0$, f is strictly decreasing on [0, a],
(iii) $x^2 + (f(x))^2$ is strictly increasing on $[0, a]$,

has proved the following: λ_2 is simple and $N(u_2) = \Omega \cap \{x_2 = 0\}$. We show here the same under weaker geometric (but higher regularity) assumptions on the boundary of Ω, namely, Ω must be convex in x_2 and expand from $\{x_1 = 0\}$ to $\{x_2 = 0\}$ (see Definition 3.2).

2. Domains with an axial symmetry

We first need the following observation.

LEMMA 2.1. *Suppose that Ω is a domain in $\mathbb{R}^2$, $\lambda \in \mathbb{R}$ and that $u \in C^2(\Omega)\backslash\{0\}$ solves $\Delta u + \lambda u = 0$ in Ω. Let x_0 be a point in Ω with $u(x_0) = 0$.*

Then u changes sign near x_0, i.e. in each neighbourhood U of x_0, u assumes positive and negative values.

Proof. An easy consequence of the strong maximum principle for subharmonic functions and the fact that u actually is analytic in Ω.

For the domains under consideration we now reformulate the property of convexity with respect to a point. Set

$$H^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}, \qquad \bar{H}^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\}$$

and let $D^{90} \in O(2, \mathbb{R})$ be the rotation by 90 degrees in the positive sense.

LEMMA 2.2. *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded, simply connected domain of class C^1. Assume that Ω is symmetric about the axis $\{x_2 = 0\}$ and that $a \cdot e_1, a \in \mathbb{R}$, is a point on this axis. We then have:*

Ω is convex with respect to $a \cdot e_1$ iff

$$\begin{cases} \forall x \in \partial\Omega \cap \bar{H}^2 : \langle D^{90}(x - a \cdot e_1), \nu(x)\rangle \geq 0 \\ \text{or} \\ \forall x \in \partial\Omega \cap \bar{H}^2 : \langle D^{90}(x - a \cdot e_1), \nu(x)\rangle \leq 0. \end{cases} \tag{2.1}$$

Here, for $x \in \partial\Omega$, $\nu(x)$ is the outer normal to $\partial\Omega$ at x.

Proof. Since Ω is simply connected, we may parametrize $\partial\Omega \cap \bar{H}^2$ by a regular curve $c \in C^1([0, 1], \bar{H}^2)$. We orient c in such a way that $\nu(c(t)) = D^{90}\dot{c}(t)$ for all t. We then have

$$\frac{d}{dt}|c(t) - a \cdot e_1|^2 = 2\langle c(t) - a \cdot e_1, \dot{c}(t)\rangle = 2\langle D^{90}(c(t) - a \cdot e_1), \nu(c(t))\rangle.$$

Hence (2.1) is equivalent to $|c(t) - a \cdot e_1|$ being monotone on $[0, 1]$. This condition is violated if and only if there is an $r > 0$ such that, for $K(t) = r(\cos t, \sin t) + a \cdot e_1$, the set $\{K(t) : t \in [0, \pi]\} \setminus (\partial\Omega \cap \bar{H}^2)$ decomposes into at least three connected components. Since Ω is symmetric in the axis $\{x_2 = 0\}$, we have that $\{K(t), t \in [0, 2\pi)\} \setminus \partial\Omega$ decomposes into at least four connected components. This is equivalent to the condition that Ω is not convex with respect to $a \cdot e_1$.

Before proving the main result of this section we introduce the following terminology: For $x \in \partial\Omega$, we say that u is positive near x if there is an open ball B around x such that u is positive on $B \cap \Omega$. For $\Gamma \subset \partial\Omega$, we say that u is positive near Γ if u is positive near x for each $x \in \Gamma$.

THEOREM 2.3. *Suppose Ω is a bounded, simply connected domain in $\mathbb{R}^2$ of class $C^{2,\alpha}$. Suppose further that Ω is symmetric about the axis $\{x_2 = 0\}$ and convex with respect to a point $a \cdot e_1, a \in \mathbb{R}$. Then if $a \cdot e_1 \notin \Omega$,* (1.4) *holds.*

Proof. We proceed by contradiction.

Assume v is a solution of (1.1) with $\lambda = \lambda_2$, that is, a second eigenfunction, and $\overline{N(v)}$ intersects $\partial\Omega$ in at most one point p. We may then suppose that v is positive near $\partial\Omega \backslash \{p\}$.

