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Autor(en): **Vogel, Pierre**

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Representation of links by braids: A new algorithm

PIERRE VOGEL

§0.1 Introduction

If σ is a braid with n components, the closure of σ, denoted $\hat{\sigma}$, is constructed by connecting the endpoints at the top level to the bottom endpoints with n standard curves. This procedure yields an oriented link $\hat{\sigma}$ having the same number of crossings as σ. A classical result of Alexander [1], [2], [3] states that every oriented link is isotopic to a closed braid $\hat{\sigma}$. In his proof Alexander modifies the diagram of an oriented link by a sequence of elementary operations to obtain a closed braid. During this transformation the "geometry of the picture" is completely changed. In many applications of Alexander's algorithms links with few crossings yield closed braid with a large number of crossings.

On the other hand many algebraic invariants of links are first defined on braids. If we wish to compute these invariants for a "small" link L, it will be very useful to have the following principle:

"A 'small' oriented link is isotopic to a 'small' closed braid".

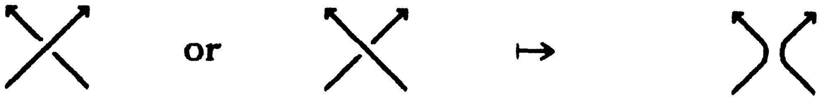
Unfortunately Alexander's proof cannot be used to check this principle. Recently Yamada [4] proposed another proof which is much more economical. He uses two types of elementary operations which don't change the number of Seifert circles and for which the change in the number of crossings is not too large.

In this paper we will give another proof which use only one type of elementary operation. This operation is very easy to describe. It preserves the number of Seifert circles and adds only two crossings to the diagram (by a type II Reidemeister move). Moreover there is an explicit (small) bound for the total number of operations. This procedure is more economical as Yamada's construction. It is much simpler and can be easily programmed on a computer.

§1. Description of the elementary operation

Let L be an oriented link in $\mathbb{R}^3$ represented by a regular projection D of L in the plane. Near each crossing x of D, the diagram D has one of the following form:

depending on the signe of x (positive in the first case and negative in the second one). By making the following transformation near all crossings of D:



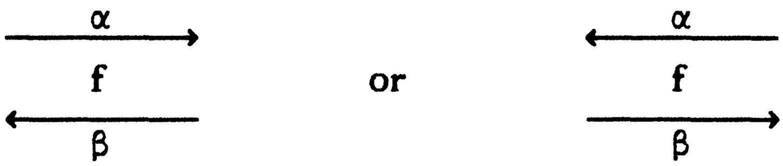
one obtains a new picture S which is a union of disjoint circles in the plane. These circles are called the Seifert circles of D and S will be called the Seifert picture of D.

Consider D as a graph in the plane. The vertices of D are the crossings and the edges are arcs in L (and in S). Let f be a face of D (i.e. a component of the complement of D in the plane) and α and β two edges of D contained in ∂f. Suppose that (f, α, β) satisfy the following conditions:

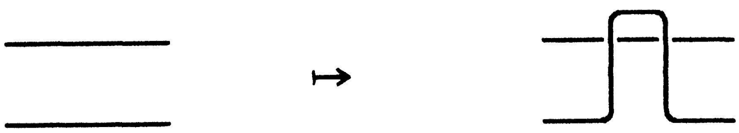
(i) α and β are contained in different Seifert circles
(ii) α and β have the same orientation with respect to any orientation of ∂f

Such a triple will be called an *admissible* triple.

So we have one of the following picture:



In this situation the elementary transformation $T(f, \alpha, \beta)$ will transform D by a type II Reidemeister elementary move as in the following picture:



THEOREM 1-1. *There exists a function χ from the set of isotopy classes of link diagrams to N, with the following properties*:

(i) *If D is a diagram of an oriented link, with n Seifert circles, then*:

$$2n+1 \leq \chi(D) \leq \frac{(n+1)(n+2)}{2}$$

Moreover, if D is connected then $\chi(D)$ is not less than $3n$.

(ii) *If D' is obtained from a diagram D by an elementary transformation T, then*:

$$\chi(D) < \chi(D')$$

(iii) *If D is a connected diagram with n Seifert circles such that*:

$$\chi(D) < \frac{(n+1)(n+2)}{2}$$

an elementary operation T can be performed on D.

