On the algebraic hull of an automorphism group of a principal bundle.

Comment. Math. Helvetici 65 (1990) 375–387
0010-2571/90/030375-13$1.50 + 0.20/0

On the algebraic hull of an automorphism group of a principal bundle

ROBERT J. ZIMMER[1]

1. Introduction

Suppose a locally compact group G acts by principal bundle automorphisms of a (continuous) principal H-bundle $P \to M$ where H is a real algebraic group and M is separable and metrizable. Then under the further assumption that the G-action on M is ergodic with respect to some quasi-invariant measure, there is a natural (conjugacy class of an) algebraic subgroup $L \subset H$ associated to the G-action. Namely, there is a smallest algebraic subgroup $L \subset H$, unique up to conjugacy in H, such that there is a measurable G-invariant reduction of P to L, i.e., a measurable G-invariant section of $P/L \to M$. This group L is called the algebraic hull of the action of G on $P \to M$, and has proven to be a quite useful invariant for studying smooth transformation groups. We refer the reader to [9] for an introduction to and discussion of this notion, and to [1], [2], [6], [9], [10] for some examples of applications. In the special situation in which G acts transitively on M, say with stabilizer $G_1 \subset G$, there is an isotropy homomorphism $G_1 \to H$, and the algebraic hull of the action of G on $P \to M$ is simply the algebraic hull (in the usual sense) of the image in H.

The main point of this paper is to prove the following result.

THEOREM 1.1. *Suppose M is compact and G preserves a finite measure on M with respect to which the G-action is ergodic. Suppose further that G is a semisimple group of higher rank (i.e., a finite product $\Pi\, G_i$ where each G_i is the set of k_i-points of a k_i-simple connected k_i-group of k_i-rank at least 2, where k_i is a local field of characteristic 0. For $k_i = \mathbb{R}$, we may also take G_i to be a connected semisimple Lie group with finite center and all simple factors of $\mathbb{R}$-rank at least 2.) If $P \to M$ is a principal H-bundle on which G acts by principal bundle automorphisms, where H is a real algebraic group, then the algebraic hull of the action is a reductive group with compact center.*

---

[1]Partially supported by NSF Grant.

COROLLARY 1.2. *Let G, H be as in Theorem* 1.1 *and $\Gamma \subset G$ a cocompact lattice. Suppose $P \to M$ is a principal H-bundle on which Γ acts by principal bundle automorphisms. If M is compact and the Γ-action on M is ergodic with respect to a finite Γ-invariant measure, then the algebraic hull of the Γ-action on $P \to M$ is reductive with compact center.*

Special cases of Theorem 1.1 have been known before. If H is amenable, then the result follows fairly easily from Kazhdan's property [7, Theorem 9.11]. (In fact, in this case the algebraic hull is compact.) If H does not locally contain any of the simple factors of G, then the result follows by the argument of [10, Theorem 4.5] (which again shows the algebraic hull is compact). If H does not locally contain G itself (which holds automatically if one of the p-adic factors is non-trivial), the conclusion of the theorem was obtained by Stuck [5] under one further assumption. Namely, Stuck assumed that the action of G on M is irreducible, i.e., each simple factor of G acts ergodically. This also enabled Stuck to deduce Corollary 1.2 (assuming the same relation between G and H) for actions of Γ on M which induce to irreducible actions of G, e.g., isometric Γ-actions, or mixing Γ-actions. If the action of G on M is transitive, then as we remarked above, we are dealing with the algebraic hull of a representation of the stabilizer for this transitive action, and since M has a finite invariant measure the fact that the algebraic hull is reductive is established for example in [3]. However, in the non-transitive case where H locally contains a copy of G (e.g., if G is real and $H = SL(n, \mathbb{R})$ for n sufficiently large), the conclusions of Theorem 1.1 and Corollary 1.2 have not been previously established. The technique of proof will depend heavily on the ideas of [3], [7].

If we measurably trivialize the bundle $P \to M$, the action of G on P is given by a measurable cocycle $\alpha : G \times M \to H$. That is, writing $P \cong M \times H$, we have $g(m, h) = (gm, \alpha(g, m)h)$. One can define the algebraic hull for any measurable cocycle [9] and the results of Stuck and those preceding it that we mentioned above in fact hold for all such cocycles. The proof we give of Theorem 1.1, however, depends upon boundedness properties of the cocycle deriving from the fact it is obtained from a continuous action on principal bundle over a compact base. It would be of interest to obtain the conclusion of Theorem 1.1 for measurable cocycles in general. Such a result should, for example, have application to the study of measurable orbit equivalence [7] for actions of semi-direct product groups. (Cf. [7, Chap. 4] and Theorem 4.1 below.)

If the algebraic hull L is semisimple with no compact factors, then the superrigidity theory of [3], [7] implies that the cocycle is essentially given (up to measurable equivalence) by a rational surjection $G \to L$. Thus, Theorem 1.1 combined with superrigidity yields very precise information on the measurable structure for the action of G on P. On the other hand, as in [3] in the transitive case, the superrigidity theorem (applied to the cocycle obtained by composing with the projection of the algebraic hull onto its maximal semisimple factor) is in fact used in the proof of Theorem 1.1.

