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Locally flat 2-spheres in simply connected 4-manifolds

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Introduction

Let N^{2k} be a closed $(k-1)$-connected manifold. It is known that for $k \geq 3$, each integral homology class $x \in H_k(N)$ can be presented by a locally flat (differentiable, piecewise linear, or topological) embedding $f: S^k \to N, f_*[S^k] = x$, and that for $k \geq 4$, any two such embeddings are isotopic (see [H], [K–S]). Both of these facts have important consequences for the classification of closed manifolds (for the classification of $(k-1)$-connected $2k$-manifolds see [W_1]). It is also known that both statements fail to be true when $k = 2$. Knot theory in dimension 4 provides of course counterexamples to the isotopy statement. First examples of homology classes that cannot be represented by locally flat 2-spheres were found by Kervaire and Milnor [K–M], Tristram [T], Rochlin [R], and Hsiang and Szczarba [H–S]. Several authors investigated the embedding problem in the case of some specific manifolds like $S^2 \times S^2$ and $CP^2 \# \overline{CP}^2$. In these cases it was possible to determine precisely which classes can be represented by embedded 2-spheres and a complete analysis was carried out by Freedman [F_1], Kuga [K], Suciu [S], Fintushel and Stern [F–S], Lawson [L], and Luo [Lu]. In particular, it follows from their results that different classes are representable in the differentiable and topological categories. Among results showing a particular method of embedding a 2-sphere in a simply connected 4-manifold we should also mention papers by Wall [W_3], Boardman [B], and Freedman and Kirby [F–K]. Some aspects of the embedding problem in relation to 4-dimensional surgery have also been considered by Cappell and Shaneson [C–S], Quinn [Q_1], and Freedman [F_2].

The purpose of the present paper is to discuss both questions of existence and uniqueness up to isotopy for locally flat 2-spheres topologically embedded in a simply connected 4-manifold. Concerning the existence, we shall show that two well known obstructions for embedding 2-spheres in 4-manifolds (op. cit.) are essentially the only obstructions for this problem in the topological category. (In the differentiable category, Donaldson theory provides additional obstructions.) The embeddings $f: S^2 \to N^4$ that we are going to construct in the process of proving this statement (Theorem 1.1) will have one additional property: they have an abelian fundamental group of the complement. We shall refer to them as "simple" embeddings.

The significance of this additional condition on the fundamental group becomes clear in the context of the isotopy classification of embedded 2-spheres representing the same homology class. Indeed, given a locally flat embedding $f: S^2 \to N^4$ and a knotted, locally flat 2-sphere $K \subseteq S^4$, we can vary the fundamental group $\pi_1(N - f(S^2))$ by means of the connected sum operation $(N, f(S^2)) \# (S^4, K) = (N, f'(S^2))$. Many nonisotopic embeddings f', representing the same homology class, can be created in this way. Simple embeddings, however, often enjoy the following rigidity property: they are topologically ambient isotopic iff they represent the same homology class. This is roughly the contents of our Theorem 1.2.

The paper is organized in four sections. Our main results are stated in Section 1; there we also introduce some notation and terminology. In Section 2, as a preparation for the results of the next two sections, we prove stable versions of Theorems 1.1 and 1.2. In Section 3, we reformulate our problems in terms of certain finite group actions on 4-manifolds. Working in this equivariant context, we make a reduction of our topological statements to questions about certain hermitian pairings. It is then a purely algebraic task to decide when a simple embedding representing given homology class exists or whether it is unique up to isotopy. The remaining algebra is carried out in Section 4, where we also prove Theorems 1.1 and 1.2.

Though in principle the methods of this paper apply to all homology classes, our results here concern mainly homology classes of odd divisibility. There are some additional complications in the case of even divisibility; these classes must be given special consideration and we plan to address this issue in a subsequent paper.

It is a pleasure to acknowledge our debts to Sylvain Cappell, Jim Davis, John Ewing, Ian Hambleton and Shmuel Weinberger. Conversations with them were important at various stages of this work.

