Complex singularities and the framed cobordism class of compact quotients of 3-dimensional Lie groups by discrete subgroups.

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Complex singularities and the framed cobordism class of compact quotients of 3-dimensional Lie groups by discrete subgroups[1]

J. A. SEADE AND B. F. STEER

Let Γ be a discrete subgroup of a 3-dimensional Lie group G with compact quotient $M_\Gamma = \Gamma \backslash G$. The tangent bundle of G may be trivialized by left translation. This descends to M_Γ and defines a trivialization $\mathscr{L}$ of its tangent bundle. Once an orientation is chosen for M_Γ the pair $(M_\Gamma, \mathscr{L})$ determines an element in Ω_3^{fr}, the cobordism group of stably framed 3-manifolds, which is isomorphic to the stable homotopy group of the spheres, π_3^s, via the Pontryagin construction [25]. In this article we determine the elements in π_3^s so obtained in terms of invariants of the embedding of Γ in G, thus completing the work started in [38].

We use the Adams e-invariant [1]. It is a monomorphism

$$e : \pi_3^s \to \mathbb{Q}/\mathbb{Z},$$

expressible in terms of spin-cobordism [6]. However, in the cases relevant here we have $\Gamma \backslash G = \partial X$ and X is seldom spin. So we use a formula of [36], derived from Rochlin's theorem [12], [31], [35], that expresses e in terms of complex cobordism plus a correction term for the lack of a spin-structure.

Up to isomorphism, there are six different 3-dimensional, simply-connected Lie groups that admit discrete subgroups with compact quotient. A complete list of these is given in [34], where a list of the discrete subgroups and the corresponding quotients $\Gamma \backslash G$ is also given. The Lie groups are $SU(2)$, $\widetilde{Sl_2\mathbb{R}}\backslash\mathbb{R}^3$, $\widetilde{E^+(2)}$, the Lorentz group $E(1, 1)$ (or SOLV) and the Heisenberg group H. From work initiated by F. Klein and continued by F. Hirzebruch and others [20], [15], [9], [10], [26], [29] we know that if G is $SU(2)$, $\widetilde{Sl_2\mathbb{R}}$, H or $E(1, 1)$, then for every $\Gamma \subset G$ the quotient $M_\Gamma = \Gamma \backslash G$ is diffeomorphic to the link of a normal, Gorenstein, surface singularity. The first three correspond to singularities with $\mathbb{C}^*$-action and $E(1, 1)$ to

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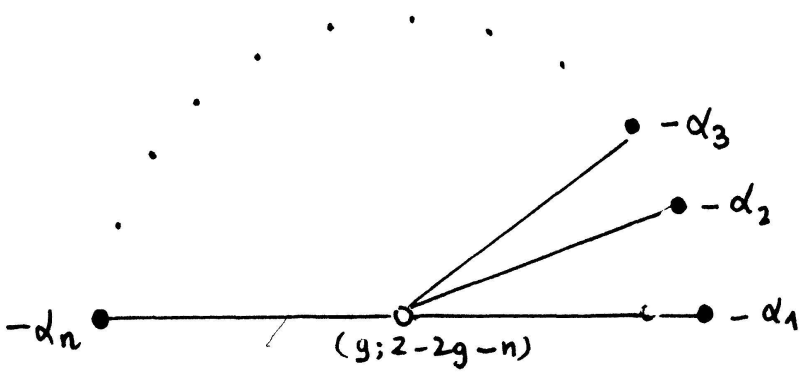
the cusp singularities. Thus, in all these cases M_Γ can be regarded as the boundary of a resolution $\tilde{V} = \tilde{V}_\Gamma$ of the corresponding singularity. Moreover, the framing $\mathscr{L}$ on M_Γ is compatible with the complex structure on $\tilde{V}$, hence the formula [36] for the e-invariant says

$$e(M_\Gamma, \mathscr{L}) = \frac{\chi(\tilde{V}) + K^2}{24} + \tfrac{1}{2}\operatorname{Arf}(K) \quad \bmod \mathbb{Z}$$
$$= \frac{\mu(\tilde{V}) - \delta(\tilde{V})}{16} \qquad \bmod \mathbb{Z}$$

where $\chi(\tilde{V})$ is the Euler–Poincaré characteristic, K is the canonical class of $\tilde{V}$, Arf (K) is the Arf invariant of K for a suitable quadratic form on $H_1(K, \mathbb{Z}_2)$ [12], [35], $\mu(\tilde{V})$ is Rochlin's invariant [12], [16], [22], [41] and $\delta(\tilde{V})$ is the signature defect so much studied [11], [15], [17], [21], [27].

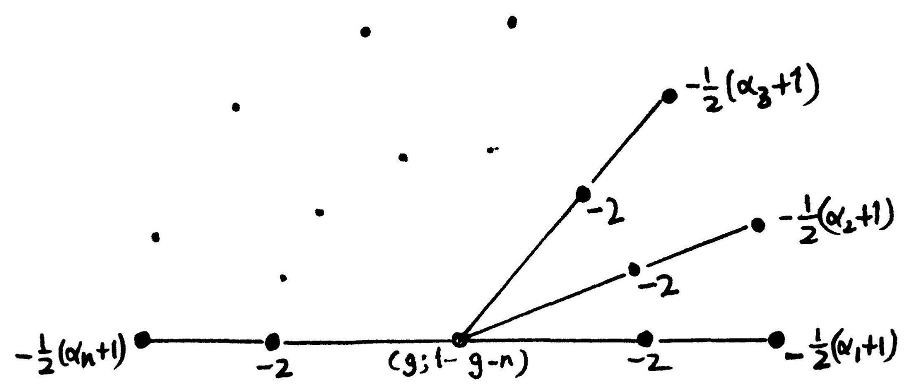
In §2 of this article we make explicit computations of these invariants for the semi-simple groups $SU(2)$ and $\widetilde{Sl_2\mathbb{R}}$. The case of $\widetilde{Sl_2\mathbb{R}}$ is the most interesting and it takes most of our energy. Let us illustrate the situation with two examples: $\Gamma \subset PSl_2\mathbb{R}$ and $\Gamma \subset Sl_2\mathbb{R}$. In the first case the corresponding singularities are the quotient singularities of Dolgachev [9]. The resolution of these is of the form



where $\{g; \alpha_1, \ldots, \alpha_n\}$ is the signature of the Fuchsian group Γ and all the vertices represent spheres, except the centre E_0 which has genus g. The canonical class K is $-2E_0 - \Sigma_{i=1}^n E_i$, the invariant Arf (K) is $(g-1)$ mod (2) and the e-invariant is

$$e(\Gamma \backslash PSl_2\mathbb{R}, \mathscr{L}) = \frac{1}{24}\left(2g - 2 + n - \sum_{i=1}^{n} \alpha_i\right) \quad \bmod \mathbb{Z}.$$

Now, if $\Gamma \subset Sl_2\mathbb{R}$ and $-I \notin \Gamma$, so that $\pi : Sl_2\mathbb{R} \to PSl_2\mathbb{R}$ is an isomorphism over G, then the corresponding singularity has graph



where $\{g; \alpha_1, \ldots, \alpha_n\}$ is the signature of $\pi(\Gamma)$, the α_i's are (automatically) odd integers and all vertices represent 2-spheres, except the centre $E_0 \cong \Gamma\backslash\mathscr{H}$. In this case K is $(-3E_0 - 2\sum_{i=1}^n E_{i,1} - \sum_{i=1}^n E_{i,2})$ and $\operatorname{Arf}(K)$ is

$$(g-1) + \dim H^0(E_0; L^{-1}) \mod (2)$$

where $L = \nu \otimes L_1 \otimes \cdots \otimes L_n$, ν is the normal bundle of E_0 and L_i, $i = 1, \ldots, n$, is the holomorphic line bundle over E_0 determined by its intersection point with the curve $E_{i,1}$. Then we find

$$e(\Gamma\backslash Sl_2\mathbb{R}, \mathscr{L}) = \frac{1}{48}\left(2g - 2 + n - \sum_{i=1}^n \alpha_i + 24 \dim H^0(E_0; L^{-1})\right) \mod \mathbb{Z}.$$

These two cases exemplify the general situation. Given $\Gamma \subset \widetilde{Sl_2}\mathbb{R}$ with compact quotient, there is an integer $r \geq 1$, the index of Γ in $\pi^{-1}(\pi(\Gamma))$, where π is the projection onto $PSl_2\mathbb{R}$; this integer r is determined by the Seifert invariant of M_Γ and $-(r+1)$ is the coefficient of the central curve E_0 in the canonical class K of the resolution in [32]. If r is odd, then we can drop E_0 from the expression for a characteristic submanifold W and we may determine $e(M_\Gamma, \mathscr{L})$ in terms of the Seifert invariants: indeed, simply in terms of r and the 'signature' $\{g; \alpha_1, \ldots, \alpha_n\}$ of $\pi(\Gamma)$ as W. D. Neumann's improvement (2.7) clearly shows. However, when r is even the central curve E_0 has a canonical spin-structure and the corresponding Arf invariant depends on the embedding of Γ in G. Thus, to determine the class of $(\Gamma\backslash G, \mathscr{L})$ in π_3^s one must specify the Seifert invariants of $\Gamma\backslash G$ *plus* an additional invariant when r is even: the mod (2)-index of the Dirac operator on the central curve E_0.

