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Polynomial algebras over the Steenrod algebra

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0. Introduction

The most significant recent developments in homotopy theory have been the proofs of the Segal Conjecture and Sullivan Conjecture by G. Carlsson [C] and H. Miller [M] respectively. A key step in the proof of each of these theorems was a splitting, first proved by Carlsson in the case $p = 2$, which Miller observed is equivalent to the statement that H, the mod p cohomology of $B(\mathbb{Z}/p\mathbb{Z})$, is injective in the category $\mathcal{U}$ of unstable modules over the mod p Steenrod algebra. This was later generalized by J. Lannes and S. Zarati [LZ], who showed that in this category, I injective implies that $I \otimes H$ is injective. Thus $H^{\otimes s}$ is injective in $\mathcal{U}$ for each s. Then J. Lannes and L. Schwartz [LS] showed that the collection of indecomposable injectives in $\mathcal{U}$ consists of all modules of the form $J(n) \otimes Q$, where Q is an indecomposable summand of $H^{\otimes s}$, and $J(n)$ is one of the modules described in Section 2.

J. Harris, N. Kuhn, S. Mitchell, and S. Priddy have studied the problem of finding a stable decomposition of the classifying space BG of a finite group G. The case $G = (\mathbb{Z}/p\mathbb{Z})^s$ was solved by Harris and Kuhn [HK] who gave a method for finding the indecomposable summands and showed that such decompositions correspond to decompositions of $H^{\otimes s}$ in $\mathcal{U}$.

In theory the results of Lannes and Schwartz combined with those of Harris and Kuhn give a classification of the indecomposable $\mathcal{U}$-injectives, although it is impractical to carry out the computations involved in the Harris–Kuhn method for s greater than 4 or 5. One of the purposes of this paper is to give a common framework for consideration of both the injectivity of $H^{\otimes s}$ and its decomposition.

The bulk of this paper deals with the case $p = 2$. The modifications required for odd primes are considered in Section 3. Let A denote the mod 2 Steenrod algebra. In Section 1 we consider the A-algebra $\mathbb{F}_2[x_0, x_1, \ldots, x_{s-1}]$ with the twisted action given by $Sq^1 x_j = x_{j-1}^2$ for $j > 0$, and $Sq^1 x_0 = x_{s-1}^2$. Somewhat surprisingly this A-algebra turns out to be isomorphic as an A-module to $\mathbb{F}_2[t_0, t_1, \ldots, t_{s-1}]$ with the standard action. By means of this redescription of the A-module

$H^{\otimes s} \cong \mathbb{F}_2[t_0, t_1, \ldots, t_{s-1}]$ we give a splitting theorem which simultaneously demonstrates the injectivity of $H^{\otimes s}$ while producing A-module decompositions of it. We do not obtain a complete decomposition of $H^{\otimes s}$ into indecomposable summands, but from certain points of view our summands, although further decomposable, are more interesting than the indecomposable summands. At least they are relatively straightforward to describe. Initially we find $2^s - 1$ summands of $H^{\otimes s}$ labelled $M_s(0), \ldots, M_s(2_s - 2)$. $M_s(0)$ is an A-algebra which can be described as the invariants of $H^{\otimes s}$ under an action of the group $\mathbb{Z}/(2^s - 1)\mathbb{Z}$. The decomposition gives $H^{\otimes s}$ the structure of an augmented graded $M_s(0)$-algebra. According to W. Henn and L. Schwartz [HS] the indecomposable summands of $H^{\otimes s}$ are A-algebras only in a few isolated cases. The splitting theorem mentioned above can be applied to give a further decomposition of the modules $M_s(n)$, although not into indecomposable pieces. Our methods also exhibit the injective modules $K(n)$ used by Carlsson, Miller, and Lannes–Zarati (see Section 2) as direct limits of modules $M_s(n)$.