Consider $u \in C^{2,\alpha}(\Omega)$ defined by

$$u(x_1, x_2) = \tfrac{1}{2}(v(x_1, x_2) + v(x_1, -x_2)). \tag{2.2}$$

Then u is also a second eigenfunction on Ω and we have:

$$u \text{ is positive near } \partial\Omega \text{ with the possible exception of two points.} \tag{2.3}$$

Define $u_\Theta \in C^{1,\alpha}(\Omega)$ by

$$u_\Theta(x) = -x_2\, \partial_1 u(x) + (x_1 - a)\, \partial_2 u(x) = \langle D^{90}(x - a \cdot e_1), \nabla u(x) \rangle.$$

Then we have $u_\Theta \not\equiv 0$, since otherwise, u would be rotationally symmetric around $a \cdot e_1$ and $\partial\Omega$ a circle with center $a \cdot e_1$, which is impossible because $a \cdot e_1 \notin \Omega$. Set $\Omega^+ = \{x \in \Omega : x_2 > 0\}$.

We now claim:

$$\forall x \in \Omega^+ : u_\Theta(x) \neq 0. \tag{2.4}$$

As the differential operators $\partial_\Theta = -x_2\, \partial_1 + (x_1 + a)\, \partial_2$ and Δ commute, we have

$$\Delta u_\Theta + \lambda_2 u_\Theta = 0 \quad \text{in } \Omega. \tag{2.5}$$

(2.2) implies that u is even in x_2, and so

$$u_\Theta(x_1, x_2) = -u_\Theta(x_1, -x_2) \quad \text{for all } x \in \Omega. \tag{2.6}$$

The condition $u = 0$ on $\partial\Omega$ implies

$$\nabla u(x) = \partial_\nu u(x) \cdot \nu(x) \quad \text{for all } x \in \partial\Omega, \tag{2.7}$$

and, by virtue of (2.3),

$$\partial_\nu u \leq 0 \quad \text{on } \partial\Omega. \tag{2.8}$$

Furthermore

$$\begin{aligned}\partial_\Theta u(x) &= \langle D^{90}(x - a \cdot e_1), \nabla u(x)\rangle \\ &= \partial_\nu u(x)\langle D^{90}(x - a \cdot e_1), \nu(x)\rangle \forall x \in \partial\Omega. \qquad (2.9)\end{aligned}$$

Since Ω is convex with respect to $a \cdot e_1$, we may assume by Lemma 2.2 that $\langle D^{90}(x - a \cdot e_1), \nu(x)\rangle \geq 0$ for all $x \in \partial\Omega \cap \bar{H}^2$.

Together with (2.8) and (2.9) this implies

$$\partial_\Theta u \leq 0 \quad \text{on } \partial\Omega \cap \bar{H}^2. \qquad (2.10)$$

Set $\Gamma_0 = \Omega \cap \{x_2 = 0\}$, $\Omega^- = \{x \in \Omega : x_2 < 0\}$.

From (2.6) we obtain

$$u_\Theta = 0 \quad \text{on } \Gamma_0. \qquad (2.11)$$

We now show (2.4). Assume $u_\Theta(x) = 0$ for an $x \in \Omega^+$. According to Lemma 2.1

$$u_\Theta \text{ changes sign near } x. \qquad (2.12)$$

Hence $V^+ := \{y \in \Omega^+ : u_\Theta(y) > 0\}$ is non-empty. (2.8) and (2.11) imply that $u_\Theta = 0$ on ∂V^+. Hence V^+ is the union of one or more nodal domains of u_Θ. By (2.6) $V^- := \{y \in \Omega^- : u_\Theta(y) < 0\}$ also contains one or more nodal domains of u_Θ. Thus $\lambda_2 = \lambda_2(\Omega) \geq \lambda_2(V^+ \cup V^-)$. (2.12) implies $\operatorname{int}(\Omega \backslash (V^+ \cup V^-)) \neq \emptyset$. The monotonicity principle for eigenvalues (see [2]) now yields that $\lambda_2(V^+ \cup V^-) > \lambda_2(\Omega)$, a contradiction, and (2.4) is proved.

To achieve a final contradiction, choose $x \in \Omega \cap \{x_2 \geq 0\}$ with $u(x) = 0$. Taking into account that u is a second eigenfunction and symmetric in x_2, such an x must exist. Set $r = |x - a \cdot e_1|$ and consider the curve $\sigma : [0, \pi) \to \mathbb{R}^2$, $\sigma(t) = r(\cos t, \sin t) + a \cdot e_1$. There exist $t_1, t_2 \in [0, \pi)$, $t_1 < t_2$, such that $\sigma((t_1, t_2)) \subset \Omega^+$, $\sigma(t_1) = x$, $\sigma(t_2) \in \partial\Omega$. Thus for $\varphi(t) = u(\sigma(t))$ we have $\varphi(t_1) = \varphi(t_2) = 0$. Hence there is a $t_0 \in (t_1, t_2)$ such that $\varphi'(t_0) = 0$, that is $u_\Theta(\varphi(t_0)) = 0$. This a contradiction to (2.4) and the theorem is proved.