In the case $G = E(1, 1)$, §3, our work depends upon the basic article of F. Hirzebruch [15]: all that is left for us is the evaluation of $\operatorname{Arf}(K)$. This we do 'by hand', using the topological definition as given in [12]. In §3 we also study the other solvable, non-nilpotent group: $\widetilde{E^+(2)}$, the universal cover of $E^+(2)$, the orientation preserving affine motions of $\mathbb{R}^2$. In this case, the quotients Γ/G are no longer singularity links as in the other cases, but they are boundaries of 'big neighbourhoods' containing 3 or 4 singular points, and the same method applies. The nilpotent case has already appeared in [8], but we include (in §4) a short and quite different proof, for completeness, together with a proof in the abelian case, which is similar. In §1 we first recall some definitions and results about the e-invariant and the Arf invariant. We then define the invariant $\operatorname{Arf}(K)$ for singularities and we relate it with the e-invariant and the spin cobordism invariant of Atiyah [4], as well as with other invariants of the singularity. Our computations also determine the Rochlin μ-invariant of Γ/G in all cases.

We are happy to thank Igor Dolgachev, Nigel Hitchin and Elmer Rees for a number of conversations and suggestions. Walter Neumann we especially thank for showing us how to simplify and improve the statements in several instances. At his suggestion we include the values of the Rochlin invariant μ and the signature defect δ. We are grateful to the Universities of Geneva and Neuchâtel, to IMPA at Rio de Janeiro, the Tata Institute at Bombay and to colleagues at these institutions whilst working on this paper.

The table below summarizes the classification of the elements in π_3^s represented by $(\Gamma/G, \mathscr{L})$, for the 3-dimensional Lie groups with compact quotients.

Group G	**Subgroup** Γ	**Element represented in** $\pi_3^s \cong \{1, 2, \ldots, 24\}$
Abelian		
$\mathbb{R}^3$	$\Gamma \cong \mathbb{Z}^3, \quad M_\Gamma = T^3$	12
Nilpotent		
H	$\Gamma = \Gamma_k \cong \begin{pmatrix} 1 & a & c/k \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$; $a, b, c \in \mathbb{Z}$, $k \in \mathbb{Z}$ fixed	$12 - k$
Solvable		
$E(1, 1)$	$\Gamma \cong M \rtimes V$, with $M = \underline{\mathbb{Z}.v + \mathbb{Z}.1}$, $w = [[d_1, \ldots, d_r]]$, $V \subset U_M^+ \cong \mathbb{Z}$ of index $l \geq 1$	$3lr - l\,\Sigma_{i=1}^r d_i + 12$
$\widetilde{E^+}(2)$	$\pi(\Gamma) \subset E^+(2)$ *in* $\mathbb{Z}^2$, a triangle group $(2, 3, 6)$, $(2, 4, 4)$ or $(3, 3, 3)$ or the quadrangle group $(2, 2, 2, 2)$.	0, if $\pi(\Gamma) \cong \mathbb{Z}^2$; $2 - n + \alpha_1 + \cdots + \alpha_n$, if $\pi(\Gamma) = (\alpha_1, \ldots, \alpha_n)$

Semi-simple

$SU(2)$	$\Gamma \cong \mathbb{Z}/r$, $r \geq 1$, or Γ is a (binary) triangle group $\langle 2, 2, r\rangle$, $r \geq 2$, $\langle 2, 3, 3\rangle$, $\langle 2, 3, 4\rangle$ or $\langle 2, 3, 5\rangle$.	r, if $\Gamma \cong \mathbb{Z}/r$; $-1 + \alpha_1 + \alpha_2 + \alpha_3$, if $\Gamma = \langle \alpha_1, \alpha_2, \alpha_3\rangle$
$\widetilde{Sl_2\mathbb{R}}$	Γ is given by a cocompact Fuschian group $\Gamma_1 \subset PSl_2\mathbb{R}$, and an embedding of Γ_1 in some finite cyclic cover G_r of $PSl_2\mathbb{R}$	See (2.8) and (2.10) in the text.

§1 Surface singularities and Rochlin's invariant

(a) Let X be a **closed**, C^∞, 4-manifold with a complex structure on its (stable) tangent bundle, and let $-K \subset X$ be an oriented submanifold of X representing the first Chern class $c_1(X)$. An oriented 2-submanifold W of X is **characteristic** if it represents the same class as K in $H_2(X; \mathbb{Z}_2)$. Thus, in particular, K is characteristic. Rochlin's theorem [12], [30], [31], [35] says that

$$\sigma(X) - W^2 \equiv 8\mathrm{Arf}(W) \mod (16),$$

where $\sigma(X)$ denotes the signature of X. Taking $W = K$ we have

$$\mathrm{Td}\,(X) \equiv \mathrm{Arf}\,(K) \mod (2),$$

where Td (X) is the Todd genus [14].

Let us now suppose that X is compact but has non-empty boundary $M = \partial X$. Assume that the complex bundle TX is trivial over M and $\beta : TX|_M \xrightarrow{\cong} M \times \mathbb{C}^2$ is a specific trivialisation. Then β defines a complex vector bundle $\tau(\beta)$ over the quotient X/M, whose pull-back to X is TX. By definition [19], the Chern classes of $\tau(\beta)$, $c_i(\beta) \in H^{2i}(X, M; \mathbb{Z})$, $i = 1, 2$, are the **Chern classes of X relative to β**. Their image in $H^*(X; \mathbb{Z})$ are the usual Chern classes of X, but as relative classes they depend on β. The following results of [36] is a consequence of Rochlin's theorem and, indeed, provides an extension of Rochlin's theorem to manifolds with boundary. Recall that W, an oriented submanifold of Int (X), is **characteristic relative to β** if it represents a homology class in $H_2(X; \mathbb{Z}_2) \cong H^2(X, M; \mathbb{Z}_2)$ dual to the reduction mod (2) of $c_1(\beta)$.

(1.1) THEOREM. *If $W \subset X$ is characteristic relative to β, then the e-invariant of (M, β) is,*

$$\begin{aligned} e(M, \beta) &= \tfrac{1}{48}(2c_2(\beta) - c_1^2(\beta))[X] + \tfrac{1}{16}(W^2 - 8\,\mathrm{Arf}(W)) \quad \mathrm{mod}\ \mathbb{Z} \\ &= \tfrac{1}{16}(\mu(M, \beta) - \delta(M, \beta)) \quad \mathrm{mod}\ \mathbb{Z}, \end{aligned}$$

where $[X]$ is the orientation cycle, $\mathrm{Arf}(W) \in \{0, 1\}$ *is as in Rochlin's theorem and $\mu(M, \beta) = \mu(X)$ is Rochlin's* mod (16) *invariant* [12], [16].

Note that if we know $\mathrm{Arf}(W)$ for some characteristic W, then we can determine $\mathrm{Arf}(W')$ for another characteristic W' by the formula

$$(1.2) \quad \mathrm{Arf}(W') = \mathrm{Arf}(W) + \frac{(W')^2 - W^2}{8} \quad \mathrm{mod}\ 2.$$

(b) Let us denote by $(\mathscr{V}, P)$ the germ of a normal, Gorenstein, surface singularity, and we let $\pi : \tilde{\mathscr{V}} \to \mathscr{V}$ be a (good) resolution of P. If $\tilde{V} \subset \tilde{\mathscr{V}}$ denotes a compact tubular neighbourhood of the exceptional set $E = \pi^{-1}(P)$, then $\tilde{V}$ is an almost complex 4-manifold with boundary M, the link of P in $\mathscr{V}$. If ω is a nowhere zero holomorphic 2-form on $\mathscr{V}\backslash\{P\}$ (which exists because P is Gorenstein), then [36], [38] ω defines a canonical trivialization $\mathscr{C}$ of TM, compatible with the complex structure on $\mathscr{V}$. Thus $\mathscr{C}$ defines a complex trivialization of $T\tilde{V}|_M$. Moreover, the Chern class $c_1(\mathscr{C})$ of $\tilde{V}$ relative to $\mathscr{C}$ is indeed independent of $\mathscr{C}$ and it is dual to $-K$, the anti-canonical class. By [28], the class K is uniquely characterized by the adjunction formula

$$2g - 2 = E \cdot (K + E)$$

for every non-singular curve E in $\mathrm{Int}(\tilde{V})$ of genus g. Also [19], [36], if β is a trivialization of $T\tilde{V}|_M$ induced from one of TM and, like $\mathscr{C}$, compatible with the complex structure then $c_2(\beta)[X]$ is the (topological) Euler characteristic of $\tilde{V}$, $\chi(\tilde{V})$. Thus we have,

(1.3) THEOREM [36]. *Let β be some trivialization of TM compatible with the complex structure on $\tilde{V}$. Then*

$$\begin{aligned} e(M, \beta) &= \tfrac{1}{48}(2\chi(\tilde{V}) - K^2) + \tfrac{1}{16}(W^2 - 8\,\mathrm{Arf}(W)) \quad \mathrm{mod}\ \mathbb{Z} \\ &= \tfrac{1}{16}(\mu(\tilde{V}) - \delta(\tilde{V})) \quad \mathrm{mod}\ \mathbb{Z}, \end{aligned}$$

where W is a characteristic submanifold of $\tilde{V}$. In particular if $W = \tilde{K}$, a representative of the canonical class K, then

$$e(M, \beta) = \tfrac{1}{24}(\chi(\tilde{V}) + K^2) + \tfrac{1}{2}\operatorname{Arf}(\tilde{K}) \quad \text{mod } \mathbb{Z}.$$

(If D is a (possibly) singular divisor on $\tilde{V}$, then we can always find a smooth $\tilde{D}$ representing the same homology class as D, hence $\tilde{D}^2 = D^2$. It thus follows from (1.2) that if $[D]|_2 = [K]|_2$, then we may define $\operatorname{Arf}(\tilde{D}) = \operatorname{Arf}(\tilde{D})$ mod (2), and this definition does not depend on the choice of $\tilde{D}$. In this case we say that D is a **characteristic divisor** and $\operatorname{Arf}(D)$ its associated **Arf invariant**. Moreover, if all components of D are non-singular, with multiplicities equal to 1 and normal corssings, then there is a canonical way of smoothing D and obtaining a smooth $\tilde{D}$ in the same class as D: at each crossing point, we choose local coordinates so that D is given by $z_1 z_2 = 0$, and replace this with $z_1 z_2 = t$, for some small t.) Because the Todd genus is invariant under blowing up [14], [18] the following result is an immediate consequence of (1.3).