(iv) *If D is a connected diagram with n Seifert circles, then the diagram D is isotopic in the Riemann sphere to the closure of a braid if and only if*:

$$\chi(D) = \frac{(n+1)(n+2)}{2}$$

REMARK 1-2. Let D be a connected diagram of a link L. To modify D in order to obtain a closed braid, it suffices to perform an elementary transformation $T(f, \alpha, \beta)$ each time we can find an admissible triple (f, α, β). When no operations are possible we have a diagram of a closed braid. If D has n Seifert circles and p crossings, the number of elementary operations we must do is at most

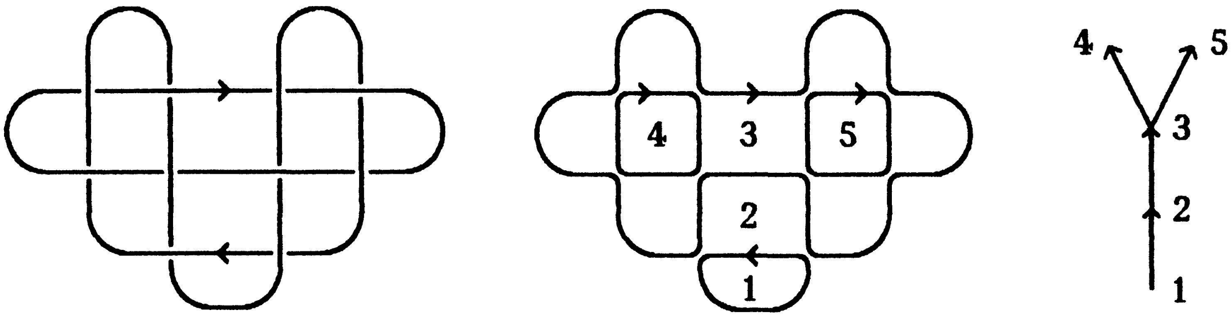
$$\frac{(n+1)(n+2)}{2} - 3n = \frac{(n-1)(n-2)}{2}$$

and we obtain a word in the braid group B_n of length at most $p + (n-1)(n-2)$. And, if the number $\chi(D)$ is greater than $3n$, the number of elementary transformations needed will be smaller.

§2. Construction of the map χ

Let D be the diagram of a link L. The oriented Seifert circles of D separate the plane into many components which we will call the faces of the Seifert picture S.

Moreover each oriented Seifert circle C bounds two faces of S: a face f_0 on the left hand side of C and a face f_1 on the right hand side of C. We construct an oriented graph Γ as follows: Each vertex of Γ corresponds to a face of S and each Seifert circle C represents an oriented edge from the vertex corresponding to f_0 to the vertex corresponding to f_1. The graph Γ is clearly a tree.



D S Γ

An oriented tree isomorphic to a subdivision of an oriented interval will be called a chain. A chain has n edges ($n \geq 0$) and $n+1$ vertices.

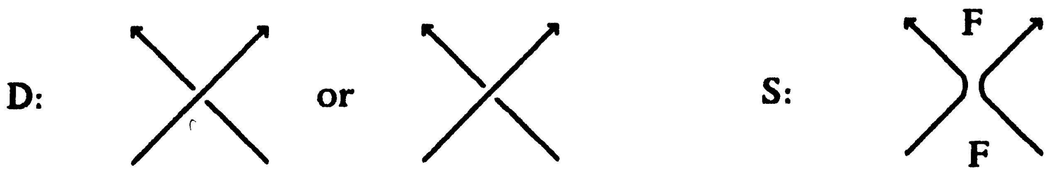
Define $\chi(\Gamma)$ to be the number of chains included in Γ and let $\chi(D) = \chi(\Gamma)$.

§3. Properties of χ

If the diagram D has n Seifert circles, the picture S has n circles and the tree Γ has n edges and $n+1$ vertices. Hence Γ contains $2n+1$ chains of length less than 2 and we have:

$$2n + 1 \leq \chi(D)$$

Now suppose that D is connected. Then the boundary of each face of D is connected. Let F be a face of S which is not a disk. The boundary of F is disconnected and F is not a face of D. Hence F is the connected sum of, at least, two faces of D and somewhere in the diagram we have the following picture:



Therefore there are two circles with opposite orientations in the boundary of F. Furthermore the picture S has this property for each face with disconnected boundary. This means that Γ satisfies the following property (P): each vertex of Γ which is not an isolated vertex (i.e. a vertex of valence 1) is contained in the interior of a chain.