One of the interesting facets of the proof that $H^{\otimes s}$ is isomorphic to $\mathbb{F}_2[x_0, x_1, \ldots, x_{s-1}]$ with the twisted action is that it takes place over $\mathbb{F}_{2^s}$, although the statement is over $\mathbb{F}_2$. In fact after tensoring with $\mathbb{F}_{2^s}$, the two sides become isomorphic not just as A-modules but as A-algebras. Lannes and Zarati define an unstable ρ-A-algebra to be an A-algebra, unstable as an A-module, together with a map ρ such that $Sq^{|x|}x = \rho(x)^2 = (\rho x)^2$. Our polynomial algebra with twisted action is thus an unstable ρ-A-algebra. However the initially mysterious ρ becomes much more natural when one passes to $\mathbb{F}_{2^s}$-coefficients—it becomes the inverse of the Frobenius automorphism.

The authors would like to give special thanks to Bill Singer, whose unique insights led him to suggest to us that we study the twisted action described above. Also we would like to thank Ian Hughes and Joe Repka for helpful conversations concerning this work. Additionally we would like to thank John Harris and Tom Hunter for their comments and suggestions on preliminary versions of the manuscript.

Added in proof

Since the first draft of this paper was submitted, J. Harris, T. Hunter, and J. Shank have applied and expanded upon the ideas in this paper. The reader is referred to [HHS], [H], and [S] for details.

1. A twisted Steenrod action on polynomial algebras

Let $\phi(a) = a^2$ denote the Frobenius automorphism of $\mathbb{F}_{2^s}$. It is a generator of the Galois group Gal $(\mathbb{F}_{2^s}/\mathbb{F}_2)$. Extend ϕ to an automorphism of the polynomial algebra

$\mathbb{F}_{2^s}[t_0, t_1, \ldots, t_{s-1}]$ by setting its restriction to $\mathbb{F}_2[t_0, t_1, \ldots, t_{s-1}]$ to be the identity. By the primitive normal basis theorem (see Davenport [D]) it is possible to choose an element $\omega \in \mathbb{F}_{2^s}$ so that ω generates the cyclic multiplicative group of units in $\mathbb{F}_{2^s}$ and $\{\omega, \phi(\omega), \ldots, \phi^{s-1}(\omega)\}$ forms a basis for $\mathbb{F}_{2^s}$ over $\mathbb{F}_2$. Let $p(x) = a_0 + a_1 x + \cdots + a_{s-1}x^{s-1} + x^2$ be the minimum polynomial of ω. Let

$$T = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{s-1} \end{bmatrix}$$

be the $s \times s$ matrix over $\mathbb{F}_2$ representing multiplication by ω in the basis $\{1, \omega, \ldots, \omega^{s-1}\}$. Since ω is a generator of $\mathbb{F}_{2^s}^*$, we see that T has order $2^s - 1$ in $Gl_n(\mathbb{F}_2)$. Let L denote the linear subspace of $\mathbb{F}_{2^s}[t_0, t_1, \ldots, t_{s-1}]$. Regard T as a linear transformation on L using $\{t_0, t_1, \ldots, t_{s-1}\}$ as basis. Extend this map multiplicatively to a self-map of $\mathbb{F}_2[t_0, t_1, \ldots, t_{s-1}]$. Since the characteristic polynomial of T is $p(x)$, ω becomes an eigenvalue of T over $\mathbb{F}_{2^s}$. Let $x_0 \in L$ be a nonzero eigenvector corresponding to the eigenvalue ω. Set $x_j = \phi^j x_0$ for $j = 0, 1, \ldots, s-1$. The entries of T lie in $\mathbb{F}_2$ so T commutes with ϕ. Therefore $Tx_j = \omega^{2^j} x_j$. So $\{x_0, x_1, \ldots, x_{s-1}\}$ is a basis for L consisting of eigenvectors of T. Extending multiplicatively the map taking one basis for L to the other given an isomorphism of polynomial algebras

$$B : \mathbb{F}_{2^s}[t_0, t_1, \ldots, t_{s-1}] \cong \mathbb{F}_{2^s}[x_0, x_1, \ldots, x_{s-1}].$$