(1.4) PROPOSITION. *The invariant* $\operatorname{Arf}(K)$ *depends only on* $(\mathscr{V}, P)$ *and not on any of the choices involved in its definition.*

(1.5) THEOREM. *Let β be any trivialization of TM (i.e. $\beta : TM \xrightarrow{\simeq} M \times \mathbb{R}^3$) compatible with the complex structure and let $\tilde{V}$ be a resolution of P.*

(1) *If $\mu(M) \in \mathbb{Q}/\mathbb{Z}$ is the Rochlin μ-invariant of M*[16], [41], *with respect to the spin-structure defined by β, then*

$$\mu(M) = \tfrac{1}{2}\operatorname{Arf}(K) - \tfrac{1}{16}(K^2 + b_2) \quad \text{mod } \mathbb{Z}$$

where b_2 is the rank of $H_2(\tilde{V}; \mathbb{Z})$.

(2) *If $(\mathscr{V}, P)$ is smoothable then*

$$\operatorname{Arf}(K) = \rho_g \quad \text{mod}\,(2)$$

where ρ_g is the geometric genus of P, $\rho_g = \dim H^1(\tilde{V}, \mathcal{O})$.

Proof: Clause 2 is immediate from the Laufer–Steenbrink formula [13], [23], [40], (1.3), and the observation that if $\mathscr{V}'$ is a smoothing with Milnor number d then $e(M, \beta) = (d + 1)/24$ mod $\mathbb{Z}$. (It admits a rather better proof, independent of the existence of a smoothing, in several cases.)

Clause 1 follows directly from Rochlin's theorem [12], [35] if we bear in mind Novikov additivity for the signature and Mumford's theorem [28] that $b_2 = -\sigma(\tilde{V})$. $\square$

Let us now assume that the characteristic submanifold W is, in fact, a Riemann surface in $\mathrm{Int}(\tilde{V})$, i.e. W is an effective, non-singular divisor. Then W is given by the zeroes of a holomorphic section of a bundle of the form $\mathscr{L}_W = \mathscr{K}_{\tilde{V}}^{-1} \otimes \mathscr{D}^2$, for some line bundle $\mathscr{D}$ which is holomorphically trivial near $\partial\tilde{V}$, where $\mathscr{K}$ is the canonical bundle of $\tilde{V}$, $\mathscr{K}_{\tilde{V}} = \Lambda^2 T^*\tilde{V}$. Since $H^1(\tilde{V}, \partial\tilde{V}; \mathbb{Z}_2) \cong H^3(\tilde{V}; \mathbb{Z}_2) = 0$, $\mathscr{D}$ is uniquely determined.

(1.6) PROPOSITION.

$$\mathrm{Arf}\,(W) = \dim H^0(W; \mathscr{D}|_W) \quad \mathrm{mod}\,(2).$$

Proof. The normal bundle of W in $\tilde{V}$ is isomorphic to $\mathscr{L}_W|_W$, by the adjunction formula, thus

$$\mathscr{K}_{\tilde{V}}|_W \cong \mathscr{K}_W \otimes \mathscr{L}_W^{-1}|_W, \quad \text{where } \mathscr{K}_W \cong T^*W.$$

Therefore,

$$\mathscr{K}_W \cong \mathscr{K}_{\tilde{V}}|_W \otimes \mathscr{L}_W|_W \cong \mathscr{D}|_W^2$$

by the definition of $\mathscr{L}_W$. This means [4] that $\mathscr{D}$ determines spin-structure on W, as described in [31], so (1.6) follows from the fact that both sides are (non-trivial) spin-cobordism invariants by [4], [31] and $\Omega_2(\mathrm{Spin}) \cong \mathbb{Z}/2$ by [2]. $\square$

We note [4] that $\dim H^0(W; \mathscr{D}|W)$ is the mod (2)-index of the Dirac operator on W, for the spin-structure. If we denote this index by $h(W)$, then (1.6) can be stated as

$$\mathrm{Arf}\,(W) = h(W) \quad \mathrm{mod}\,(2).$$

In this form (1.6) holds in the smooth category, i.e. even if W is not complex analytic.

§2 The Semi-simple groups

(a) We consider first the case $G = SU(2)$, though the calculations are in [38]. The discrete subgroups of $SU(2)$ are the cyclic groups $\mathbb{Z}/r$, $r \geq 1$, and the triangle groups $\langle 2,2,r\rangle$, $r \ge 2$, $\langle 2,3,3\rangle$, $\langle 2,3,4\rangle$ and $\langle 2,3,5\rangle$. The quotient $\Gamma\backslash G$ is diffeomorphic to the link of the singularity $\Gamma/\mathbb{C}^2$. The resolution of these singularities is well-known: it is given by the famous Dynkin diagrams A_{r-1}, D_r, E_6, E_7 and E_8, see [3], so that the canonical class K is 0 in all cases, by the Adjunction Formula. Since G acts on $\mathbb{C}^2$ by holomorphic transformations, the framing $\mathscr{L}$ on $\Gamma\backslash G$ is compatible with the complex structure on $\Gamma\backslash\mathbb{C}^2$. Thus we may use (1.3) to determine the e-invariant of $(\Gamma\backslash G, \mathscr{L})$ and we find

$$\delta(\Gamma\backslash G, \mathscr{L}) = 4 - \tfrac{2}{3}\chi(\tilde{V}_\Gamma), \quad \mu(G\backslash G, \mathscr{L}) = 4 \mod (16);$$

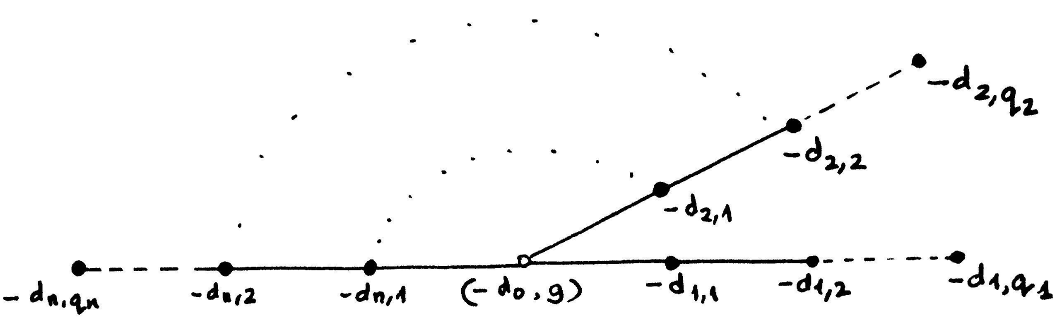
$$e(\Gamma\backslash G, \mathscr{L}) = \tfrac{1}{24}\chi(\tilde{V}_\Gamma) \mod (\mathbb{Z})$$

where $\tilde{V}_\Gamma$ is the resolution.

(2.1) THEOREM. *Let $\Gamma \subset SU(2)$ be a discrete subgroup. If $\Gamma \cong \mathbb{Z}/r$, then $(\Gamma\backslash G, \mathscr{L})$ has e-invariant $r/24 \bmod \mathbb{Z}$. If Γ is a triangle group $\langle p,q,r\rangle$, then the e-invariant is*

$$\tfrac{1}{24}(p+q+r-1) \mod (\mathbb{Z}).$$

(b) **The case of $\widetilde{Sl_2\mathbb{R}}$.** For a moment consider in general a germ $(\mathscr{V}, P)$ of a complex surface with a good $\mathbb{C}^*$-action. Let M denote the link of P in $\mathscr{V}$. Then M is a Seifert manifold and its Seifert invariants $\{g : d_0 : (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$ can be determined, as in [32], from the weights of P. It is shown in [32] that these singularities have a canonical, equivariant resolution $\tilde{V}$ with graph



where the centre E_0 is a non-singular curve of genus $g \ge 0$ with self intersection $-d_0$ and all other vertices $E_{i,j}$ are copies of $\mathbb{C}P^1$. The number of branches corresponds to the exceptional fibres of M and the weights $d_{i,j}$ corresponding to the i^{th}-branch are determined by (α_i, β_i) as follows: set $\alpha_{i,0} = \alpha_i$, $\alpha_{i,1} = \beta_i$ and then define inductively

$$\begin{aligned}(2.2)\quad \alpha_{i,0} &= d_{i,1}\alpha_{i,1} - \alpha_{i,2}\\ \alpha_{i,1} &= d_{i,2}\alpha_{i,2} - \alpha_{i,3}\\ &\vdots\\ \alpha_{i,q_i-2} &= d_{i,q_i-1}\alpha_{i,q_i-1} - 1\\ \alpha_{i,q_i-1} &= d_{i,q_i}.\end{aligned}$$

We set $\alpha_{i,q_i} = 1$. All the $d_{i,j}$ are then ≥ 2, but d_0 may be 1, see [32], [16]. So $[[d_{i,1}, \ldots, d_{i,q_i}]]$ is the continued fraction for (α_i/β_i), in the notation of [15].