LEMMA 3-1. *If Γ is a tree with n edges ($n \geq 1$) satisfying the property (P) then*:

$$\chi(\Gamma) \geq 3n$$

Proof. (Induction on n). The inequality is obvious if $n = 1$. Suppose $n \geq 2$ and Γ_1 is the tree obtained from Γ by removing all free edges and their isolated vertices. Suppose that Γ_1 is not a point and σ is a free edge of Γ_1 with a and n the vertices of σ and a is the isolated vertex of σ. By reversing the orientation of Γ, if necessary, we may assume that σ is oriented from a to b. In this situation Γ has p edges with terminal vertex a ($p \geq 0$) and $q + 1$ edges with initial vertex a ($q \geq 0$). Since Γ satisfies the property (P), we have:

$$q \neq 0 \Rightarrow p \neq 0$$

Now let Γ' be the tree obtained from Γ by removing the free edges of Γ containing a. It is easy to see that Γ' has $n - p - q$ edges and satisfies the property (P). On the other hand Γ has exactly $p + q$ vertices and $p + q$ edges not contained in Γ', and $p(q + 1)$ chains of length 2 passing through a. Therefore we have:

$$\chi(\Gamma) \geq \chi(\Gamma') + 2(p + q) + p(q + 1) \geq 3(n - p - q) + 3p + 2q + pq$$
$$\Rightarrow \chi(\Gamma) \geq 3n + (p - 1)q \geq 3n$$

If the graph Γ_1 consists of a single vertex a, Γ has p edges with terminal vertex a and q edges with initial vertex a. Because Γ satisfies the property (P), p and q are positive. We have:

$$\chi(\Gamma) = p + q + 1 + p + q + pq = 3n + (p - 1)(q - 1) \geq 3n$$

§4. The map χ and the operation T

LEMMA 4-1. *Let (f, α, β) be an admissible triple of a diagram D, Γ the tree associated to D and σ and τ be the edges of T corresponding to the Seifert circles containing a and b (respectively). Let u be the vertex of Γ corresponding to the face* *of S containing f. Denote by D' the diagram obtained from D by the operation $T(f, \alpha, \beta)$ and by Γ' the tree corresponding to D'.*

Then σ and τ have both u as initial or as terminal vertex and Γ' is obtained from Γ by identifying σ and τ and by adding a new free edge θ such that:

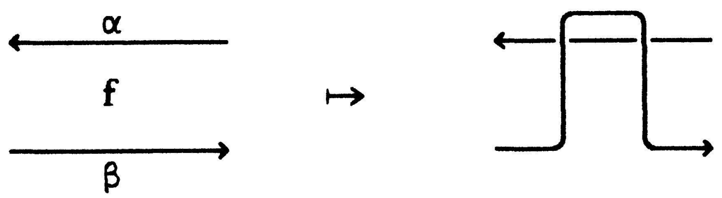
- *if σ and τ have u as initial vertex in Γ, the initial vertex of θ is the terminal vertex of both σ and τ.*
- *if σ and τ have u as terminal vertex in Γ, the terminal vertex of θ is the initial vertex of both σ and τ.*

REMARK 4-2. This elementary operation on Γ depends only on σ and τ and can be defined on every tree. The only condition on σ and τ is the following:

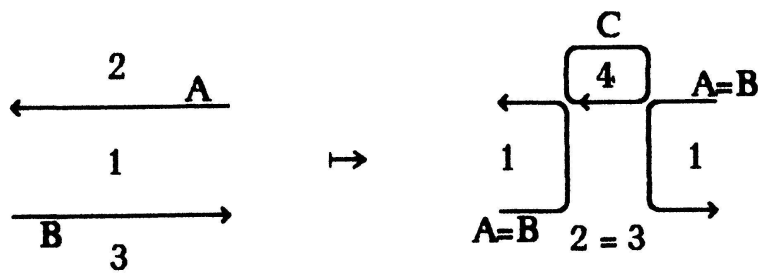
$\sigma \cap \tau \neq \varnothing$ and $\sigma \cup \tau$ is not a chain

Such an operation will be denoted by $T(\sigma, \tau)$.