Define an A-algebra structure on $\mathbb{F}_2[t_0, t_1, \ldots, t_{s-1}]$ by regarding it as $H^*((\mathbb{Z}/2\mathbb{Z})^s; \mathbb{F}_2)$. Extend the action to $\mathbb{F}_{2^s}$ $[t_0, t_1, \ldots, t_{s-1}]$ by requiring that the action of each element of A be $\mathbb{F}_{2^s}$-linear. Next define an A-algebra structure on $\mathbb{F}_{2^s}[x_0, x_1, \ldots, x_{s-1}]$ by means of the isomorphism B. In this way $\mathbb{F}_{2^s}[t_0, t_1, \ldots, t_{s-1}] \cong \mathbb{F}_{2^s}[x_0, x_1, \ldots, x_{s-1}]$ becomes an A-algebra which is unstable as an A-module. Observe that the action of each element of A on $\mathbb{F}_{2^s}[t_0, t_1, \ldots, t_{s-1}]$ commutes with both ϕ and with multiplication by ω which are thus each A-module homomorphisms. Also observe that if x belongs to L then $Sq^1\phi x = x^2$. So $Sq^1 x_j = x_{j-1}^2$ for $j = 0, 1, \ldots, s-1$, where we conventionally let $x_j = x_{j'}$ when $j \equiv j' \pmod{s}$. In particular $\mathbb{F}_2[x_0, x_1, \ldots, x_{s-1}]$ is a sub-A-algebra of $\mathbb{F}_{2^s}[x_0, x_1, \ldots, x_{s-1}]$.

THEOREM 1. $\mathbb{F}_2[x_0, x_1, \ldots, x_{s-1}] \cong \mathbb{F}_2[t_0, t_1, \ldots, t_{s-1}]$ *as A modules.*

Proof. Define $\lambda : \mathbb{F}_{2^s}[t_0, t_1, \ldots, t_{s-1}] \to \mathbb{F}_{2^s}[t_0, t_1, \ldots, t_{s-1}]$ by $\lambda(x) = \mathrm{tr}(\omega x) \equiv \omega x + \phi(\omega x) + \cdots + \phi^{s-1}(\omega x)$. By earlier observations, λ is an A-module homomorphism. Since $\phi(\lambda x) = \lambda x$, λx is invariant under the action of $\mathrm{Gal}(\mathbb{F}_{2^s}/\mathbb{F}_2)$ on $\mathbb{F}_{2^s}[t_0, t_1, \ldots, t_{s-1}]$. So $\mathrm{Im}\,\lambda \subset \mathbb{F}_2[t_0, t_1, \ldots, t_{s-1}]$. Define θ to be the composite

$$\mathbb{F}_2[x_0, \ldots, x_{s-1}] \hookrightarrow \mathbb{F}_{2^s}[x_0, \ldots, x_{s-1}] \xrightarrow{B^{-1}} \mathbb{F}_{2^s}[t_0, \ldots, t_{s-1}] \xrightarrow{\lambda} \mathbb{F}_2[t_0, \ldots, t_{s-1}].$$

Since $\{\omega, \phi(\omega), \ldots, \phi^{s-1}(\omega)\}$ forms a basis for $\mathbb{F}_{2^s}$ over $\mathbb{F}_2$, θ is a monomorphism. However the Euler–Poincaré series of the two sides are identical, so this implies that θ is an isomorphism. The inverse to θ can be described explicitly as follows. For $x \in \mathbb{F}_2[t_0, t_1, \ldots, t_{s-1}]$, write $B(x) = \omega q_1 + \phi(\omega)q_2 + \cdots + \phi^{s-1}(\omega)q_s$, with q_j in $\mathbb{F}_2[x_0, x_1, \ldots, x_{s-1}]$, and set $\psi(x) = q_1$. The fact that x is invariant under the action of $\mathrm{Gal}(\mathbb{F}_{2^s}/\mathbb{F}_2)$ on $\mathbb{F}_{2^s}[t_0, t_1, \ldots, t_{s-1}]$ shows that $q_j = \phi^{j-1}(q_1)$ so $B(x) = \mathrm{tr}(\omega q_1)$. Thus $\psi = \theta^{-1}$. □

Tom Hunter has asked the following related question. Let M be a matrix representing an element of $\mathrm{End}(L)$. For $x \in L$, set $Sq^1 x = (Mx)^2$. Does this yield an action of the Steenrod algebra on $\mathbb{F}_2[x_0, x_1, \ldots, x_{s-1}]$, and if so, is the resulting action isomorphic to the standard one (coming from the identity) in the case where M is invertible? Using the preceding theorem and change of basis it is easy to see that this is true if M is conjugate to a permutation matrix. Joe Repka has shown us that if M is invertible then there exists N such that $M \otimes N$ is conjugate to a permutation matrix. This can be used to show that we do get an action of the Steenrod algebra in this case, but we do not know if it is isomorphic to the standard one.