The **Euler number** of M, as a Seifert manifold, is defined by

$$E(M) = -d_0 + \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} \in \mathbb{Q}.$$

It is always a negative number [29], because M is the link of a singularity. The number

$$X(M) = 2 - 2g - n + \sum_{i=1}^{n} \frac{1}{\alpha_i} \in \mathbb{Q}$$

is the Euler characteristic of the base orbifold.

If K is the canonical class of the resolution of $(\mathscr{V}, P)$ described above, then K is a linear combination of the form

$$K = m_0 E_0 + \sum_{i=1}^{n} \sum_{j=1}^{q_i} m_{i,j} E_{i,j}, \quad m_0, m_{i,j} \in \mathbb{Q}.$$

Let us define $m_{i,0}$ to be m_0 for $1 \leq i \leq n$, and set $m_{i,q_i+1} = 0$. The following proposition tells us what m_0 and the $m_{i,j}$'s are.

(2.3) PROPOSITION. *If we set* $\lambda_0 = m_0 + 1$ *and* $\lambda_{i,j} = m_{i,j} + 1$, *then*

$$\lambda_{i,j+1} + \lambda_{i,j-1} = \lambda_{i,j} d_{i,j}; \tag{1}$$

$$\alpha_{i,j}\lambda_{i,j+1} - \alpha_{i,j+1}\lambda_{i,j} = 1; \tag{2}$$

$$\lambda_0 = -X(M)/E(M) \in \mathbb{Q}. \tag{3}$$

To prove (2.3) one applies the adjunction formula to each $E_{i,j}$ and to E_0 and obtains the following system of equations:

$$d_{i,j} - 2 = -d_{i,j}m_{i,j} + m_{i,j-1} + m_{i,j+1}, \quad \text{for all } i, j \geq 1; \tag{1'}$$

$$2g - 2 + d_0 = -d_0 m_0 + \sum_{i=1}^{n} m_{i,1}. \tag{4'}$$

Replacing the $m_{i,j}$ by $\lambda_{i,j} - 1$ yields

$$\lambda_{i,j}d_{i,j} = \lambda_{i,j-1} + \lambda_{i,j+1}, \quad \text{for all } i, j \geq 1; \tag{1}$$

$$2g - 2 + n = -d_0\lambda_0 + \sum_{i=1}^{n} \lambda_{i,1}. \tag{4}$$

Equation (2) follows from a trivial induction on decreasing j.

Again by decreasing induction on j we can show that if we specify λ_{i,q_i} to be s_i then $\lambda_{i,j} = \alpha_{i,j}s_i - \gamma_{i,j}$, where $0 < \gamma_{i,j} < \alpha_{i,j}$ and $\gamma_{i,j}\alpha_{i,j+1} \equiv 1(\alpha_{i,j})$; $1 \leq j < q_i$. So if $\gamma_i\tilde{\beta}_i \equiv 1(\alpha_i)$, $0 < \gamma_i < \alpha_i$, and $r \equiv \gamma_i(\alpha_i)$ then system (1) has a solution with $\lambda_0 = -r$ and $\lambda_{i,q_i} = -[(r - \gamma_i)/\alpha_i]$. The general one can be obtained from this particular solution and we see that to satisfy equation (4) we must have $\lambda_0 = -X(M)/E(M)$.

If the germ is Gorenstein then [10], [29] its link M is of the form $\Gamma\backslash G$ where G is $SU(2)$, $\widetilde{Sl_2\mathbb{R}}$, or the Heisenberg group H, according as $X(M)$ is positive, negative, or zero. In these cases the canonical class K is integral and so the $\lambda_{i,j}$ are integers.

Now we turn to $Sl_2\mathbb{R}$. It is an infinite central extension of $PSl_2\mathbb{R}$ so that, if we denote $PSl_2\mathbb{R}$ by G_1 and $\widetilde{Sl_2\mathbb{R}}$ by G_∞, we have an exact sequence

$$0 \to \mathbb{Z} \to G_\infty \xrightarrow{\pi} G_1 \to 1.$$

For every integer $r \geq 1$ there are central extensions

$$0 \to \mathbb{Z}/r \to G_r \xrightarrow{\pi_r} G_1 \to 1$$

$$0 \to \mathbb{Z} \to G_\infty \xrightarrow{p_r} G_r \to 1;$$

moreover, given $\Gamma_\infty \subset G_\infty$, a discrete subgroup with compact quotient, there exists an integer $r \geq 1$ and a discrete cocompact subgroup Γ_r of G_r such that $\ker(\pi_r|_{\Gamma_r}) = 1$ and $\Gamma_r\backslash G_r \cong \Gamma_\infty\backslash G_\infty$ [34]. (The number r is the index of Γ_∞ in $\pi^{-1}(\pi(\Gamma_\infty))$. The group $\pi\Gamma_\infty$ is determined up to quasiconformal equivalence by its 'signature' [7].)

The group $G_1 = PSl_2\mathbb{R}$ acts on the upper half plane $\mathscr{H}$ via the Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

$a, b, c, d \in \mathbb{R}$, $ad - bc = 1$ and $\operatorname{Im}(z) > 1$. This action extends to one, also by holomorphic maps, of G on $T\mathscr{H} \cong \mathscr{H} \times \mathbb{C}$, the holomorphic tangent bundle of $\mathscr{H}$. It is defined by differentiation:

$$g \cdot (z, w) = (g(z), g'(z) \cdot w)$$

where $(z, w) \in \mathscr{H} \times \mathbb{C}$. So, choosing a base point $(z_0, 1) \in T\mathscr{H}$, we map G into $T\mathscr{H}$ as the orbit of $(z_0, 1)$. In fact, this map is an embedding and identifies G with $T_1\mathscr{H}$, the **unit** sphere bundle of $\mathscr{H}$. (G acts transitively on $T_1\mathscr{H}$ with trivial isotropy group.) The situation for G_r is similar.

In [26], J. Milnor gives an explicit description of G_∞. If we adopt his notation, we can identify G_r with the r-labelled biholomorphic maps of $\mathscr{H}$ to itself, where such an object is a biholomorphic map of $\mathscr{H} = \{\operatorname{Im}(z) > 0\}$, $f: \mathscr{H} \to \mathscr{H}$, together with a lift $\tilde{f}': \mathscr{H} \to \mathbb{C}_r^* \cong \mathbb{C}^*$ of the derivative to the r-fold cover $\mathbb{C}_r^*$ of the multiplicative group $\mathbb{C}^*$. If we denote by $\mathbb{C}_r \cong \mathbb{C}$ the r-fold cyclic cover of $\mathbb{C}$ branched at 0, we then have an action of G_r on $\mathscr{H} \times \mathbb{C}_r$, which embeds G_r as the orbit of $(i, 1)$, where 1 is a selected point in $\mathbb{C}_r$ lying above $1 \in \mathbb{C}$. Let $\Gamma \subset G_r$. We have $\Gamma \backslash G_r = M_\Gamma = \partial V_\Gamma$, where V_Γ is an analytic surface with a single normal singularity at a point P [9], [10] obtained from $\mathscr{H} \times \mathbb{C}_r/\Gamma$. Moreover, it is shown in [38] that there is a holomorphic 2-form $\tilde{\omega}$ on $\mathscr{H} \times \tilde{\mathbb{C}}^*$, invariant under G_∞. This induces the canonical framing $\mathscr{C}$ on the link of P, which is M_Γ, and by the G_∞-invariance of $\tilde{\omega}$ we have that $\mathscr{L}$ and $\mathscr{C}$ are the same framing on M_Γ, up to homotopy. The surface V_Γ has a $\mathbb{C}^*$-action, so the graph of its canonical equivariant resolution $\tilde{V}_\Gamma$ is a star and it is fully determined by the Seifert invariants of $M_\Gamma = \Gamma_\infty \backslash G_\infty \cong \Gamma \backslash G_r$.

(2.4) PROPOSITION. *Let K denote the canonical class of $\tilde{V}_\Gamma$ and let m_0 be the coefficient of the central curve E_0 in K. Then*

$$r = -(m_0 + 1) = X(M_\Gamma)/E(M_\Gamma), \tag{1}$$

$$-K \text{ is an effective divisor.} \tag{2}$$

Statement (1) follows from [34] or [29] and (2.3)(2) by direct computation, whilst statement (2) is immediate from (2.3)(2) since $-\lambda_0 \geq 1$ and all $\alpha_{i,j}$ are positive, $1 \leq j \leq q_i$.

(2.5) LEMMA. *If K is the canonical class of $\tilde{V}_\Gamma$ then*

$$K^2 = (r+2)(2-2g-n) - d_0 + n + \sum_{i=1}^{n}\left(\sum_{j=1}^{q_i}(2-d_{i,j}) - m_{i,q_i}\right),$$

where $\{g : d_0 : (\alpha_1, \bar{\beta}_1), \ldots, (\alpha_n, \bar{\beta}_n)\}$ are the Seifert invariants [32] of M_Γ and $d_{i,j}$ are the entries in the continued fraction expansion of $\alpha_i/\bar{\beta}_i$.