1. Main results

Let N be a closed, oriented, simply connected, topological 4-manifold, and let $\lambda : H_2(N; \mathbf{Z}) \times H_2(N; \mathbf{Z}) \to \mathbf{Z}$ denote the unimodular, symmetric pairing defined by the algebraic intersection number of 2-cycles, $\lambda(x, y) = x \cdot y$. It is a well known result of J. H. C. Whitehead that the oriented homotopy type of N is completely determined by this intersection pairing λ. Moreover, by the work of Freedman [F_1], each unimodular form λ on a free abelian group of finite rank occurs as the intersection pairing of exactly one or two simply connected 4-manifolds N, depending on whether λ is even or odd. In the case of an odd intersection pairing λ, the two nonhomeomorphic 4-manifolds are distinguished by their Kirby–Siebenmann invariants. Recall that the Kirby–Siebenmann invariant of N,

$$KS(N) \in H^4(N; \mathbf{Z}_2) = \mathbf{Z}_2,$$

is the stable obstruction to a differentiable structure on N (cf. [K–S]).

As already observed by Wall [W_3], for the purpose of representing homology classes in $H_2(N)$ by embedded 2-spheres, one has to make a distinction between characteristic and ordinary classes. A homology class x is said to be *characteristic* if its mod 2 reduction $[x]_2 \in H_2(N; \mathbf{Z}_2)$ is Poincaré dual to the Stiefel–Whitney class $w_2(N)$ (i.e. $x \cdot z \equiv \langle w_2(N), z\rangle \bmod 2$ for each $z \in H_2(N)$) and *ordinary* otherwise. For a characteristic class x, $x \cdot x \equiv \sigma(N) \bmod 8$ (see [M–H], Lemma 5.2), where $\sigma(N)$ denotes the signature of N (=signature of λ).

Since $H_2(N)$ is torsion free, each $x \neq 0$ is an integral multiple of some primitive (=indivisible) class $y \in H_2(N)$, $x = dy$. If $x \neq 0$ and $d > 0$, d is called the *divisibility* of x.

THEOREM 1.1. *Let $x \in H_2(N)$ be a class of odd divisibility d. There exists a locally flat, simple embedding $f: S^2 \to N$ representing x if and only if*

(i) $KS(N) = \frac{1}{8}[\sigma(N) - x \cdot x]$ *mod* 2, *when x is a characteristic class, and*
(ii) $b_2(N) \geq \max_{0 \leq j < d} |\sigma(N) - 2j(d-j)(1/d^2)x \cdot x|$.

Both conditions in Theorem 1.1 are known to be necessary (also for d even). Condition (i) is a topological version of a result due to Kervaire and Milnor [K–M]. The proof is essentially the same as that in [K–M], the main ingredient being Rochlin's formula relating the signature of a closed spin 4-manifold to its Kirby–Siebenmann invariant. Inequality (ii) is due to Rochlin [R] (cf. [H–S]) in the differentiable category, and it follows from the G-signature formula applied to a ramified covering of N branched over the embedded 2-sphere. By an argument of Wall [W_4], (ii) holds also for a locally flat topological embedding. The proof of Theorem 1.1 will be completed in Section 4, where assuming (i) and (ii) we construct the required embedding. It should also be pointed out that the existence of a simple embedding representing a primitive class x in Theorem 1.1 can be deduced directly from Freedman and Quinn's results.

THEOREM 1.2. *Let $x \in H_2(N)$ be a class of odd divisibility d, and if $d > 1$ assume that $b_2(N) \neq 2$. If*

$$b_2(N) > \max_{0 \leq j < d} |\sigma(N) - 2j(d-j)(1/d^2)x \cdot x|,$$

then any two locally flat, simple embeddings $S^2 \to N$ representing x are ambient isotopic.