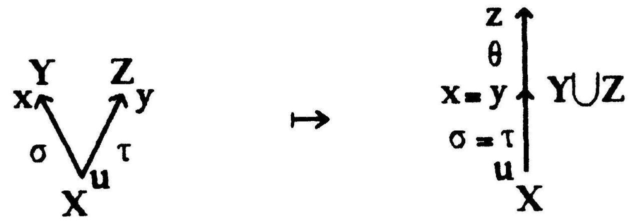
Proof of 4-1. Let (f, α, β) be an admissible triple of D. By reversing the orientation of D and Γ, if necessary, we may assume that the orientations of α and β are compatible with the orientation of ∂f. So the transformation $T(f, \alpha, \beta)$ is as follows:



The transformation $T(f, \alpha, \beta)$ modifies the Seifert picture S in the following way:



Since α and β are not in the same Seifert circles A and B, the new picture S' has the same number of components. Moreover the circles A and B and faces 2 and 3 become the same circle and the same face. However, we create a new circle C and a new face 4. The corresponding transformation on the tree Γ is:



In the picture X, Y and Z are subtrees of Γ and u, x, y and z are vertices corresponding to faces 1, 2, 3 and 4. The tree Γ is the union of X, Y, Z, σ and τ and the tree Γ' is the union of X, Y, Z, $\sigma = \tau$ and θ. In this manner, we have constructed the equivalent transformation $T(\sigma, \tau)$ on Γ.

LEMMA 4-3. *If a tree Γ' is obtained from a tree Γ by a transformation $T(\sigma, \tau)$, $\chi(\Gamma')$ is greater than $\chi(\Gamma)$.*

Proof. Using the preceding notations with u the common vertex of σ and τ and x and y the other vertices of σ and τ (in Γ). The extra edge in Γ' is θ with isolated vertex z and the images of x and σ by the obvious map $\varphi : \Gamma \to \Gamma'$ will be denoted by x' and σ' respectively. Let C and C' be the set of chains contained in Γ and Γ'. As usual we may suppose that u is the initial vertex of σ and τ. We have a map ψ from C to C' defined by:

$$\psi(U) = \begin{cases} \varphi(U) \cup \theta & \text{if } y \text{ is the terminal vertex of } U \\ \varphi(U) & \text{otherwise} \end{cases}$$

It is easy to see that ψ is injective and that $\{z\}$ is not in the image of ψ. Therefore $\chi(\Gamma')$ is greater than $\chi(\Gamma)$.

REMARK 4-4. Suppose that the trees Y and Z have no edges with terminal vertex x or y. In this case $\chi(\Gamma') = \chi(\Gamma) + 1$. In particular this occurs if Γ satisfies:

(P1)—every subtree of Γ with only two isolated vertices is the union of one or two chains.

Moreover if Γ satisfies (P1) then so does Γ'.

COROLLARY 4-5. *If Γ is a tree with n edges, then:*

$$\chi(\Gamma) \leq \frac{(n+1)(n+2)}{2}$$

Equality holds if and only if Γ is a chain.

Proof. If Γ is not a chain, an elementary operation $T(\sigma, \tau)$ is possible on Γ. We obtain a sequence of trees Γ_p such that $\Gamma_0 = \Gamma$ and each Γ_{p+1} is obtained from Γ_p by an elementary operation T. This procedure stops at the pth stage if Γ_p is a chain. Therefore

$$\chi(\Gamma) < \chi(\Gamma_p) = \frac{(n+1)(n+2)}{2}$$

with equality if Γ is a chain.

§5. End of the proof

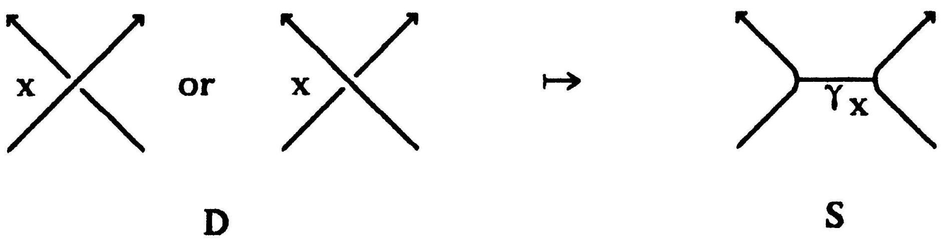
This section proves parts (iii) and (iv) of the main theorem.

5-1. *Proof of* (*iii*)

Let D be the diagram of a link L. The diagram is assumed to be connected with n Seifert circles. If $\chi(D)$ is less than $((n+1)(n+2))/2$, then the associated tree Γ is not a chain and Γ has two edges σ and τ with a common initial or terminal vertex u. Equivalently the Seifert picture S has a face F and two Seifert circles in ∂F with the orientations agreeing with the induced orientation of ∂F.