Proof. Since the graph of V_Γ is a star we may write

$$K^2 = -m_0^2 d_0 - 2\sum_{i=1}^{n} m_0 m_{i,1} + D^2,$$

where $D^2 = \Sigma_{i=1}^{n} D_i^2$ is the contribution of the branches. From (2.3)(1) we deduce

$$\begin{aligned} D^2 &= \sum_{i=1}^{n}\left\{\sum_{j=1}^{q_i} m_{i,j}(d_{i,j}-2) - m_0 m_{i,1}\right\} \\ &= \sum_{i=1}^{n}\left\{\sum_{j=1}^{q_i}(2-d_{i,j}) - m_{i,1} - m_{i,q_1} + m_0(1-m_{i,1})\right\}. \end{aligned}$$

Thus we obtain

$$K^2 = -m_0^2 d_0 + m_0\sum_{i=1}^{n}(m_{i,1}+1) - \sum_{i=1}^{n}(m_{i,1}+m_{i,q_1}) + \sum_{i=1}^{n}\sum_{j=1}^{q_i}(2-d_{i,j}).$$

Using this equation together with (2.3) again we obtain

$$K^2 = -(r+1)^2 d_0 - (r+2)\{2g-2+n-d_0 r\} + n - \sum_{i=1}^{n} m_{i,q_i}\sum_{i=1}^{n}\sum_{j=1}^{q_i}(2-d_{i,j}),$$

from which (2.5) is immediate. □

The expression in (2.5) can be written much more neatly, as W. D. Neumann has pointed out to us. First note that from the proof of (2.3) we know that

$$m_{i,q_i} + 1 = \lambda_{i,q_i} = -\frac{r-\gamma_i}{\alpha_i},$$

where $\bar{\beta}_i\gamma_i \equiv 1(\alpha_i)$ and $0 < \gamma_i < \alpha_i$. Substituting for m_{i,q_i} we reach his version of (2.5):

$$K^2 = rX(M_\Gamma) - d_0 + 2(2-g) + \sum_{i=1}^{n} \left\{ \sum_{j=1}^{q_i} (2 - d_{i,j}) - \frac{\gamma_i}{\alpha_i} \right\}.$$

(This he deduces directly using §20 of the Annals of Mathematics study number 101 "Three-dimensional link theory and invariants of plane curve singularities" by D. Eisenbud and himself.)

Now he reminds us that, given a pair of coprime integers $(\alpha, \tilde{\beta})$ with $0 < \tilde{\beta} < \alpha$ and $\gamma\tilde{\beta} \equiv 1(\alpha)$ and $1 \leq \gamma < \alpha$,

$$\sum_{j=1}^{q} (3 - d_j) - \frac{\gamma + \tilde{\beta}}{\alpha} = 3d(\alpha, \gamma);$$

where

$$d(\alpha, \gamma) = \frac{1}{\alpha} \sum_{\substack{\zeta^\alpha = 1 \\ \zeta \neq 1}} \frac{\zeta + 1}{\zeta - 1} \cdot \frac{\zeta^r + 1}{\zeta^r - 1}$$

and is a form of the classical Dedekind sum [17]. So we have that

$$K^2 = rX(M_\Gamma) + 2(2 - 2g) - d_0 + \sum_{i=1}^{n} \frac{\tilde{\beta}_i}{\alpha_i} - \left(1 + \sum_{i=1}^{n} q_i\right) + \sum_{i=1}^{n} 3d(\alpha_i, r),$$

giving a very neat expression for $\delta(M_\Gamma)$.

(2.6) THEOREM (W. D. Neumann's version):

$$\delta(M_\Gamma) = \frac{1}{3}\left(r + \frac{1}{r}\right)X(M_\Gamma) + 1 + \sum_{i=1}^{n} d(\alpha_i, r).$$

It remains for us to compute W^2 and Arf(W) for a characteristic W or—what amounts to the same thing—the Rochlin μ-invariant of M_Γ. To do this, choose

$$W = [m_0]E_0 + \sum_{i=1}^{n} \sum_{j=1}^{q_i} [m_{i,j}]E_{i,j},$$

where $[m]$ denotes the reduction of m mod (2). Note that $[m_0] = 0$ if and only if r is odd (2.4) and that $[m_{i,j}] \neq 0$ implies $[m_{i,j-1}] = 0 = [m_{i,j+1}]$, so that W is automatically smooth. We give W. D. Neumann's formula for W^2, because it is a decided improvement on ours, but our derivation.

(2.7) PROPOSITION. (W. D. Neumann)

$$W^2 = \begin{cases} \sum_{i=1}^{n} \{2d(2\alpha_i, r) - d(\alpha_i, r) - q_i\} & \textit{if } r \textit{ is odd;} \\ \sum_{i=1}^{n} \left\{2d\left(\alpha_i, \frac{r}{2}\right) - d(\alpha_i, r) - q_i\right\} + E(M_\Gamma) & \textit{if } r \textit{ is even.} \end{cases}$$

When r is odd $[m_0] = 0$, so that W is the disjoint union of copies of $\mathbb{C}P^1$. Consequently $H_1(W, \mathbb{Z}/2) = 0$ and $\mathrm{Arf}(W) = 0$.

(2.8) THEOREM. *If $\Gamma = \Gamma_\infty \subset \widetilde{Sl_2\mathbb{R}}$ and r, the index of Γ in $\pi^{-1}(\pi\Gamma)$, is odd (where $\pi : \widetilde{Sl_2\mathbb{R}} \to PSl_2\mathbb{R}$ is the canonical projection) then*

$$e(\Gamma\backslash\widetilde{Sl_2\mathbb{R}}, \mathscr{L}) = -\frac{1}{48}\left(r - \frac{1}{r}\right)X(\Gamma\backslash\widetilde{Sl_2\mathbb{R}}) + \frac{1}{8}\sum_{i=1}^{n} \{d(2\alpha_i, r) - d(\alpha_i, r)\};$$

$$\mu(\Gamma\backslash\widetilde{Sl_2\mathbb{R}}, \mathscr{L}) = 1 + \sum_{i=1}^{n} \{2d(2\alpha_i, r) - d(\alpha_i, r)\}$$

where $\{g, d_0; \alpha_1, \ldots, \alpha_n\}$ is the signature of $\pi(\Gamma)$.

If r is even W is again smooth, but now $\mathrm{Arf}(W) = \mathrm{Arf}(E_0)$. Let us write $r = 2s$, let ν denote the normal bundle of the central curve E_0 in $\tilde{V}_\Gamma$ and let $L_1, \ldots, L_n$ be the line bundles over E_0 determined by the intersection points of E_0 with the curves $E_{i,1}$, $1 \le i \le n$. We know from (2.3) that each $E_{i,1}$ appears in K with negative even multiplicity, say $-2s_i$.

(2.9) THEOREM. *Let $L = \nu^s \otimes L_1^{s_1} \otimes \cdots \otimes L_n^{s_n}$. Then*

$$\mathrm{Arf}(E_0) = \dim H^0(E_0, L^{-1}) \mod (2).$$

From Theorems (2.9), (2.6) and (2.7) we have the computation of $e(\Gamma\backslash\widetilde{Sl_2\mathbb{R}})$ when r is even.

(2.10) THEOREM. *If $\Gamma = \Gamma_\infty \subset \widetilde{Sl_2\mathbb{R}}$ and r, the index of Γ in $\pi^{-1}(\pi\Gamma)$, is even then*

$$e(\Gamma\backslash\widetilde{Sl_2\mathbb{R}}, \mathscr{L}) = -\frac{1}{48}\left(r - \frac{2}{r}\right)X(\Gamma\backslash\widetilde{Sl_2\mathbb{R}}) + \frac{1}{8}\sum_{i=1}^{n} \left\{d\left(\alpha_i, \frac{r}{2}\right) - d(\alpha_i, r)\right\}$$
$$+ \dim H^0(E_0, L^{-1});$$

$$\mu(\Gamma\backslash \widetilde{Sl_2}\mathbb{R}, \mathscr{L}) = 1 + E(\Gamma\backslash \widetilde{Sl_2}\mathbb{R}) + \sum_{i=1}^{n} \left\{2d\left(\alpha_i, \frac{r}{2}\right) - d(\alpha_i, r)\right\}$$
$$+ 8 \dim H^0(E_0, L^{-1}).$$

Computations of μ for special cases may be found in [30].

We shall prove proposition (2.7) in a rather dull way by using the lemma below. If α and r are coprime positive integers with $\tilde{\beta}r \equiv 1(\alpha)$ and $0 < \tilde{\beta} < \alpha$, let us write

$$d'(\alpha, r) = \begin{cases} 2d(2\alpha, r) - d(\alpha, r), & \text{if } r \text{ is odd} \\ 2d\left(\alpha, \dfrac{r}{2}\right) - d(\alpha, r) + \dfrac{\tilde{\beta}}{\alpha}, & \text{if } r \text{ is even.} \end{cases}$$

Define $d'(\alpha, 0) = 0$.

(2.11) LEMMA. *If* $\alpha = d_1\tilde{\beta} - \alpha_2$, *with* $\alpha_2 > 0$, *and* $r\tilde{\beta} - s\alpha = 1$ *then* $d'(\alpha, r) = 1 - [s + 1]d_1 + d'(\tilde{\beta}, s)$.