We expect this result to be of general use in a number of questions regarding transformation groups. Here we indicate a few such applications. In [6], Stuck used his result concerning algebraic hulls and ideas of Hurder and Katok [2] to deduce a vanishing theorem for characteristic classes of foliations with symmetric leaves. Theorem 1.1 allows one to establish Stuck's vanishing theorem more generally.

COROLLARY 1.3. (*via* [6]: *cf.* [2]) *Let $\mathcal{F}$ be an ergodic codimension q foliation of a compact Riemannian manifold M with a holonomy invariant transverse volume density. Assume all leaves are locally isometric to a fixed symmetric space X of non-positive curvaturae each of whose irreducible factors in the de Rham decomposition has rank at least 2. Let $\chi : H^*(\mathcal{G}(q, \mathbb{R}), O(q)) \to H^*(M, \mathcal{F})^*$ be the Weil homomorphism defined by $\mathcal{F}$. Then there is a subgroup $G \subset GL(q, \mathbb{R})$ locally isomorphic to a factor of the isometry group of X such that χ factors through the map $H^*(\mathcal{G}\ell(q, \mathbb{R}), O(q)) \to H^*(\mathcal{G}, G \cap O(q))$ induced by restriction.*

The second geometric application we give is to manifolds admitting a connection preserving action of a semi-direct product. For actions of semisimple groups, obstructions to the existence of such actions in terms of the representation theory of the fundamental group of the manifold are given in [1], [11]. Via Theorem 1.1 we establish in Section 4 below the following result

THEOREM 1.4. *Suppose G is a simple Lie group with finite center and $\mathbb{R}$-rank $(G) \geq 3$, and V_1, V_2 are real vector spaces on which G acts irreducibly, with* $\dim(V_i) \neq 1$ for $i = 1, 2$, and $V_1 \neq (0)$. *Suppose M is a compact manifold with a connection and a volume density and that $G \ltimes V_1$ acts smoothly on M, preserving the connection and the volume. If there is an embedding $\pi_1(M) \hookrightarrow G \ltimes V_2$ with discrete image, then V_1 and V_2 are equivalent G-modules.*

Other results along this line can be derived using Theorem 1.1 and the techniques of [11]. We leave this to the interested reader.

It is a natural conjecture that every volume preserving ergodic action of a semisimple Lie group of higher rank or of a lattice in such a group actually preserves a connection. As a consequence of Theorem 1.1 it follows that there is always a measurable invariant connection. There are a number of situations in which one can deduce that the presence of a measurable invariant geometric structure implies the presence of a smooth one. (See [10], e.g.) Thus one may hope Theorem 1.1 will be useful in making progress on the above conjecture.

2. Exponential cocycles

In this section we discuss some consequences of the multiplicative ergodic theorem [3], [4]. We shall organize the material in a way that will prove convenient for the proof of Theorem 1.1. We refer the reader to [3] for proofs and further discussion.

We recall that if G acts on M and H is a group, a cocycle is a measurable function $\alpha : G \times M \to H$ such that for each g_1, $g_2 \in G$, $\alpha(g_1 g_2, m) = \alpha(g_1, g_2 m)\alpha(g_2, m)$ for a.e. $m \in M$. Two such cocycles α, β are called equivalent ($\alpha \sim \beta$) if there is a measurable $\varphi : M \to H$ such that for each g, $\alpha(g, m) = \varphi(gm)\beta(g, m)\varphi(m)^{-1}$.

DEFINITION 2.1. (i) $\alpha : G \times M \to GL(n, \mathbb{R})$ is called integrable if for each $g \in G$, $\log^+ \|\alpha(g, m)\| \in L^1(M)$. (This is clearly independent of the norm on $\mathbb{R}^n$.)

(ii) α is called quasi-integrable if it is equivalent to an integrable cocycle.

EXAMPLE 2.2. (i) If $P \to M$ is a principal $GL(n)$-bundle on which G acts by automorphisms, then after choosing a measurable trivialization the action is given by a $GL(n)$-valued cocycle. If M is compact we may choose a bounded measurable trivialization. Letting α be the corresponding cocycle we have $x \mapsto \log^+ \|\alpha(g, x\|$ is bounded, and hence L^1. Since any two measurable trivializations define equivalent cocycles, if M is compact any measurable trivialization of P defines a quasi-integrable cocycle.

(ii) If $\alpha : G \times M \to GL(n)$ is a cocycle, a measurable field of linear subspaces on M, $x \mapsto Y(x) \subset \mathbb{R}^n$ is called α-invariant if for each g, $\alpha(g, x)Y(x) = Y(gx)$ for a.e. $x \in M$. Assuming ergodicity of G on M, dim $Y(x)$ will be essentially constant, say r. We may then measurably choose an orthonormal basis $w_1(x), \ldots, w_n(x)$ of $\mathbb{R}^n$ such that $\{w_i(x) \mid 1 \leq i \leq r\}$ is a basis of $Y(x)$. Writing α with respect to this basis we obtain a cocycle $G \times M \to Gl(r)$ by restricting α to $\{Y(x)\}$, and which we denote, somewhat ambiguously, by $\alpha|_Y$. Since the $\{w_i(x)\}$ are an orthonormal basis, it is clear that $\alpha|_Y : G \times M \to GL(r)$ is integrable if α is integrable. Similarly, the quotient cocycle say $\tilde{\alpha} : G \times M \to GL(n-r)$ (representing the mapping $\mathbb{R}^n/Y(x) \to \mathbb{R}^n/Y(gx)$) is also integrable. It follows that for any quasi-integrable β and any β-invariant field of subspaces $\{Y(x)\}$, the cocycles $\beta|_Y$ and $\tilde{\beta}$ are quasi-integrable.