2. A-module splittings and injectivity of polynomial algebras

In this section we prove the splitting theorem referred to in the introduction. We begin by recalling some of the history and setting notation.

Let $\mathscr{U}$ be the category of unstable left A-modules with degree raising action and let $\mathscr{U}_*$ be the category of unstable right A-modules (or equivalently left A^{opp}-modules) with degree lowering action. Let $G(n)$ be the free $\mathscr{U}_*$-object generated by ι_n in degree n. Thus $\mathrm{Hom}_{\mathscr{U}_*}(G(n), M) = M_n$. Let $J(n) = G(n)^*$ in $\mathscr{U}$. Let $J = \bigoplus_{n \geq 0} J(n)$. If $i + j = n$, define $G(n) \to G(i) \otimes G(j)$ by sending ι_n to $\iota_i \otimes \iota_j$. Dualizing gives maps $J(i) \otimes J(j) \to J(i + j)$ which turn J into an A-algebra which is unstable as an A-module.

THEOREM 2 (Miller). *$J \cong \mathbb{F}_2[\{x_j\}_{j \ge 0}]$ with $x_j \in J(2^j)$ and*

$$Sq^1 x_j = \begin{cases} x_{j-1}^2 & \text{if } j > 0 \\ 0 & \text{if } j = 0 \end{cases}. \qquad \square$$

Define $G(n) \to G(2n)$ by sending ι_n to $Sq_*^n \iota_{2n}$. Let

$$G_n = \varinjlim_k G(n) \to G(2n) \to \cdots \to G(2^k n) \to \cdots$$

So $G_n \cong G_{2n}$. For $0 \le n = m/(2^r) \in \mathbb{Z}[\frac{1}{2}]$ define $G_n = G_m$, and set $G_n = 0$ for $n < 0$. Let $K(n) = G_n^*$. For all $n \in \mathbb{Z}[\frac{1}{2}]$, G_n is projective in $\mathcal{U}_*$ so $K(n)$ is injective in $\mathcal{U}$. Let $K = \bigoplus_{n \in \mathbb{Z}[\frac{1}{2}]} K(n)$. The maps $G(i+j) \to G(i) \otimes G(j)$ induce maps $G_{i+j} \to G_i \otimes G_j$ for i and j in $\mathbb{Z}[\frac{1}{2}]$. Dualizing gives maps $K(i) \otimes K(j) \to K(i+j)$ which turn K into an A-algebra which unstable as an A-module.

THEOREM 3 (Lannes–Zarati). *$K \cong \mathbb{F}_2[\{x_j\}_{j \in \mathbb{Z}}]$ with $x_j \in K(2^j)$ and $Sq^1 x_j = x_{j-1}^2$ for all j.* $\square$

To define the maps used in our splittings it is convenient to enlarge the algebra K to an algebra M which we now describe.

Define the weight $w(m) \in \mathbb{Z}[\frac{1}{2}]$ of a monomial m in K by $w(x_j) = 2^j$ and $w(yz) = w(y) + w(z)$. So $w(m) = j$ if and only if $m \in K(j)$.

For $n \ge 0$ in $\mathbb{Z}$ let $\alpha(n)$ be the number of 1's in the diadic expansion of n. Noting that $\alpha(n) = \alpha(2n)$, extend the domain of α to $\mathbb{Z}[\frac{1}{2}]$ by setting $\alpha(m/(2^r)) = \alpha(m)$. Observe that $K^n(j) = 0$ unless $\alpha(j) \le n$. Let

$$M = \prod_{j \in \mathbb{Z}[\frac{1}{2}]} K(j),$$

where the product is taken in $\mathcal{U}$. Define an algebra structure on M as follows.