Proof. A trivial calculation establishes that when $r = 1$ then s is zero and $d'(\alpha, r) = 1 - \alpha$.

When r is 2 we may write $\alpha = 2k + 1$ and then $\tilde{\beta} = k + 1$, $s = 1$, $d_1 = 2$, $\alpha_2 = 1$ and $d_2 = k + 1$. Direct computation shows that $d'(\alpha, 2) = 1 - k - (k + 1)/\alpha + (k + 1)/\alpha = 1 - k = 1 + d'(k + 1, 1)$.

The general result now follows by induction upon r and breaks up into three cases: (i) r odd, s even, (ii) r odd, s odd and, (iii) r even, s odd. In the course of the proof one notes that $s = -\lambda_1$ and that if s is odd then the parity of λ_2 depends upon that of d_1. (It is, of course, essential that $0 \le s < r$—which follows from the relation $r\tilde{\beta} - s\alpha = 1$.) $\square$

(2.12) COROLLARY.

$$d'(\alpha, r) = -\sum_{j=1}^{q} [\lambda_j + 1]d_j + q,$$

where $\tilde{\beta}r \equiv 1(\alpha)$, $0 < \tilde{\beta} < \alpha$, $\tilde{\beta}r - \alpha s = 1$ *and* $[[\overline{d_1, \ldots, d_q}]]$ *is the continued fraction expansion for* $\alpha/\tilde{\beta}$, *and* $\lambda_0 = -r$, $\lambda_1 = -s$ *with* $\lambda_{j+1} = d_j\lambda_j - \lambda_{j-1}$ *if* $j \ge 1$.

The formulae of Proposition 2.7 are now direct consequences of this corollary and the definition of $d'(\alpha, r)$, since

$$W^2 = -[r+1]d_0 - \sum_{i=1}^{n} \sum_{j=1}^{q_i} [\lambda_{i,j}+1]d_{i,j}.$$

We are left with establishing Theorem 2.9. The proof is implicit in the article by S. Ochanine [31]. Any hermitian manifold has natural spin^c-structure with the 'determinant' bundle given by the dual of the canonical bundle K_M. The complete set of spin^c-structures is in bijective correspondence with $H^2(M, \mathbb{Z})$; the group of topological line bundles over M. If L is such a bundle the 'determinant' of the spin^c-structure is $K_M \otimes L^{-2}$. When M is complex analytic we may argue holomorphically. Suppose that W is a smooth divisor (without multiplicities) which is characteristic for M. Then $K - W = 2D$ and, taking corresponding line-bundles, we have

$$\mathscr{K}_m = \mathscr{L}_W \otimes \mathscr{L}_D^{-2}.$$

The manifold W is characteristic for the spin^c-structure with determinant bundle $\mathscr{L}_W$. It has a spin^c-structure with zero determinant: that is a spin-structure. This spin-structure is determined by K_M and $[\mathscr{L}_D]$. When $\dim_{\mathbb{C}} M = 2$ this is easy to see, for spin-structures on W correspond to square roots of T_W [4]. By the adjunction formula,

$$\nu_W \cong \mathscr{L}_W$$

and hence $TW = L^2TM| \otimes \mathscr{L}_W| = \mathscr{K}_M^*| \otimes \mathscr{L}_W| = \mathscr{L}_D|^2$, where the vertical bar denotes restriction to W.

This argument applies to open manifolds or manifolds with boundary provided K and W are compact and do not cut the boundary. This is the situation with $\tilde{V}_\Gamma$. Let $\mathscr{L}_0$ and $\mathscr{L}_i$, $1 \le i \le n$ be the line bundles defined over $\tilde{V}_\Gamma$ by the divisors E_0 and $E_{i,1}$, $1 \le i \le n$. Write $r = 2s$ so that $m_0 = -(2s+1)$, and let $\mathscr{D}$ be the bundle corresponding to the outer divisor $\cup_{i=1}^{n} \cup_{j=2}^{q_i} E_{i,j}$. From the adjunction formula we have

$$\Lambda^2 T^* \tilde{V}_\Gamma \cong \mathscr{L}_0^{-(2s+1)} \otimes \mathscr{L}_1^{-2s_1} \otimes \cdots \otimes \mathscr{L}_n^{-2s_n} \otimes \mathscr{D},$$

where we write $m_{i1} = -2s_i$, and

$$TE_0 \cong \Lambda^2 T\tilde{V}_\Gamma| \otimes \mathscr{L}_0^{-1}.$$

From these we see immediately that $TE_0 \cong \mathscr{L}_0^{2s} \otimes \mathscr{L}_1^{2s_1} \otimes \cdots \otimes_n^{2s_n}|$ since $\mathscr{D}$ is canonically trivial on E_0. The bundle ξ determines a spin^c-structure on $\tilde{V}_\Gamma$ with determinant $\Lambda^2 T^* \tilde{V}_\Gamma \otimes \xi^{-2}$ and, restricted to E_0, we have $\eta| \cong \mathscr{L}_0| \cong \nu$, the normal bundle. Consequently the spin^c-structure determined by ξ gives a holomorphic square root of TE_0: namely $L = \xi^{-1} \otimes \mathscr{L}_0|$. The result now follows from [31] since $\dim H^0(E_0, L^{-1}) = \mathrm{Arf}\,(E_0)$.

§3 The solvable groups

(a) Up to isomorphism, there are two simply connected, solvable, non-nilpotent Lie groups of dimension 3, which admit cocompact discrete subgroups. These are $\tilde{E}^+(2)$, the universal cover of the group $E^+(2)$ of orientation preserving isometrics of the euclidean plane, and the inhomogeneous Lorentz group $E(1, 1)$. We consider the latter first. The computations to be made in this case depend upon the work of F. Hirzebruch in [15], where he identifies the quotients $\Gamma \subset G$ with the boundaries of the cusps (p. 194), constructs a resolution of the singularities and calculates several numerical invariants. The group $G = E(1, 1)$ is a semi-direct product of $\mathbb{R}^2$ and $\mathbb{R}$, multiplication being defined by

$$(x, t)(x', t') = (x + x'X^t, t + t');$$

where $x, x' \in \mathbb{R}^2$, $t, t' \in \mathbb{R}$ and X is a suitably chosen matrix in $SL_2(\mathbb{Z})$. Further, X has real eigenvalues e^α, $e^{-\alpha}$ for some $\alpha > 1$ and X^t denotes X to the power t, so it has eigenvlaues $e^{\pm\alpha t}$. Since $w = e^\alpha$ is an eigenvalue of $X \in SL_2(\mathbb{Z})$, it is an algebraic integer in the quadratic field $k = \mathbb{Q}(w)$: and not just an integer, but in fact a generator of U_k^+ the positive units. Now $M = \mathbb{Z} \cdot w + \mathbb{Z} \cdot 1$ is a complete $\mathbb{Z}$—module of k. Let $U_M^+ \cong \mathbb{Z}$ be the totally positive units in k that preserve M, see [15: pp. 214, 215], and let $V \subset U_M^+$ be a subgroup of finite index $l \geq 1$. Then V acts on M in the natural way and we may form the semi-direct product $\Gamma = M \rtimes \mathrm{V}$; Γ is a discrete subgroup of $E(1, 1)$. Up to isomorphism, every cocompact discrete subgroup of $E(1, 1)$ is of this form.

The number w above is a quadratic irrational, see [15: p. 215], and its expression as an infinite continued fraction is purely periodic: if we express w as

$$w = d_1 - \frac{1}{d_2} - \ddots$$

then $d_i = d_{i+r}$ for every $i \geq 1$ and for some (smallest) $r \geq 1$. We thus write $w = [\overline{d_1, \ldots, d_r}]]$ following [15].

(3.1) THEOREM. *Let $\Gamma = M \times V$ be a cocompact discrete subgroup of the solvable group $E(1, 1) = G$, with $M = \mathbb{Z} \cdot W + \mathbb{Z} \cdot 1$, $w = [[\overline{d_1, \ldots, d_r}]]$, and $V \subset U_M^+ \cong \mathbb{Z}$ a subgroup of index $l \geq 1$. Then $(\Gamma\backslash G, \mathscr{L})$ represents the homotopy element,*

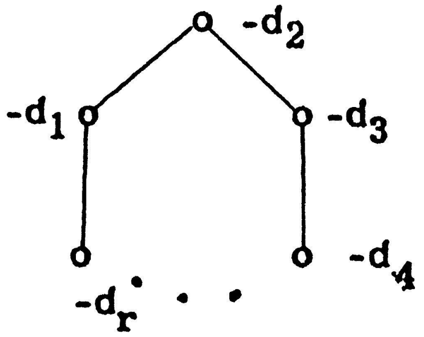
$$\left[3lr - l\sum_{i=1}^{r} d_i + 12\right] \nu \in \pi_3^s.$$

Also $\delta(\Gamma\backslash G, \mathscr{L}) = \frac{1}{3}l(3r - \Sigma_{i=1}^r d_i)$ [15] *and* $\mu(\Gamma\backslash G, \mathscr{L}) = l(3r - \Sigma_{i=1}^r d_i) - 8$.