Suppose now $G = \mathbb{Z}$. If $A \subset M$ and $x \in M$, we set $S_A(x) = \{n \in \mathbb{Z} \mid n \cdot x \in A\}$.

DEFINITION 2.3. Suppose $\mathbb{Z}$ acts ergodically on M, preserving a finite measure. A cocycle $\alpha : \mathbb{Z} \times M \to GL(n)$ is called exponential if there are:

(a) for each $x \in M$ a direct sum decomposition $\mathbb{R}^n = W_-(x) \oplus W_0(x) \oplus W_+(x)$ so that $\{W_-(x)\}$, $\{W_0(x)\}$, $\{W_+(x)\}$ are measurable α-invariant fields of subspaces; and

(b) an increasing sequence of measurable subsets, $M_i \subset M_{i+1}$, with $\cup M_i = M$, such that if $A \subset M_i$ for some i and $x \in A$, we have:

$$\text{(i)} \quad \begin{cases} W_0(x) = \left\{ v \in \mathbb{R}^n \;\middle|\; \lim\limits_{\substack{n \in S_A(x) \\ |n| \to \infty}} \frac{1}{n} \log\|\alpha(n, x)v\| = 0 \right\}, \\ W_-(x) = \left\{ v \in \mathbb{R}^n \;\middle|\; \lim\limits_{\substack{n \in S_A(x) \\ |n| \to \infty}} \frac{1}{n} \log\|\alpha(n, x)v\| < 0 \right\}, \\ W_+(x) = \left\{ v \in \mathbb{R}^n \;\middle|\; \lim\limits_{\substack{n \in S_A(x) \\ |n| \to \infty}} \frac{1}{n} \log\|\alpha(n, x)v\| > 0 \right\}. \end{cases}$$

$$\text{(ii)} \quad \begin{cases} W_+(x) = \left\{ v \in \mathbb{R}^n \;\middle|\; \lim\limits_{\substack{n \in S_A(x) \\ n \to -\infty}} \frac{1}{|n|} \log\|\alpha(n, x)v\| < 0 \right\}, \\ W_-(x) = \left\{ v \in \mathbb{R}^n \;\middle|\; \lim\limits_{\substack{n \in S_A(x) \\ n \to +\infty}} \frac{1}{n} \log\|\alpha(n, x)v\| < 0 \right\}. \end{cases}$$

$$\text{(iii)} \quad \begin{cases} W_0(x) \oplus W_+(x) = \left\{ v \in \mathbb{R}^n \;\middle|\; \lim\limits_{\substack{n \in S_A(x) \\ n \to -\infty}} \frac{1}{|n|} \log\|\alpha(n, x)v\| \le 0 \right\}, \\ W_-(x) \oplus W_0(x) = \left\{ v \in \mathbb{R}^n \;\middle|\; \lim\limits_{\substack{n \in S_A(x) \\ n \to +\infty}} \frac{1}{n} \log\|\alpha(n, x)v\| \le 0 \right\}, \end{cases}$$

REMARK 2.4

(i) From (i) and (ii) in 2.3 we see that W_-, W_0, W_+ are uniquely determined if they exist; in particualr, they are independent of the expresison $M = \cup M_i$, for any $\{M_i\}$ which satisfy the conditions of the definition.

(ii) If $\alpha : \mathbb{Z} \times M \to GL(n)$ is integrable, then the multiplicative ergodic theorem implies that α is exponential, where we may take each $M_i = M$.

(iii) Suppose $\alpha \sim \beta$, say $\beta(n, x) = \varphi(nx)\alpha(n, x)\varphi(x)^{-1}$, and that α is exponential. Let $M = \cup M_i$ such that the conditions in Definition 2.3 hold for α. Then we can write $M = \cup N_j$ where $\varphi \mid N_j$ is bounded in $GL(n)$, $N_j \subset N_{j+1}$, and for each j,

$N_j \subset M_{i(j)}$ for some $i(j)$. If $A \subset N_j$, $x \in A$, $n \in S_A(x)$, and $n \in \mathbb{R}^n$, then

$$c_1 \|\alpha(n, x)n\| \leq \|\beta(n, x)\varphi(x)n\| \leq c_2 \|\alpha(n, x)n\|$$

where $c_1, c_2 > 0$ depend only on the bound of $\|\varphi\|$, $\|\varphi^{-1}\|$ on N_j. It follows from (ii), (iii) in Definition 2.3 that β is exponential with the corresponding invariant fields, say V_-, V_0, V_+, being given by

$$V_-(x) = \varphi(x)W_-(x), \quad V_0(x) = \varphi(x)W_0(x), \quad V_+(x) = \varphi(x)W_+(x).$$

(iv) It follows from (ii), (iii) that any quasi-integrable cocycle is exponential.