$$M \otimes M = \bigoplus_{n \in \mathbb{Z}} (M \otimes M)^n = \bigoplus_{p,q \in \mathbb{Z}} (M^p \otimes M^q),$$

so to describe $\mu : M \otimes M \to M$ we must define $\mu^{p,q}(j) : M^p \otimes M^q \to K^{p+q}(j)$ for each $p, q \in \mathbb{Z}$ and $j \in \mathbb{Z}[\frac{1}{2}]$. Let $\mu^{p,q}(j)$ be the map induced from the bilinear map $M^p \times M^q \to K^{p+q}(j)$ taking $((a_s)_{s \in \mathbb{Z}[\frac{1}{2}]}, (b_t)_{t \in \mathbb{Z}[\frac{1}{2}]})$ to $\Sigma_{s+t=j} a_s b_t$, where $a_s \in K^p(s)$ and $b_t \in K^q(t)$. This makes sense because although $(a_s)_{s \in \mathbb{Z}[\frac{1}{2}]}$ and $(b_t)_{t \in \mathbb{Z}[\frac{1}{2}]}$ may have infinitely many nonzero terms, the sum $\Sigma_{s+t=j} a_s b_t$ is finite. This is because for given p, q, and j there are only finitely many pairs $s, t \in \mathbb{Z}[\frac{1}{2}]$ such that $s + t = j$ and $\alpha(s) \le p$, $\alpha(t) \le q$. Thus M becomes an A-algebra which is unstable as an A-module.

Where convenient we will use the following alternate notations.

$$M_s \equiv \begin{cases} \mathbb{F}_2[x_0, x_1, \ldots, x_{s-1}] & \text{if } s < \infty \\ M & \text{if } s = \infty \end{cases}, \quad K_s \equiv \begin{cases} \mathbb{F}_2[x_0, x_1, \ldots, x_{s-1}] & \text{if } s < \infty \\ K & \text{if } s = \infty \end{cases},$$

where the Steenrod action on $\mathbb{F}_2[x_0, x_1, \ldots, x_{s-1}]$ is the twisted one described in Section 1. We also set $M_\infty(j) \equiv K(j)$, and write $\Sigma_{t \in \mathbb{Z}[\frac{1}{2}]} a_t$ for $(a_t)_{t \in \mathbb{Z}[\frac{1}{2}]}$ in M.

For $s < \infty$ let G be $\mathbb{Z}/(2^s - 1)\mathbb{Z}$. Define weight $w(m) \in G$ for monomials m in M_s by $w(x_j) = 2^j$ and $w(yz) = w(y) + w(z)$. Let $M_s(j)$ be the subspace of M_s having the monomials of weight j as basis. Observe that Sq^1 and thus all Steenrod operations preserve weight. So $M_s(j)$ is an A-module for each j and $M_s(0)$ is an A-algebra. The vector space decomposition $M_s = \bigoplus_{j \in G} M_s(j)$ is thus a decomposition of M_s as A-modules and exhibits an augmented graded $M_s(0)$-algebra structure on M_s. The map sending x_j to x_{j+1} induces A-module isomorphisms $M_s(k) \cong M_s(2k)$ for all k. Returning to the original description of the x_j's as eigenvectors of the matrix T in Section 1 shows that after tensoring with $\mathbb{F}_{2^s}$ this decomposition becomes the decomposition of $\mathbb{F}_{2^s}[t_0, t_1, \ldots, t_{s-1}]$ into eigenspaces of T. In particular $M_s(0)$ is the algebra of invariants of $\mathbb{F}_2[t_0, t_1, \ldots, t_{s-1}]$ under the group G acting by means of T. This allows $M_s(0)$ to be described without reference to the isomorphism of $H^{\otimes s}$ with M_s. The algebra $M_s(0)$ also appears in one of Carlsson's papers [C1]. Although the descriptions of the summands differs considerably, the above decomposition of $H^{\otimes s} \cong M_s$ bears a strong resemblance to one obtained by C. Witten [W] and it would not surprise us to learn that they were isomorphic. We have not attempted to verify this however*. $M_s(0)$ can also be described as the cohomology of a group as follows. Let G act on the additive group $\mathbb{F}_{2^s}$ by letting the generator act as multiplication by ω and form the semidirect product $\mathbb{F}_{2^s} \ltimes G$. In general, if N is a normal subgroup of a group Q with $|Q/N|$ relatively prime to p, then $H^*(Q; \mathbb{F}_p) \cong H^*(N; \mathbb{F}_p)^{Q/N}$. (cf. [CE, pages 257–258].) Applying this to the inclusion $\mathbb{F}_{2^s} \lhd \mathbb{F}_{2^s} \ltimes G$ gives $H^*(\mathbb{F}_{2^s} \ltimes G; \mathbb{F}_2) \cong H^*(\mathbb{F}_{2^s}; \mathbb{F}_2)^G$. However as an additive group $\mathbb{F}_{2^s} \cong (\mathbb{Z}/2\mathbb{Z})^s$, so