The proof of this result is along the same lines as for the semi-simple groups. Let H denote the upper half plane of $\mathbb{C}$ and $H^2 = H \times H$. We identify $G = E(1, 1)$ with the group of affine transformations of H^2 of the form

$$(z, w) \to (t_1 z + a_1, t_2 w + a_2)$$

where $a_1, a_2, t_1, t_2 \in \mathbb{R}$ and $t_1 t_2 = 1$. Then G acts freely on H^2, [15, p. 194] and we can identify G with the orbit of $(i, i) \in H^2$. The quotient $\Gamma\backslash \bar{H}^2 = \mathscr{V}_\Gamma$ is a complex analytic surface with a cusp singularity at ∞, and it is normal [15; p. 202] and Gorenstein [15; p. 233]. Moreover, the action of G on H^2 is by holomorphic transformations, so the framing $\mathscr{L}$ on $\Gamma\backslash G$ (or on G) is compatible with the complex structure on $\mathscr{V}_\Gamma$. Hence the e-invariant of $(\Gamma\backslash G, \mathscr{L})$ is given by formula (1.3), where $\tilde{V}$ is a resolution of the cusp singularity. Such is described in [15]: if $l = 1$, the graph of $\tilde{V}$ is a cycle of rational curves E_j, where the d_i's are given by the continued fraction expansion of w.



(There are two special cases, corresponding to $r = 1, 2$.) If $l > 1$, the cycle just repeats itself l times before closing. The canonical class K of $\tilde{V}$ was determined in [15: p. 224] and so were the self-intersection number K^2 and the Euler characteristic:

$$(3.2)\quad K = \sum_{j=1}^{r} E_j; \qquad K^2 = 2lr - l\sum_{j=1}^{r} d_j; \qquad X(\tilde{V}) = lr.$$

We are left with evaluating the invariant Arf (K) for the cusp singularities.

(3.3) LEMMA. *For the cusp singularities,*

$$\mathrm{Arf}\,(K) = 1$$

(*i.e.* Arf (K) *equals the geometric genus* [23]).

Proof. We know (3.2) that K is $-E = -\Sigma E_j$, so that

$$\mathrm{Arf}\,(K) = \mathrm{Arf}\,(E) \bmod (2)$$

by definition [35]. To evaluate Arf (E) we must first smooth E as in §1.(b): at each crossing point $p_1, \ldots, p_r$ of E, we choose local coordinates so that E is given by $z_1 z_2 = 0$, we remove $\{z_1 z_2 = 0\} \cap D_\epsilon$, where D_ϵ is a small disc in $\tilde{V}$ around p_i, and we attach back $\{z_1 z_2 = t\} \cap D_\epsilon$. Topologically, we are removing two discs from E that intersect at p_i and replacing these by an embedded $S^1 \times I$. Doing this for $i = i = 1, \ldots, r-1$ we obtain a space E' which is (topologically) S^2 with one self-intersection point, p_r. After smoothing E' we get a manifold $\tilde{E}$, diffeomorphic to the 2-torus and Arf $(E) =$ Arf $(\tilde{E})$ by definition. Now, $H_1(\tilde{E}; \mathbb{Z}_2) \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$, so that Arf $(\tilde{E}) = 0$ if and only if q, the corresponding quadratic form, has at least two zeroes (cf. [4]). Thus, we will prove (3.3) by finding a suitable basis $\{C_1, C_2\}$ of $H_1(\tilde{E}; \mathbb{Z}_2)$ and showing that $q(C_1) = q(C_2) = 1$. Choose $\epsilon' > \epsilon > 0$ small and spheres $S_{\epsilon'}$, S_ϵ around the point $p_r \in E'$. We use S_ϵ to smooth E' and obtain $\tilde{E}$, and we look at the intersection $\tilde{E} \cap S_{\epsilon'} \cong E \cap S_{\epsilon'}$. This is the Hopf link, i.e. it consists of two linked circles S_1, S_2, which are homotopic in $\tilde{E}$. We choose $C_1 = S_1$ as one of our generators of $H_1(\tilde{E}; \mathbb{Z}_2)$, and we let D_1 be a small disc in $S_{\epsilon'}$ bounded by C_1. Then the interior of D_1 will intersect $\tilde{E}$ in one point, because S_1 and S_2 have linking number 1. If τ denotes the 'inwards' normal field of C_1 in E, then C_1 is transversal to $S_{\epsilon'}$, so that τ extends to a normal vector field of D_1 in $\tilde{V}$. Hence $q(C_1) = 1$, by definition of q (see [12], [35]). We are thus left with finding the generator C_2 and showing $q(C_2) = 1$. For this we let C be the obvious circle in $E \subset \tilde{V}$ that generates $H_1(\tilde{V}) \cong H_1(E) \cong \mathbb{Z}$. This defines r arcs in $\tilde{E} \cong T^2$ after smoothing. We then join these arcs in $\tilde{E}$ by attaching small intervals, in such a way that the resulting curve $\tilde{C} = C_2$ is a smooth unknotted circle in $\tilde{E}$, generates $H_1(\tilde{V}, \mathbb{Z})$ and restricted to $\tilde{C}$ the tangent bundle of $\tilde{E}$ is a complex subbundle of $T\tilde{V}$. (This we can always assume because $\mathbb{C}P^1$ is 1-connected.) Moreover, with no loss of generality we may assume that the normal bundle of $\tilde{E}$ in $\tilde{V}$ is trivial restricted to $\tilde{C}$. Let v be a nowhere-zero section of $N(\tilde{E})$, the normal bundle, defined on $\tilde{C}$ and push $\tilde{C}$ away with v. Then we 'kill' $[v(\tilde{C})]$ in $H_1(\tilde{V}; \mathbb{Z})$ by performing framed surgery along $v(\tilde{C})$. If $\tilde{V}'$ is the modified manifold then $H_1(\tilde{V}', \mathbb{Z}) = 0$ and $\tilde{C}$ bounds an oriented disc D in $\tilde{V}'$, whose interior does not meet $\tilde{E}$ and it is transversal to $\tilde{E}$ on $\tilde{C}$, i.e. D is a *membrane* for $\tilde{C}$. Let $N(D)$ be the normal bundle of D and let $T(D)$ be its tangent bundle. If τ denotes a unit, normal vector field on $\tilde{C}$ in $\tilde{E}$, then τ defines a trivialization of $N(D)$ over $\tilde{C}$. By definition [12], $q(C) \in \mathbb{Z}/2$ is the obstruction for extending to $N(D)$ the spin-structure on $N(D)|\tilde{C}$ defined by τ. That is, $q(C) = 0$ if and only if $w_2(N(D); \tau) = 0$, where this is the 2nd Stiefel-Whitney class of ND relative to τ. Now, $N(D)$ is isomorphic to TD, and multiplication by the complex number i maps τ into a section $i\tau$ of $TD|\tilde{C}$. Hence $w_2(N(D); \tau) = w_2(TD; i\tau)$. The result now follows because $i\tau$ is tangent to $\tilde{C}$, since $T\tilde{E}|\tilde{C}$ is a complex subbundle; so $w_2(TD; i\tau) = 1$. □

(b) We now consider $G = \tilde{E}^+(2)$. Every discrete subgroup of G with compact quotient is the lifting of some such subgroup in $E^+(2)$. These are $\mathbb{Z}^2$, the triangle groups (2, 3, 6), (2, 4, 4), (3, 3, 3) and the quadrangle group (2, 2, 2, 2). The corresponding quotient M_Γ is a Seifert manifold with Seifert invariants $\{g; 0\}$ for $\mathbb{Z}^2$, and $\{0; 2; (\alpha_1, \alpha_1 - 1), \ldots, (\alpha_n, \alpha_{n-1})\}$, where n is 3 or 4 and $(\alpha_1, \ldots, \alpha_n)$ is one of the above triples (if $n = 3$) or quadruple (if $n = 4$). (When Γ is $\mathbb{Z}^2$, the corresponding quotient $M_\Gamma = \Gamma\backslash G$ is the torus T^3, but its framing $\mathscr{L}$ induced from G is not its framing $\mathscr{L}$ as a Lie group. The latter corresponds to the abelian case and represents the element in π_3^s with e-invariant $\frac{1}{2}$, while T^3 represents 0 with its framing coming from $E^+(2)$, the difference being determined by the difference in their corresponding Arf invariants.) The value of the δ-invariant is again well-known [21] [27].

(3.4) THEOREM. *Let $\Gamma \subset G$ be as above. Then $(M_\Gamma, \mathscr{L})$ represents the element*

$$\left\{2 - 2g - n + \sum_{i=1}^{n} \alpha_i\right\}\nu \in \pi_3^s.$$

(*Here* $8(M_\Gamma) = \frac{1}{3}\chi(\tilde{V}_\Gamma) - 1$ *and* $\mu(M_\Gamma, \mathscr{L}) = \chi(\tilde{V}_\Gamma) - 1$.)