COROLLARY 1.3. *Any two locally flat, simple embeddings $S^2 \to N$ representing a homology class of odd divisibility are ambient isotopic in $N \# (S^2 \times S^2)$.*

We remark that the isotopy statement of Theorem 1.2 holds true in the differentiable category as well provided

$$b_2(N) > \max_{0 \le j < d} |\sigma(N) - 2j(d-j)(1/d^2)x \cdot x| + \varepsilon(d, m)$$

for some integer $\varepsilon(d, m) \ge 0$ which depends only on d and m. The proof is essentially the same; in addition, one only needs to know that the smooth s-cobordism theorem and Quinn's isotopy theorem hold stably, up to connected sum with copies of $S^2 \times S^2$ [Q₁], [Q₂].

2. Stable embeddings

The purpose of this section is to discuss locally flat embeddings $S^2 \to N^4$ in the stable category. Let N be a closed, oriented, simply connected 4-manifold as in Section 1 and let $x \in H_2(N)$. We say that x is *stably represented* if $x \oplus 0 \in H_2(N \# k(S^2 \times S^2))$ is represented by a locally flat embedding $f: S^2 \to N \# k(S^2 \times S^2)$ for some $k \ge 0$. Two stable embeddings $f, f': S^2 \to N_k = N \# k(S^2 \times S^2)$ are *stably homeomorphic* if there is an orientation preserving homeomorphism $h: (N_{k+r}, f(S^2)) \to (N_{k+r}, f'(S^2))$, $r \gg 0$.

For $x \in H_2(N)$, we define $\Theta(x) \in \mathbf{Z}_2$ according to the following rule. If x is characteristic $\Theta(x) = KS(N) + \frac{1}{8}[\sigma(N) - x \cdot x] \bmod 2$, and we set $\Theta(x) = 0$ otherwise.

THEOREM 2.1. *For each homology class $x \in H_2(N)$, x can be stably represented by a (simply embedded) locally flat 2-sphere if and only if $\Theta(x) = 0$.*

As pointed out in Section 1, the necessity of the condition follows from an argument of Kervaire and Milnor [K–M]. Before proving the other direction, let us recall some terminology from [F–K] (see also [K] for a somewhat different treatment of some of these topics).

Let Ω_4^{char} be the topological characteristic bordism gorup of characteristic pairs (N^4, K^2), where N and K are closed and oriented, $K \subseteq N$ is a locally flat surface and $[K] \in H_2(N; \mathbf{Z})$ is characteristic. Two pairs (N, K) and (N', K') are said to be characteristically bordant if there exists a compact oriented 5-manifold $\bar{N}$ and an oriented, locally flat 3-submanifold $\bar{K}^3$ with $[\bar{K}]_2 \in H_3(\bar{N}, \partial\bar{N}; \mathbf{Z}_2)$ dual to $w_2(\bar{N})$ and $\partial(\bar{N}, \bar{K}) = (N, K) \cup -(N', K')$.

Given a characteristic pair (N, K), we can try to perform ambient surgery on K to get an embedded 2-sphere in N. The "obstruction" to such a surgery problem can be briefly described as follows (see [F–K] for details). Let $A_1, \ldots, A_{2s}$ be embedded circles representing the generators of a symplectic basis of $H_1(K) = \mathbf{Z}^{2s}$, and let v_i be a normal vector field to A_i which is tangent to K. Embed a 2-disk D_i in N transversely to K with $\partial D_i = A_i$. Let $d_i \in \mathbf{Z}$ denote the algebraic intersection number of int D_i with K. The obstruction to extending v_i to a normal vector field over D_i is another integer $e_i \in \mathbf{Z} = \pi_1(SO(2))$. Associating $d_i + e_i$ (mod 2) to each A_i, we obtain a quadratic form $q : H_1(K; \mathbf{Z}_2) \to \mathbf{Z}_2$. Let $\varphi(N, K)$ denote the Arf invariant of q. It turns out that φ determines a well-defined homomorphism $\Omega_4^{\text{char}} \to \mathbf{Z}_2$ ([F–K] Lemma 5).

LEMMA 2.2. $\varphi(N, K) = \Theta([K])$.