$$H^*(\mathbb{F}_{2^s} \ltimes G; \mathbb{F}_2)^G \cong \mathbb{F}_2[t_0, t_1, \ldots, t_{s-1}]^G \cong M_s(0).$$

We will now define the maps used to produce our the splittings.

For $s < \infty$, let $s' = ls$ where $l \leq \infty$. The algebra homomorphism $f_{s,s'} : M_s \to M_{s'}$ is defined by defining it on the polynomial generators by

$$f_{s,s'}(x_j) = \sum_{\{n \in P \mid n \equiv j (\text{mod } s)\}} x_n,$$

* *Added in proof.* John Harris [H] has subsequently checked that the decompositions are equivalent.

where $P = \mathbb{Z}$ if $s' = \infty$ and $P = \mathbb{Z}/s'\mathbb{Z}$ if $s' < \infty$. Since $f_{s,s'}$ commutes with Sq^1 on generators, $f_{s,s'}$ is an A-algebra homomorphism. For each $j \in \mathbb{Z}$, let $f_{s,s'}(j)$ denote the composite A-module homomorphism

$$M_s(j) \hookrightarrow M_s \xrightarrow{f_{s,s'}} M_{s'} \xrightarrow{\pi_j} M_{s'}(j),$$

where π_j is the projection onto the jth component. Define an A-algebra homomorphism $\gamma_{s,s'} : K_{s'} \to M_s$ by defining it on polynomial generators by $\gamma_{s,s'}(x_j) = x_j$. Let $\gamma_{s,s'}(j)$ denote the composite A-module homomorphism

$$M_{s'}(j) = K_{s'}(j) \hookrightarrow K'_s \xrightarrow{\gamma_{s,s'}} M_s \xrightarrow{\pi_j} M_s(j).$$

Let $\overline{M_s(j)}$ be the supspace of elements of degree greater than zero in $M_s(j)$. Thus $\overline{M_s(j)} = M_s(j)$ for $1 \leq j < 2^s - 1$ and $\overline{M_s(2^s - 1)} = \overline{M_s(0)}$, the augmentation ideal in $M_s(0)$.

THEOREM 4. *The composite* $\gamma_{s,s'}(j) \circ f_{s,s'}(j)$ *is the identity on* $\overline{M_s(j)}$ *for* $1 \leq j \leq 2^s - 1$.

Proof. Monomials $x_j^{2^k}$ will be called atoms. Every monomial in M_s has a unique expression as a product of distinct atoms. Define $\tau : M_s \to M_s$ by extending linearly the map defined on monomials by $\tau(x_{i_1}^{2^{r_1}} \cdots x_{i_k}^{2^{r_k}}) = x_{i_1 + r_1} \cdots x_{i_k + r_k}$, when the monomial is written as a product of distinct atoms.

LEMMA 5.

(1) $\tau(xy) = \tau(x)\tau(y)$ *if no monomial in* x *has an atom in common with any monomial in* y.

(2) *For a monomial* m, $w(\tau m) = w(m)$.

(3) $\pi_j(\tau x) = \tau(\pi_j(x))$ *for all* j.

(4) $f_{s,s'}(\tau x) = \tau(f_{s,s'}(x))$.

(5) $\gamma_{s,s'}(\tau x) = \tau(\gamma_{s,s'}(x))$ *if for each monomial in* x *the images under* $\gamma_{s,s'}$ *of the distinct atoms comprising that monomial are distinct.*

(6) $f_{s,s'}(j)(\tau x) = \tau(f_{s,s'}(j)(x))$ *for all* j.