LEMMA 2.5. [3] *Suppose α is exponential and $Y \subset \mathbb{R}^n$ is α-invariant, i.e., $\alpha(n, x)Y = Y$ for all n, x. Suppose further that $\tilde{\alpha}$, the induced cocycle on $\mathbb{R}^n/Y$, is exponential. (For example, suppose α is quasi-integrable.) Let W_-, W_0, W_+ be the invariant fields for α, $\tilde{W}_-$, $\tilde{W}_0$, $\tilde{W}_+$ the fields for $\tilde{\alpha}$. Then for (a.e.) $x \in M$, there is an exact sequence*

$$0 \to (W_0(x) \cap Y) \to W_0(x) \to \tilde{W}_0(x) \to 0.$$

Proof. [3] Let $p : \mathbb{R}^n \to \mathbb{R}^n/Y$ be the projection. From (ii), (iii) of 2.3, we see that

$$p(W_-(x)) \subset \tilde{W}_-(x), \quad p(W_+(x)) \subset \tilde{W}_+(x),$$

$$p(W_-(x) \oplus W_0(x)) \subset \tilde{W}_-(x) \oplus \tilde{W}_0(x),$$

$$p(W_+(x) \oplus W_0(x)) \subset \tilde{W}_+(x) \oplus \tilde{W}_0(x).$$

It follows that

$$p(W_-(x) \oplus W_0(x)) = \tilde{W}_-(x) \oplus \tilde{W}_0(x)$$

and

$$p(W_+(x) \oplus W_0(x)) = \tilde{W}_+(x) \oplus \tilde{W}_0(x)$$

Therefore, $p(W_0) = \tilde{W}_0$.

LEMMA 2.6. *Suppose A is an abelian group acting on M and $a_0 \in A$. Let $\mathbb{Z}$ act by powers of a_0, and assume that the action is ergodic. Let $\alpha : A \times M \to GL(n)$ be a cocycle such that $\alpha \mid \{a_0^n\} \times M$ is exponential, with invariant fields W_-, W_0, W_+. Then these fields are α-invariant.*

Proof. Let $M = \cup M_i$ be as in Definition 2.3. Fix $a \in A$. Then we can write $M = \cup N_j$, $N_j \subset N_{j+1}$, such that $x \mapsto \alpha(a, x)$ is bounded on each N_j, and for each j, $N_j \subset M_{i(j)}$ for some $i(j)$. For any $x \in M$ we have

$$\alpha(a_0^n a, x) = \alpha(a a_0^n, x),$$

that is,

$$\alpha(a_0^n, ax)\alpha(a, x) = \alpha(a, a_0^n x)\alpha(a_0^n, x)$$

Thus, if $Y \subset N_j$ for some j, $x \in Y$, $n \in S_Y(x)$ and $v \in \mathbb{R}^n$ we have

$$c_1 \|\alpha(a_0^n, x)v\| \leq \|\alpha(a_0^n, ax)(\alpha(a, x)v)\| \leq c_2 \|\alpha(a_0^n, x)n\|$$

where $c_1, c_2 > 0$ depend only on the bound of $\|\alpha(a, x)\|$, $\|\alpha(a, x)^{-1}\|$ for $x \in N_j$. The lemma then follows from (ii), (iii) in Definition 2.3.

3. Proof of Theorem 1.1

The basic use we make of the notion of exponential cocycle is the following.

LEMMA 3.1. *Let H be a connected non-compact semisimple algebraic Lie group and $\pi : H \to Gl(V)$ a finite dimensional non-trivial irreducible (real) representation. Let A be an abelian Lie group and $\sigma : A \to H$ be a homomorphism onto a maximal $\mathbb{R}$-split torus in H. Choose $a_0 \in A$ such that $\pi(\sigma(a_0))$ has all positive eigenvalues with at least one of these being strictly greater than 1. Suppose A acts ergodically and with a finite invariant measure on M, and assume that the restriction of the action to $\{a_0^n\}$ is still ergodic. Let $\alpha : A \times M \to H \ltimes_\pi V$ be a cocycle such that, if $p : H \ltimes_\pi V \to H$ is projection, then $(p \circ \alpha)(a, m) = \sigma(a)\eta(a, m)$, where $\eta(a, m) \in K$ and $K \subset H$ is a compact subgroup that centralizes $\sigma(A)$. Finally, suppose $\mathrm{Ad} \circ \alpha_0$ is exponential where $\alpha_0 = \alpha\,|\{a_0^n\} \times M$ and Ad is the adjoint representation of $H \ltimes V$. Then the algebraic hull of α does not contain V.*

Proof. The Lie algebra of $H \ltimes V$ can be identified (as a vector space) with $\mathfrak{h} \oplus V$, and clearly V is $\mathrm{Ad} \circ \alpha$-invariant. Choose W_-, W_0, W_+ as in Definition 2.3 for the exponential cocycle $\mathrm{Ad} \circ \alpha_0$. We can view W_0 as a measurable map $M \to \mathrm{Gr}\,(\mathfrak{h} \oplus V)$ (the Grassmann variety), and by Lemma 2.6, W_0 is $\mathrm{Ad} \circ \alpha$-invariant. Since $H \ltimes$ V acts tamely on $\mathrm{Gr}\,(\mathfrak{h} \oplus V)$ (cf. [9]) and A acts ergodically on M, the image of W_0 lies in a single $H \ltimes V$-orbit, and thus W_0 can be interpreted as an

α-invariant map $M \to (H \ltimes V)/H_1$ where H_1 is the stabilizer of a point in the orbit containing (almost all of) $W_0(M)$. This is equivalent [9] to saying that the algebraic hull of α is contained in H_1. Therefore, to prove the lemma, it suffices to see that $H_1 \not\supset V$.