Let C be a Seifert circle in ∂F. This circle will be called positive (or negative) when the orientation of C is compatible (not compatible) with the orientation of ∂F. Denote by p (resp. q) the number of positives (resp. negatives) Seifert circles in ∂F. By assumption either p or q is greater than 1. Up to a change of orientation of D, we may assume that $p > 1$.

Let x be a crossing of D. Denote by γ_x a line segment near the crossing joining the two Seifert circles as in the following picture:



These line segments are all disjoint and each segment joins, in a face of S, two Seifert circles with opposite orientations. Let K be the set of line segments γ_x contained in the face F. If we cut F along these segments we get a subspace $\hat{F}$ of F with many components, each corresponding to a face of D.

Suppose that, for each such face f, there is no admissible triple (f, α, β). That means that each component f of $\hat{F}$ meets exactly one positive and one negative Seifert circle in ∂F. Let f_+ be the positive Seifert circle meeting f. If two components f and f' intersect, the associated Seifert circles f_+ and f'_+ are the same. Therefore the map $f \mapsto f_+$ is locally constant, hence constant. This means that ∂F has only one positive Seifert circle contradicting the assumption $p > 1$.

Therefore an admissible triple exists and an elementary operation T can be performed on D.

5-2. *Proof of* (*iv*)

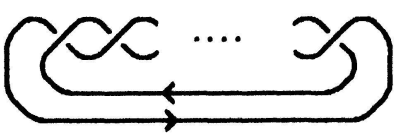
If $\chi(D) = ((n+1)(n+2))/2$ the tree Γ is a chain and the Seifert picture S is, up to an isotopy in the sphere, the union of n circles with standard orientation and same center. With a second isotopy, we may also assume that each segment γ_x is contained in a radius. Now each edge of the diagram D is transverse to every radius and D is the closure of a braid.

REMARK 5-3. The number of elementary operations T needed to transform a diagram D in a closure of a braid is not completely clear. It depends on the sequence of admissible triples. The same situation holds for a tree. Another problem is the fact that not every elementary operation on the tree Γ can be lifted to an operation on the diagram D. The only result we can verify is the following: for every elementary operation $T(\sigma, \tau)$ on Γ there exists an elementary operation $T(\sigma', \tau')$ corresponding to the same vertex of Γ and the same direction of edges of Γ as $T(\sigma, \tau)$ which lifts in an elementary operation $T(f, \alpha, \beta)$.

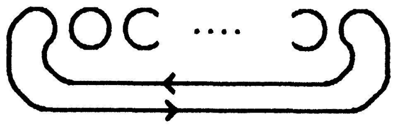
An interesting case is the following: suppose that the associated tree Γ of a link diagram D satisfies property (P1) (see Remark 4-4). Then each transformation T produces a new diagram D' with the following properties:

- the associated tree Γ' of D' satisfies the property (P1)
- $\chi(D') = \chi(D) + 1$

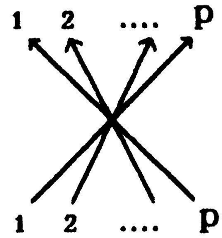
Therefore, if D has n Seifert circles, the number of transformations needed to transform D into a closed braid is exactly $((n+1)(n+2))/2 - \chi(D)$. An example of such a diagram is:



The number of crossings is $n = 2p$ and the number of Seifert circles is also $2p$. The Seifert picture is:



with p positive circles and p negative circles. The graph Γ is:



The number $\chi(\Gamma)$ is exactly $1 + 4p + p^2$. Then, after exactly $p^2 - p$ elementary transformations, we get a closed braid with $2p^2 = n^2/2$ crossings.

REFERENCES

[1] ALEXANDER, J. W., *A lemma on a system of knotted curves.* Proc. Nat. Acad. Sci. USA. *9* (1923), 93–95.

[2] BIRMAN, J. S., *Braids, links and mapping class group.* Annals of Math. Studies n° *82* Princeton Univ. Press. N.J. (1976).

[3] MORTON, H., *Threading knot diagrams.* Math. Proc. Camb. Phil. Soc. *99* (1986), 247–260.

[4] YAMADA, S., *The minimal number of Seifert circles equals the braid index of a link.* Invent. Math. *89* (1987), 347–356.

Université de Nantes
Département de Mathématiques
2 rue de la Houssinière
44072 Nantes Cedex 03

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