3. Doubly symmetric domains

We first cite the following simple lemma of Lin (see [4]).

LEMMA 3.1. *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded C^1-domain and u a Dirichlet eigenfunction of Ω.*

Then $x \in \overline{N(u)} \cap \partial\Omega$ if and only if $\partial_\nu u(x) = 0$.

In this section we shall be considering doubly symmetric domains. A crucial assumption we shall be placing upon such domains is the property of expansion from one axis to the other.

DEFINITION 3.2. Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain of class C^1. Consider the axes $T_1 = \{x_1 = 0\}$, $T_2 = \{x_2 = 0\}$ and the quadrant $Q = \{x_1 > 0$ and $x_2 > 0\}$. Suppose Ω is symmetric in T_1 and T_2. We say that Ω *expands* from T_1 to T_2 if

$$\langle D^{90}x, \nu(x)\rangle \geq 0 \; \forall x \in \partial\Omega \cap \bar{Q}. \tag{3.1}$$

Remark. If we parametrize $\partial\Omega \cap \bar{Q}$ by a regular C^1-curve $c : [0, 1] \to \bar{Q}$ with $c(0) \in T_1$ and $c(1) \in T_2$, then (3.1) is equivalent to $d/dt|c(t)|^2 \geq 0$, which motivates the defintion.

In our investigation of the eigenspace of $\lambda_2(\Omega)$, we prove first:

THEOREM 3.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected $C^{2,\alpha}$-domain. Suppose Ω is symmetric in T_1 and T_2, expands from T_1 to T_2 and is not a circular disc. Set $\Omega_2^+ := \{x \in \Omega : x_2 > 0\}$. Then we have $\lambda_1(\Omega_2^+) > \lambda_2(\Omega)$.*

Proof. Suppose u is a first eigenfunction on Ω_2^+; we may assume $u > 0$ on Ω_2^+. We reflect u antisymmetrically along T_2 and obtain an eigenfunction on the whole of Ω which we call again u. Hence

$$\Delta u + \lambda_1(\Omega_2^+)u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{3.2}$$

$$u(x_1, x_2) = -u(x_1, -x_2) = u(-x_1, x_2) \; \forall x \in \Omega. \tag{3.3}$$

The second equality in (3.3) states that, as a first eigenfunction on Ω_2^+, u is even in x_1.

Set $u_\Theta = -x_2\, \partial_1 u + x_1\, \partial_2 u = \langle D^{90}x, \nabla u\rangle$ in Ω.

We have $u_\Theta \not\equiv 0$, otherwise Ω would be a circular disc. As in (2.5) we obtain

$$\Delta u_\Theta + \lambda_1(\Omega_2^+)u_\Theta = 0 \quad \text{in } \Omega. \tag{3.4}$$

Since u is positive near $\Gamma = \partial\Omega \cap Q$ we have $\partial_\nu u \leq 0$ on $\bar{\Gamma}$. By virtue of $\nabla u = \partial_\nu u \cdot \nu$ on $\partial\Omega$ and (3.1) we obtain

$$u_\Theta = \langle D^{90}x, \nabla u\rangle = \partial_\nu u\langle D^{90}x, \nu\rangle \leq 0 \quad \text{on } \bar{\Gamma}. \tag{3.5}$$

On account of (3.3) the following holds:

$$u_\Theta(x_1, x_2) = u_\Theta(x_1, -x_2) = -u_\Theta(-x_1, x_2). \tag{3.6}$$

Set

$$T = \Omega \cap T_1, \qquad \Gamma_1 = \partial\Omega \cap \{x_1 > 0\}.$$

(3.5) and (3.6) imply

$$u_\Theta = 0 \quad \text{on } T, \qquad u_\Theta \le 0 \quad \text{on } \Gamma_1. \tag{3.7}$$