Apply Lemma 2.5 to the $\mathrm{Ad} \circ \alpha_0$-invariant subspace V. We can write $V = V_1 \oplus V_0$ where V_0 is the space of $\pi(a_0)$-invariant vectors and V_1 is the space spanned by the eigenvectors of other eigenvalues. By the choice of a_0 we have $V_1 \neq (0)$. Similarly, on $(\mathfrak{h} \oplus V)/V$, the cocycle $(\widetilde{\mathrm{Ad} \circ \alpha_0})$ is just given by

$$\mathrm{Ad}_H\,(p(\alpha(a_0, m))) = \mathrm{Ad}_H\,(\sigma(a_0))\,\mathrm{Ad}_H\,(\eta(a_0, x)).$$

It follows that this cocycle is exponential and that $\tilde{W}_0(x) = \mathfrak{h}_0$ for all x, where $\mathfrak{h}_0$ is the space of $\mathrm{Ad}_H\,(\sigma(a_0))$-invariant vectors in $\mathfrak{h}$. By Lemma 2.5 for a.e. $x \in M$, we have an exact sequence

$$0 \to V_0 \to W_0(x) \to \mathfrak{h}_0 \to 0.$$

To see that $H_1 \not\supset V$, it suffices to see that any such subspace $W_0(x)$ is not $\mathrm{Ad}\,(V)$-invariant. Since $\mathrm{ad}\,(V)(V_0) = 0$, if $W_0(x)$ were $\mathrm{Ad}\,(V)$-invariant we would have $[V, \mathfrak{h}_0] \subset V_0$ (where the bracket is in the Lie algebra $\mathfrak{h} \oplus V$). However, since a_0 is contained in a 1-parameter subgroup whose Lie algebra clearly lies in $\mathfrak{h}_0$, and $V_1 \neq (0)$, this is clearly impossible.

We will also need the following simple fact.

LEMMA 3.2. *Suppose $\alpha : G \times M \to GL(n)$ is an integrable cocycle, and that $\alpha \sim \beta$ where $\beta(G \times M) \subset H \subset GL(n)$, and H is algebraic. Suppose $N \subset H$ is a closed normal subgroup and let $\gamma : G \times M \to GL(\mathfrak{h}/\mathfrak{n})$ be the cocycle induced by* $\mathrm{Ad}_H \circ \beta$ *acting on $\mathfrak{h}$. Then γ is quasi-integrable.*

Proof. Let Ad be the adjoint representation of $GL(n)$. Then clearly $\mathrm{Ad} \circ \alpha$ is integrable. We have $\mathrm{Ad} \circ \alpha \sim \mathrm{Ad} \circ \beta$, $(\mathrm{Ad} \circ \beta)(g, x) \mid \mathfrak{h} \subset \mathcal{G}\ell(n)$ is just $(\mathrm{Ad}_H \circ \beta)(g, x)$, and the result follows by example 2.2(ii).

The proof of Theorem 1.1 now follows closely the proof of [7, Theorem 5.2.5]. We shall indicate the new ingredients, but refer to [7] for a number of points. For simplicity, we shall also assume that the real part of G acts ergodically and irreducibly. The technical modifications necessary to remove these hypotheses are the same as those in [8]

Proof of Theorem 1.1. We may suppose $\alpha : G \times M \to H$ is a quasi-integrable cocycle whose algebraic hull is H, where $H \subset GL(n)$ is a real algebraic group. By possibly passing to a finite cover of M, we may assume H is Zariski connected by

[7, Proposition 9.2.6]. Write $H = L \ltimes U$ where L is reductive and U is unipotent. The composition of α with projection of H onto $H/[L, L] \ltimes U$ is a cocycle into an abelian algebraic group whose algebraic hull is the whole group. Since G has Kazhdan's property, it follows from [7, Theorem 9.1.1] that $H/[L, L] \ltimes U$ is compact, and hence that the center $Z(L)$ is compact. To prove the theorem, we need to show $U = \{e\}$. We may assume that L itself is not compact, for otherwise [7, Theorem 9.1.1] applies again to show that H itself is compact. If $U \neq \{e\}$, we may write $U/[U, U] = V \oplus V'$ where V, V' are L-invariant vector groups, $V \neq (0)$, and the representation π of L on V is irreducible. Projecting H to $H/\ker \pi \times [U, U]V'$, we obtain a cocycle $\alpha_1 : G \times M \to L_0 \ltimes V$ whose algebraic hull is $L_0 \ltimes V$, where $L_0 = L/\ker \pi$. Let $L = ZS$ where S is semisimple with no compact factors, Z is compact and centralizes S, and the product is almost direct. The representation $\pi \mid S$ of S on V must be non-trivial, for otherwise S is normal in $L_0 \ltimes V$, and projecting α_1 to $L_0 \ltimes V/S$ we would obtain a cocycle whose algebraic hull is a non-compact amenable group, which is impossible by another application of [7, Theorem 9.1.1]. We may lift all actions and cocycles to $\tilde{G}$, which we take in the usual sense for Lie groups, and in the algebraic sense for p-adic groups. By applying the superrigidity theorem for cocycles ([7, Theorem 5.2.5]; see also [8] for the result with precisely our present hypotheses) to the projection of α_1 to L_0, we deduce that this projection is equivalent to a cocycle $\beta : \tilde{G} \times M \to ZS$ of the form $\beta(g, m) = \eta(g, m)\sigma(g)$, where $\eta(g, m) \in Z$ and $\sigma : \tilde{G} \to S$ is a surjective homomorphism (which factors to a rational homomorphism of the maximal algebraic factor of $\tilde{G}$).