(7) *If* x *is as in* (5), $\gamma_{s,s'}(j)(\tau x) = \tau(\gamma_{s,s'}(j)(x))$ *for all* j.

(8) $\gamma_{s,s'}(j) \circ f_{s,s'}(j)(\tau x) = \tau(\gamma_{s,s'}(j) \circ f_{s,s'}(j)(x))$ *for all* j.

Proof. Properties 1, 2, and 5 follow directly from the definitions. Property 3 follows from 2. To prove 4, notice that if monomials y and z have no common atoms then neither do $f_{s,s'}(y)$ and $f_{s,s'}(z)$. So 4 can be checked on monomials by use of Property 1 and induction on the number of atoms in the decomposition into a product of distinct atoms. Properties 6 and 7 follow from 3 and from 4 or 5 respectively and Property 8 follows from 6 and 7. □

Let m be a monomial in $M_s(j)$. From the definitions, $\gamma_{s,s'}(j)$ takes each monomial in $f_{s,s'}(j)(m)$ back to m. So $\gamma_{s,s'}(j) \circ f_{s,s'}(j)(m) = \varepsilon_m m$ where $\varepsilon_m \in \mathbb{F}_2$. We must show $\varepsilon_m = 1$. Applying τ and Property 8 of the Lemma gives $\varepsilon_{\tau m} = \varepsilon_m$. Thus repeated application of τ reduces the problem to showing $\varepsilon_m = 1$ when $m = x_{i_1} \cdots x_{i_k}$ with distinct i_q's. Let $m = x_{i_1} \cdots x_{i_k}$ be such a monomial. Then $j = w(m) \equiv 2^{i_1} + 2^{i_2} + \cdots + 2^{i_k} \pmod{2^2 - 1}$. The i_q's are distinct and $0 \le i_q \le s - 1$ for all q, so $1 \le 2^{i_1} + 2^{i_2} + \cdots + 2^{i_k} \le 2^0 + 2^1 + \cdots + 2^{s-1} = 2^s - 1$. Since we chose j to be the representative of the congruence class within the range $1 \le j \le 2^s - 1$, $2^{i_1} + 2^{i_2} + \cdots + 2^{i_k} = j$.

Case 1. $s' < \infty$

Monomials in $f_{s,s'}(j)(m)$ have the form $x_{i'_1} \cdots x_{i'_k}$ with $i'_q \equiv i_q \pmod{s}$ and $2^{i'_1} + 2^{i'_2} + \cdots + 2^{i'_k} \equiv j \pmod{2^{s'} - 1}$. Write $j' = 2^{i'_1} + 2^{i'_2} + \cdots + 2^{i'_k} \equiv j \pmod{2^{s'} - 1}$. The i'_q's are distinct since their reductions mod s are, and $0 \le i'_q \le s' - 1$ for all q, so as above we conclude that $1 \le j' \le 2^{s'} - 1$. Since we also have $1 \le j \le 2^s - 1 \le 2^{s'} - 1$, the congruence forces $j = j'$. So the above sums are binary expansions of $j = j'$ and so $i'_q = i_q$ for all q. So there is a unique such monomial. Thus $f_{s,s'}(j)(m)$ has only one term when m has this form, and therefore $e_m = 1$.

Case 2. $s' = \infty$

As above, monomials in $f_{s,s'}(j)(m)$ have the form $x_{i'_1} \cdots x_{i'_k}$ with $i'_q \equiv i_q \pmod{s}$, but this time the equality $2^{i'_2} + 2^{i'_2} + \cdots + 2^{i'_k} = j$ is in $\mathbb{Z}[\frac{1}{2}]$, or equivalently in $\mathbb{Z}$ since both sides are integral. As above, the i'_q's are distinct, so the sum is the binary expansion of j. Therefore $i'_q = i_q$ for all q. So again there is a unique such monomial which is the only term in $f_{s,s'}(j)(m)$. Therefore $e_m = 1$. □

COROLLARY 6. *$M_s(j)$ is injective in $\mathcal{U}$ for all j and s.* □

Partially order the positive integers by defining $s \prec s'$ if s divides s'.