Choose an r such that $\min_{x \in \partial\Omega} |x| < r < \max_{x \in \partial\Omega} |x|$, and consider the curve $\sigma : [0, \pi/2] \to \bar{Q}$, $\sigma(t) = r(\cos t, \sin t)$. There is a $t_1 \in (0, \pi/2)$ such that $\sigma((0, t_1)) \subset \Omega^* := \Omega \cap Q$, $\sigma(t_1) \in \partial\Omega$. Define $\varphi : [0, t_1] \to \mathbb{R}$, $\varphi(t) = u(\sigma(t))$. Since $\varphi(0) = \varphi(t_1) = 0$, there is a $t_0 \in (0, t_1)$ such that $\varphi'(t_0) = u_\Theta(\sigma(t_0)) = 0$. Put $\sigma(t_0) = x \in \Omega^*$. Lemma 2.1 implies that

$$u_\Theta \text{ changes sign near } x. \tag{3.8}$$

Set

$$\Omega_1^+ = \Omega \cap \{x_1 > 0\}, \qquad V^+ = \{y \in \Omega_1^+ : u_\Theta(y) > 0\}.$$

V^+ is non-empty by (3.8) and $u_\Theta = 0$ on ∂V^+ by (3.7). Hence V^+ is the union of one or more nodal domains of u_Θ. Since u_Θ is skew-symmetric in x_1, $V^- = \{(x_1, x_2) \in \Omega : (-x_1, x_2) \in V^+\}$ also contains at least one nodal domain of u_Θ. Hence u_Θ is at least a second Dirichlet eigenfunction on $V = V^+ \cup V^-$, that is $\lambda_1(\Omega_2^+) \ge \lambda_2(V)$. Again, by (3.8), we have $\operatorname{int}(\Omega \backslash V) \neq \varnothing$, and so, by domain monotonicity, $\lambda_2(V) > \lambda_2(\Omega)$, and $\lambda_1(\Omega_2^+) > \lambda_2(\Omega)$ is proved.

Put now $E = E(\lambda_2) =$ the eigenspace corresponding to $\lambda_2(\Omega)$. We proceed by decomposing E into subspaces according to the symmetry properties of the eigenfunctions.

Let $A_1, A_2 \in O(2, \mathbb{R})$ be the reflections in T_1 and T_2 respectively and define

$$E_1^+ = \{u \in E : u = u \circ A_1\}, \qquad E_1^- = \{u \in E : u = -u \circ A_1\},$$

$$E_2^+ = \{u \in E : u = u \circ A_2\}, \qquad E_2^- = \{u \in E : u = -u \circ A_2\},$$

$$E_{1,2} = E_1^+ \cap E_2^-, \qquad E_{2,1} = E_1^- \cap E_2^+, \qquad E_s = E_1^+ \cap E_2^+, \qquad E_p = E_1^- \cap E_2^-.$$

Since

$$E = E_1^+ \oplus E_1^- = E_2^+ \oplus E_2^- \quad \text{(direct sums)},$$

we have

$$E = E_s \oplus E_{1,2} \oplus E_{2,1} \oplus E_p \quad \text{(direct sum).}$$

An element of $E_p \backslash \{0\}$ would have at least four nodal domains; hence $E_p = \{0\}$ and the following decomposition holds:

$$E = E_s \oplus E_{1,2} \oplus E_{2,1}. \tag{3.9}$$

Each $u \in E_{1,2}$ is a Dirichlet eigenfunction on Ω_2^+, and hence $u \mid \Omega_2^+$ must be a first eigenfunction, otherwise u would have four nodal domains in Ω. An analogous argument holds for $E_{2,1}$. Thus

$$\dim E_{1,2} \le 1, \qquad \dim E_{2,1} \le 1. \tag{3.10}$$

THEOREM 3.4. *Suppose Ω satisfies the hypotheses of Theorem* 3.3 *and that in addition Ω is convex with respect to x_2. Then $\lambda_2(\Omega)$ is simple and $N(u_2) = \overline{\Omega \cap T_1}$.*

Proof. Assume there exists an eigenfunction $u \in E_s$. By Payne's result [5], $\overline{N(u)}$ intersects $\partial\Omega$ in exactly two points $x, y \in \partial\Omega$. By Lemma 3.1, we have $\partial_\nu u(x) = \partial_\nu u(y) = 0$. As $N(u)$ consists of one embedded arc only, $\partial_\nu u$ changes sign near x and y. But also $\partial_\nu u$ is symmetric with respect to T_1 and T_2. Hence x and y cannot lie on the axes and so there are four points on $\partial\Omega$ in which $\partial_\nu u$ vanishes. This is impossible, and so $E_s = \{0\}$ By Theorem 3.3, $E_{1,2} = \{0\}$ and we obtain $E = E_{2,1}$. Finally, $\dim E_{2,1} \le 1$ and $\dim E \ge 1$ ensure that $\lambda_2(\Omega)$ is simple.

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