Let $A \subset \tilde{G}$ be a maximal abelian subgroup with $\mathrm{Ad}_{\tilde{G}}(A)$ a maximal split torus in $\mathrm{Ad}_G(G)$. We may choose $a_0 \in A$ such that $\pi(\sigma(a_0))$ has all positive eigenvalues with at least one of these being strictly greater than 1. The action of A on M is ergodic by Moore's theorem [7]. By Lemma 3.2, $\mathrm{Ad}_{L \ltimes V} \circ \alpha_1$ is quasi-integrable. We may therefore apply Lemma 3.1. We deduce that the algebraic hull, say H_1, of $\alpha_1 \mid AZ(\tilde{G}) \times M$ does not contain V. Now replacing A by its image in G, we have deduced that there is an $\alpha_1 \mid AZ(G) \times M$-invariant map $M \to (L_0 \ltimes V)/H_1$. Let $\tilde{\alpha}_1 : G \times (M \times G/AZ(G)) \to L \ltimes V$ be the cocycle $\tilde{\alpha}_1(g, (x, y)) = \alpha_1(g, x)$ for the diagonal action of G on $M \times G/AZ(G)$. Then (cf. [7, Section 4.2]) there is an $\tilde{\alpha}_1$-invariant map $M \times G/AZ(G) \to (L_0 \ltimes V)/H_1$. Reorganizing our notation somewhat, we see that it suffices to prove the following lemma.

LEMMA 3.3. *Let G, M, A be as above. Suppose L is an algebraic group and $\pi : L \to GL(V)$ a faithful irreducible representation. Suppose $\alpha : G \times M \to L \ltimes V$ is a cocycle and that there is a measurable $\tilde{\alpha}$-invariant map $\varphi : M \times G/AZ(G) \to (L \ltimes V)/H$ where H is an algebraic subgroup that does not contain V. Then the algebraic hull of α is not equal to $L \ltimes V$.*

The proof of Lemma 3.3 is basically the same as that of [7, Theorem 5.2.5]. Namely, suppose the algebraic hull is $L \ltimes V$. Then the proof of Step 2 in [7, p. 104] shows that for a.e. $m \in M$, $\varphi_m(y) = \varphi(m, y)$ defines a rational function on $G/AZ(G)$. The proof of Step 3 in [7, p. 105], modulo one point which we discuss imminently, shows that replacing α be an equivalent cocycle we can assume φ_m is independent of m. The only additional point that needs to be seen is that the result [7, Proposition 3.3.2] on spaces of rational mappings $V \to \mathbb{P}^n$ holds for quasi-projective varieties, not just projective varieties as in [7]. This is needed here because our functions are defined on $G/AZ(G)$ which is only quasi-projective, in contrast to the proof of [7, Theorem 5.2.5] where they are defined on G/P where P is parabolic. However, any quasi-projective variety Y is determined by the pair $(\bar{Y}, \bar{Y} - Y)$ of projective varieties and hence it is easy to modify the proof of [7, Proposition 3.3.2] to cover the quasi-projective case as well. Finally, the proof of [7, Lemma 5.2.8] shows that α is in fact given by a rational homomorphism $G \to L \ltimes V$ (cf. [7, pp. 106–107] for the p-adic case) and hence would have algebraic hull contained in L, providing the contradiction. The central point here is that what is necessary for the proof of [7, Lemma 5.2.8] to work is that the intersection of the conjugates of H in $L \ltimes V$ be trivial. However, since π is irreducible any normal subgroup not containing V must be contained in L, and since π is faithful this subgroup must be trivial. This completes the proof.

4. Application to fundamental groups and semi-direct products

In this section we prove Theorem 1.4, and in fact prove a somewhat stronger assertion. We first prove a superrigidity type result for cocycles into semi-direct products. We shall use both Theorem 1.1 and the techniques involved its proof.