THEOREM 7. *For integral n, $K(n)$ is the direct limit of the modules $M_s(n)$ under the maps $f_{s,ks}$.*

Proof. Since the maps in the direct system are injections compatible with the injections $f_{s,\infty}$, it suffices to show that the induced map from the direct limit to $K(n)$ is surjective. But it is clear that $f_{s,\infty}(n)(x_{i_1} \cdots x_{i_k})$ contains only the term $x_{i_1} \cdots x_{i_k}$ when s is sufficiently large. □

Remark. Interpreted properly, the statement and proof of the preceding theorem make sense for any $n \in \mathbb{Z}[\frac{1}{2}]$.

Example. Consider the case $s=2$. In this case the work of Harris, Kuhn, Mitchell, and Priddy yields the complete stable decomposition

$$B(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) = (BA_4)_{(2)} \vee B(\mathbb{Z}/2\mathbb{Z}) \vee B(\mathbb{Z}/2\mathbb{Z}) \vee L(2) \vee L(2),$$

where A_4 is the alternating group on 4 symbols and $L(2)$ is a quotient of symmetric product spectra. Our methods give $H \otimes H \cong \overline{M_2(0)} \oplus M_2(1) \oplus M_2(2)$ with $M_2(1) \cong M_2(2)$. However by Theorem 4, $M_2(1) \cong \overline{M_1(1)} \oplus L$ for some L, and of course $M_2(0) \cong \mathbb{F}_2 \oplus \overline{M_2(0)}$. So we obtain the complete decomposition in this case.

Added in proof

Example. Consider $s=3$. Our basic decomposition gives

$$H \otimes H \otimes H \cong M_3 \cong M_3(0) \oplus M_3(1) \oplus M_3(2) \oplus M_3(3) \oplus M_3(4) \oplus M_3(5) \oplus M_3(6)$$

where $M_3(1) \cong M_3(2) \cong M_3(4)$ and $M_3(3) \cong M_3(6) \cong M_3(5)$. Theorem 4 can be used to split the summand $M_1(1)$ off of $M_3(1)$. To proceed further one must consider compositions of our maps going from M_2 to M_3 through M_6 and back. The complete decomposition will not be obtained however. The case $s=3$ has been examined in detail in the work of Harris, Hunter, and Shank ([HHS], [S]).

3. Extension to odd primes

Let p be an odd prime. Write $H^*(B(\mathbb{Z}/p\mathbb{Z})^s; \mathbb{F}_p)$ as the free graded commutative algebra on generators $\{u_0, u_1, \ldots, u_{s-1}\} \cup \{t_0, t_1, \ldots, t_{s-1}\}$ having degrees 1 and 2 respectively with $\beta u_j = t_j$. Replacing 2 with p, define ϕ, ω, T, and the new basis $\{x_0, x_1, \ldots, x_{s-1}\}$ for the linear subspace of $\mathbb{F}_{p^s}[t_0, t_1, \ldots, t_{s-1}]$ as in Section 1. Define a new basis $\{y_0, y_1, \ldots, y_{s-1}\}$ for $H^1(B(\mathbb{Z}/p\mathbb{Z})^s; \mathbb{F}_{p^s})$ so that $\beta y_j = x_j$. The definitions of the maps B and λ and the proof of Theorem 1 proceed as before.

The results of Section 2 also carry over to odd primes in a straightforward way. This time K is the free graded commutative algebra on $\{y_j\}_{j \in \mathbb{Z}} \cup \{x_j\}_{j \in \mathbb{Z}}$ where y_j lies in $K^1(2p^j)$ and x_j lies in $K^2(2p^j)$. The Steenrod algebra action is described by $\beta y_j = x_j$ and $P^1 x_j = x_{j-1}^p$. Since $K(j)=0$ if j is odd we find it convenient to set $M(j) = K(2j)$ and assign weights by $w(x_j) = w(y_j) = p^j$. Otherwise we define M_s as before for $s \le \infty$. For $s < \infty$ weights are treated modulo $p^s - 1$. The maps $f_{s,s'}$ and

$\gamma_{s,s'}$ are defined as before and extended to the y's by the same formulas. With the obvious replacements, the proof of Theorem 4 now carries over to the odd prime case.

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