Proof. The group $G + E^+(2)$ acts on $\mathbb{C} \cong \mathbb{R}^2$ in the obvious way, and this action extends by differentiation to an action on $T\mathbb{C} = \mathbb{C} \times \mathbb{C}$:

$$g \cdot (z, w) \to (g(z), g'(z) \cdot w)$$

This action is free away from the line $z = 0$, where it has fixed points (except for $\Gamma = \mathbb{Z}^2$ when there is no fixed point). The quotient $\Gamma\backslash T\mathbb{C}$ is a complex analytic surface, with 0, 3 or 4 singular points, as the case may be, all contained in the zero-section $\Gamma\backslash\mathbb{C}$, which is either a torus T^2, if $\Gamma = \mathbb{Z}^2$, or $\mathbb{C}P^1$. At each of these singular points $\mathscr{V}_\Gamma$ is of the form $\mathbb{Z}_{\alpha_i}\backslash\mathbb{C}^2$, for some appropriate α_i. We embed G in

$T\mathbb{C}$, using the G-action, and identify M_Γ with the boundary of $\Gamma\backslash DT(\mathbb{C})$, where $DT(\mathbb{C})$ denotes the unit disc bundle. If we resolve the singularities of $\mathscr{V}_\Gamma$ over $\Gamma\backslash\mathbb{C}$, we then have M_Γ expressed as the boundary of the resolution $\tilde{V}_\Gamma$, which is a complex manifold.

Now observe that the holomorphic 2-form

$$\omega = dz \wedge dw$$

on $T\mathbb{C}$ is globally defined and G-invariant; so it descends to $\mathscr{V}_\Gamma$, and it is nowhere zero there, except at the singular points. Thus, ω defines a canonical framing $\mathscr{C}$ on M_Γ, just as before, and this framing is $\mathscr{L}$, up to homotopy, because ω is G-invariant. We are in a position to apply 1.1 to $\tilde{V}_\Gamma$.

(3.5) LEMMA. *Let $K \in H_2(\tilde{V}_\Gamma, \mathbb{Z})$ represent the Chern class of $\check{V}_\Gamma$ relative to $\mathscr{L}$. Then $K = 0$.*

If we assume this lemma, (3.4) is now straightforward: since $K = 0$, $\mathrm{Arf}\,(K) = 0$, so the e-invariant is determined by $X(\tilde{V}_\Gamma)$. If $\Gamma = \mathbb{Z}^2$, $\tilde{V}_\Gamma = \Gamma\backslash\mathbb{C} \times D$, so $X(\tilde{V}_\Gamma) = 0$, and $e(M_\Gamma, \mathscr{L}) = 0$, as claimed in 3.4. For the other cases we just look at the graph of $\tilde{V}_\Gamma$. All vertices are $\mathbb{C}P^1$'s, so that $e(M_\Gamma, \mathscr{L}) = \frac{1}{24}(2 +$ number of vertices away from central curve) and we arrive at 3.4 by inspection. □

We now prove 3.5: that is, $C_1(\tilde{V}_\Gamma, \mathscr{L}) \in H^2(\tilde{V}_\Gamma, M_\Gamma : \mathbb{Z})$ vanishes. For this we use the definition of $C_1(\tilde{V}_\Gamma, \mathscr{L})$ as an obstruction class. We handle $\Gamma = \mathbb{Z}^2$ first. The form ω is nowhere zero on $\tilde{V}_\Gamma$, hence $C_1(\Lambda^2 T\tilde{V}_\Gamma, \mathscr{L}) = 0$, so $C_1(\tilde{V}_\Gamma, \mathscr{L}) = 0$. In the other cases, the form ω is nowhere-zero away from the singular points of $\mathscr{V}_\Gamma$. Thus, when we resolve these points we obtain a form on $\tilde{V}_\Gamma$ with tubular neighbourhoods of the three or four, exceptional sets removed. The class $C_1(\tilde{V}_\Gamma, \mathscr{L})$ is concentrated around these exceptional sets. But each of these is the resolution of a quotient singularity of the form $\mathbb{Z}_p\backslash\mathbb{C}^2$. Thus $C_1(\tilde{V}_\Gamma, \mathscr{L}) = 0$, because the canonical divisor of these quotient singularities is identically 0.

§4 The nilpotent and abelian cases

The result in the nilpotent case has already appeared in [8]. We now give a simple proof of it for completeness.

Let H be the 3-dimensional Heisenberg group, consisting of all real matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

As a manifold, it is $\mathbb{R}^3$. Up to isomorphism of H, any cocompact discrete subgroup $\Gamma_k = \Gamma$ consists of matrices of the form

$$\begin{pmatrix} 1 & n_1 & n_3/k \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where $n_1, n_2, n_3 \in \mathbb{Z}$, for some integer $k \geq 1$. Then $M_k = \Gamma \backslash H$ is the total space of a principal S^1-bundle over T^3 will Euler class $\pm k$, depending on orientations. If we write (a, b, c) for a matrix

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we define a function

$$\theta : H \times \mathbb{C}^2 \to \mathbb{C}^2$$

by $\theta((a, b, c), (z, w)) = (z + a + ib, e^{2\pi i c} w)$. This is not a group action, since

$$\theta[(a, b, c), \theta((\lambda, \mu, \nu), (z, w))] = (z + a + \lambda + i(\mu + b), e^{2\pi i(\nu + c)} w)$$

while

$$\theta[(a, b, c), (\lambda, \mu, \nu), (z, w)] = (z + a + \lambda + i(\mu + b), e^{2\pi i(\nu + c + \mu a)} w).$$

Thus we have an extra factor $e^{2\pi i(\mu a)}$ in the second variable. Still, θ is an action restricted to every disorder subgroup of H, because $\mu a \in \mathbb{Z}$. Moreover, we may define a map

$$\phi : H \to \mathbb{C}^2$$

by $\phi(a, b, c) = \theta((a, b, c), (0, 1)) = (a + ib, e^{2\pi i c})$. Then ϕ maps H onto $\mathbb{C} \times S^1 = T$. Then we can divide H and $\mathbb{C}^2$ by the action of $\Gamma \subset H$, ϕ induces a diffeomorphism

$$\bar{\phi} : \Gamma \backslash H \to \Gamma \backslash T$$

and $\Gamma \backslash T$ is the boundary of $\Gamma \backslash D = \tilde{V}$, $D = \mathbb{C} \times D^2$. The torus $E_0 = \Gamma \backslash \mathbb{C}$ is embedded in $\Gamma \backslash D$ with self intersection $-k$, so we can blow it down to a point and we obtain a normal singularity (V_Γ, P), which is an elliptic singularity [23]. Moreover, for every $h = (a, b, c) \in H$, the map

$$\theta_h : \mathbb{C}^2 \to \mathbb{C}^2$$

given by $\theta_h(z, w) = \theta((a, b, c)\,(z, w))$ leaves invariant the holomorphic 2-form

$$\omega = \frac{dz \wedge dw}{w}.$$

Thus ω descends to a holomorphic 2-form on $\mathscr{V}_\Gamma$ and defines the framing $\mathscr{C}$ on $M_\Gamma = \Gamma\backslash T$. The map $\bar{\phi}$ then pulls $\mathscr{C}$ to the left invariant framing $\mathscr{L}$. Now, the Euler characteristic of $\tilde{V}$ is 0 and the canonical class K is $-E_0$ by the adjunction formula; hence $K^2 = -k$. We now claim $\operatorname{Arf}(K) = 1$, which yields the following result.

(4.1) THEOREM [8]. *If we orient $\Gamma_k\backslash H$ as the boundary of $\tilde{V}$ above, then $(\Gamma_k\backslash H, \mathscr{L})$ represents the homotopy element*

$$-(12 + k)\nu \in \pi_3^s.$$

To prove that $\operatorname{Arf}(K) = 1$ is equivalent to proving that the induced spin structure on K is the product one. Since $K = -E_0$, it follows that E_0 is defined by a holomorphic section of $K^* \cong \Lambda^2 TV_\Gamma$. Thus $K^*|_{E_0} \cong N(E_0)$, the normal bundle, and we have

$$T(E_0) \otimes N(E_0) \cong N(E_0)$$

an isomorphism of holomorphic vector bundle. Hence $T(E_0)$ inherits a holomorphic trivialization, which in turn induces the spin structure on E_0. Therefore it is the product spin structure on E_0, for that is the only trivialization of $T(E_0)$ which is holomorphic. Hence $\operatorname{Arf}(K) = \operatorname{Arf}(E_0) = 1$.

We are only left with the abelian case $G = \mathbb{R}^3$, $\Gamma = \mathbb{Z}^3$. We use the same technique: Define

$$\theta : \mathbb{R}^3 \times \mathbb{C}^2 \to \mathbb{C}^2$$

by $\theta((a, b, c), (z, w)) = (z + a + ib, e^{2\pi i c} w)$. This time θ is a group action and induces a map

$$\bar{\theta} : T^3 \to \Gamma\backslash\mathbb{C}^2 = \mathscr{V}$$

which identifies T^3 with the boundary of $\tilde{V} = \Gamma\backslash D^4 \subset \tilde{\mathscr{V}}$, where D^4 is the unit disc. Again the form $\omega = dz \wedge dw/w$ descends to $\tilde{V}$ and defines a holomorphic 2-form away from the torus $T^2 \subset \tilde{V}$, where it has a pole of order 1. Thus T^2 is $-K$ and the invariance of ω identifies the framings $\mathscr{L}$ and $\mathscr{C}$. Also, $X(\tilde{V}) = 0$ and $K^2 = 0$, because T^2 has trivial normal bundle. Finally Arf (K) is 1 [21] as one can see by the same argument as above.

THEOREM. *The torus T^3, with its left invariant framing $\mathscr{L}$, represents the homotopy element*

$$12\nu \in \pi_3^s.$$

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José A. Seade
Instituto de Matemáticas, UNAM,
Mexico 04510 D.F.
México

and

Tara Institute of Fundamental Research
Homi Bhabha Road, Bombay 400 005
India

Brian Steer
Hertford College
Oxford OX1 3BW
Great Britain

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