THEOREM 4.1. *Suppose G is a connected semisimple Lie group with finite center, $\mathbb{R}$-rank$(G) \geq 3$, and all simple factors have $\mathbb{R}$-rank at least 2. Suppose (π, V) is a finite dimensional G-module (possibly with trivial action) and $G \ltimes V$ acts on M so as to preserve a finite measure. Suppose further that G acts irreducibly on M, i.e., all simple factors act ergodically. Let H be a connected semisimple real algebraic group with no compact factors and let (ρ, W) be a finite dimensional H-module. Suppose $\alpha : M \times (G \ltimes W)$ is a quasi-integrable cocycle (for some embedding of $H \ltimes W$ in $GL(n)$) and that the projection of α onto H has algebraic hull H. Then α is equivalent to a cocycle independent of M, i.e., it is given by a homomorphism $G \ltimes V \to H \ltimes W$ of the form $(g, v) \mapsto (\sigma(g), T(v))$, where σ is a smooth surjection and $T : V \to W$ is linear and intertwines the representations (π, V) and $(\rho \circ \sigma, W)$.*

Proof. By superrigidity for cocycles [7, Theorem 5.2.5] (and [8] for the exact form we need), by possibly passing to a finite cover of G we can assume α is of the form

$$\alpha(m, (g, v)) = (\sigma(g), \beta(m, g, v))$$

where $\sigma : G \to H$ is a smooth surjection. By Theorem 1.1 we can find $\varphi : M \to H \ltimes W$ such that $\varphi(m)\alpha(m, (g, v))\varphi(m(g, v))^{-1}$ has a vanishing component in W whenever $v = 0$. Writing $\varphi(m) = (\varphi_1(m), \varphi_2(m)) \in H \ltimes W$, one easily checks simply by multiplying that the same is true if we modify α by φ_2 instead of φ. The result of this is that we may assume α is of the form $\alpha(m, (g, v)) = (\sigma(g), \delta(m, g, v))$ where $\delta(m, g, v) = 0$ if $v = 0$.

From the identity

$$\alpha(m, (g, v)) = \alpha(m, (g, 0))\alpha(mg, (e, v)),$$

we obtain

$$(\sigma(g), \delta(m, g, v)) = (\sigma(g), 0)(e, \alpha(mg, v)),$$

since $\alpha(mg, v) \in W$. Therefore

$$\delta(m, g, v) = \alpha(mg, v),$$

and we have

$$\alpha(m, (g, v)) = (\sigma(g), \alpha(mg, v)).$$

We have $gv = (\pi(g)v)g$ in $H \ltimes V$, so we also have

$$\begin{aligned} \alpha(m, gv) &= \alpha(m, (\pi(g)v)g) \\ &= \alpha(m, \pi(g)v)\alpha(m \cdot (\pi(g)v), g) \\ &= \alpha(m, \pi(g)v)\sigma(g) \\ &= (\sigma(g), \rho(\sigma(g))^{-1}\alpha(m, \pi(g)v)). \end{aligned}$$

Therefore, we deduce

$$(*) \quad \alpha(mg, v) = \rho(\sigma(g))^{-1}\alpha(m, \pi(g)v).$$

Now let $A \subset G$ be the maximal $\mathbb{R}$-split torus and fix $A_1 \subset A$ a 2-dimensional sub-torus such that the centralizer $Z_G(A_1) \supset L$, where L is a non-compact simple group. Such a group A_1 exists since $\mathbb{R}$-rank$(G) \geq 3$. Let $v \in V$ be a weight vector for

A (i.e., a simultaneous eigenvector), and $\lambda : A \to \mathbb{R}^\times$ the corresponding weight. Let $A_v = \ker(\lambda) \cap A_1$, so that $\dim A_v \geq 1$, and $A_v \subset G_v$, the stabilizer of v in G. Restricting α to the G-orbit $\pi(G)v$, we obtain a G-map (by (*)) $\alpha : M \times G/G_v \to W$, and we lift this to a G-map $\alpha : M \times G/A_v \to W$. Since A_v is non-compact, G acts ergodically on $M \times G/A_v$ [7, Chapter 2]. The action of G on W (via $\rho \circ \sigma$) is tame, and hence (cf. [9]) the image of α on $M \times G/A_v$ lies in a single G-orbit in W, say G/G_1. Thus, we can view α as a G-map $\alpha : M \times G/A_v \to G/G_1$.

By the proof of [7, Lemma 5.2.9], for almost all $m \in M$ and $y \in G/A_v$, the function $g \mapsto \alpha(m, \pi(g)y)$ is rational on L (using in a basic way the fact that $L \subset Z_G(A_v)$). Fix $y \in G/A_v$ such that this rationality property holds for a.e. $m \in M$. Then α defines an L-map $\alpha : M \times L/L_y \to G/G_1$, which for a.e., $m \in M$ is rational on L/L_y. Thus, letting R be the space of rational maps $L/L_y \to G/G_1$, α defines an L-map $\theta : M \to R$. By [7, Proposition 3.3.2] (and the remarks in the final paragraph), the action of L on R is tame with algebraic stabilizers. By tameness and ergodicity of L on M, we deduce that the image of θ lies in a single L-orbit in R. Thus, θ defines an L-map $\theta : M \to L/L_1$ where L_1 is the stabilizer of a point in R, and in particular is algebraic. Since there is an L-invariant probability measure μ on M, $\theta_* \mu$ will be such a measure on L/L_1, which by the Borel density theorem implies $L = L_1$. In other words, θ is constant, or equivalently, α is independent of M as a map defined on $M \times L/L_y$. Viewing α once again as a map $M \times V \to W$, this implies that for each weight vector $v \in V$, for a.e. $g \in G$ we have that $\alpha(m, \pi(g)v)$ is independent of m.

For each $t \in \mathbb{R}$, tv is also a weight vector, and hence we deduce by Fubini's theorem that for almost all $g \in G$, we have $\alpha(m, \pi(g)(tv)) = \alpha(m, t\pi(g)v)$ is independent of m for almost all $t \in \mathbb{R}$. In particular, we can choose $y \in V$ arbitrarily close to v such that for almost all $t \in \mathbb{R}$, $\alpha(m, ty)$ is independent of m. Since there is a basis of V consisting of weight vectors, we can choose a basis $\{y_i\}$ of V such that each y_i has this property. Given any $z_1, z_2 \in V$, then $\alpha(m, z_1 + z_2) = \alpha(m, z_1) + \alpha(mz_1, z_2)$, so that if $\alpha(m, z_i)$ are both independent of m, the same is true for $\alpha(m, z_1 + z_2)$. By taking finite sums of the form $\Sigma\, t_i y_i$, we see that $\alpha(m, z)$ is independent of m for a.e. $z \in V$. However, we have also just observed that $\{z \in V \mid \alpha(m, z)$ is independent of $m\}$ is a semi-group in V, and being conull it must be equal to V [7, B.1]. Thus, $\alpha : M \times V \to W$ is independent of M, and this completes the proof.

We now prove a somewhat more general version of Theorem 1.4 (cf. [11]).

THEOREM 4.2. *Suppose G is a simple Lie group with finite center and V is a finite dimensional G-module with no invariant vectors. Let W be another finite dimensional G-module. Assume $\mathbb{R}$-rank$(G) \geq 3$ and $\pi_1(G)$ is finite. Let M be a compact manifold on which $G \ltimes V$ acts with finite kernel preserving a connection and a volume density. If $\pi_1(M)$ embeds discretely in $G \ltimes W$, then $V \subset W$ as G-modules.*

REMARKS. (i) The result is true more generally for M any "standard" topological space on which the action of $G \ltimes V$ is topologically engaging. (See [11] for discussion of these conditions.)

(ii) With the assumption of topological engaging, we may allow V to have invariant vectors if we assume that there is a finite ergodic invariant measure for $G \ltimes V$ which is still ergodic upon restriction to G, e.g., if we assume the $G \ltimes V$ action is mixing.

Proof. By [1, Lemma 6.1.A], the action of $G \ltimes V$ is essentially proper on $\tilde{M}$. Viewing $\tilde{M} \to M$ as a principal $\pi_1(M)$-bundle, we form the associated bundle Q with fiber $G \ltimes W$ given by the embedding $\pi_1(M) \hookrightarrow G \ltimes W$. Since this image is discrete, it follows that the action of $G \ltimes V$ on Q is also essentially proper. Choose a finite $G \ltimes V$-invariant and ergodic measure. Since there are no G-invariant vectors in V, the Mackey analysis of unitary representations of semi-direct products together with the Borel density theorem shows that G itself acts ergodically. (Cf. [7, 7.3.4].) By Theorem 4.1, we deduce that the V action on Q is given by a cocycle which we can take to be a linear G-map $V \to W$. Since the V-action is proper, this map must clearly be injective.

REFERENCES

[1] M. GROMOV, *Rigid transformation groups*. In *Geometrie Differentielle*. (Ed. D. Bernard, Y. Choquet-Bruhat) Herman, Paris, 1988.
[2] S. HURDER and A. KATOK, *Ergodic theory and Weil measures for foliations*. Annals of Math., *126*, 121–175 (1987).
[3] G. A. MARGULIS, *Discrete Subgroups of Lie Groups*. Springer (to appear).
[4] V. I. OSELEDEC, A multiplicative ergodic theorem. Translations Amer. Math. Soc., *19*, 197–231 (1968).
[5] G. Stuck, *Cocycles of group actions and vanishing of first cohomology for S-arithmetic groups* (preprint).
[6] G. Stuck, *On characteristic classes of actions of lattices in higher rank Lie groups*, Trans. Amer. Math. Soc., to appear.
[7] R. J. ZIMMER, *Ergodic Theory and Semisimple Groups*. Birkhauser Boston, 1984.
[8] R. J. ZIMMER, *Orbit equivalence and rigidity of ergodic actions of Lie groups*. Erg. Th. Dyn. Syst., *1*, 237–253 (1981).
[9] R. J. ZIMMER, *Ergodic theory and the automorphism group of a G-structure*. In *Group representations, ergodic theory, operator algebras, and mathematical physics*. (Ed. C. C. Moore) pp. 247–278. Springer, New York 1987.
[10] R. J. ZIMMER, *Lattices in semisimple groups and invariant geometric structures on compact manifolds*. In *Discrete groups in geometry and analysis*. (Ed. R. Howe), pp. 152–210 Birkhauser, Boston 1987.
[11] R. J. ZIMMER, *Representations of fundamental groups of manifolds with a semisimple transformation group*. Jour. A.M.S., *2*(1989), 201–213.

Received November 24, 1988