Proof. It follows from [F–K] and [Hs] that $\alpha : \Omega_4^{\text{char}} \to \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_2$, $\alpha(N, K) = (\sigma(N), \frac{1}{8}[\sigma(N) - K \cdot K], KS(N))$ is an isomorphism. The explicit generators of Ω_4^{char} are $(\mathbf{C}P^2, \mathbf{C}P^1)$, $(\mathbf{C}P^2 \# \overline{\mathbf{C}P}^2, 3\mathbf{C}P^1 \# \overline{\mathbf{C}P}^1)$ and $(|E_8|, \varnothing)$, where $|E_8|$ denotes the simply connected 4-manifold whose intersection pairing is isomorphic to the E_8-lattice. Clearly, $\varphi(|E_8|, \varnothing) = \Theta(0) = 0$. Since φ and Θ agree also on the other two generators ([F–K] Lemma 6), the result follows. □

Proof of Theorem 2.1. Let $x \in H_2(N)$ and assume $\Theta(x) = 0$. We shall show that x can be stably represented by a simply embedded locally flat 2-sphere.

Let $N = W^4 \cup V^4$ where W is smooth and simply connected, V is contractible, and $\Sigma^3 = \partial W = \partial V$ is an integral homology sphere [F$_1$]. By immersion theory, x can be represented by a smoothly embedded surface $f : K \to W$. It was shown in [F–K] that in the case of a characteristic class x, $\varphi(N, K)$ is the only obstruction to ambient surgery on K which results in a stable embedding $S^2 \to N_k$, $k \gg 0$. By Lemma 2.2, $\varphi(N, K) = 0$ and surgery can be performed.

Next assume $x \in H_2(N)$ is an ordinary class. If x is also primitive, then by [W$_3$] Theorem 3 there is a smooth embedding $S^2 \to W_k = W \# k(S^2 \times S^2))$, $k \gg 0$ stably representing x. It remains to consider the case of an imprimitive class x. As in the characteristic case we can represent a basis of $H_1(K)$ by embedded circles $A_1, \ldots, A_{2s}$. To surger some A_i inside W_k, we first find a smoothly embedded 2-disk D_i in W_k with $\partial D_i = D_i \cap K = A_i$. Let $e_i \in \mathbf{Z} = \pi_1(SO(2))$ be the obstruction to extending a normal vector field from A_i to D_i defined as before. (Notice that $d_i = 0$ in this case.)

We claim that D_i can be chosen so that $e_i \in 2\mathbf{Z}$. Suppose this done. We can form the connected sum of pairs

$$(W_k, D_i) \# (S^2 \times S^2, \text{graph of } g_i)$$

where $g_i : S^2 \to S^2$ is a smooth map of degree $-e_i/2$. This results in an embedded disk D_i^* in W_{k+1} with $e_i^* = 0$. We can then replace the normal 1-disk bundle to A_i in K by the boundary of the 1-disk bundle determined by the extended normal vector field on D_i^*. Thus, by ambient surgery on K along A_i, we can reduce the genus of K by 1. Repeating this procedure several times gives eventually an embedded 2-sphere stably representing x.

To prove the above claim, let us suppose $e_i \equiv 1 \bmod 2$. Write $x = dy$ with y a primitive class. Suppose also that λ is an even form, i.e. $z \cdot z \in 2\mathbf{Z}$ for each $z \in H_2(W_k)$. Since x is not characteristic, d must be odd, and consequently y is ordinary. For $k \geq 2$, the orthogonal group $O(\lambda)$ operates transitively on primitive ordinary elements of given square by [W_2]. Hence we can assume that there is a pair of hyperbolic elements $u, v \in H_2(W_k)$, $u \cdot u = v \cdot v = 0$, $u \cdot v = 1$, such that $y = pu + v$ for some $p \in \mathbf{Z}$. As a primitive ordinary class, u can be represented by an embedded 2-sphere S in W_k. Under the connected sum $D_i \# S$, e_i does not change because $u \cdot u = 0$, but $d_i = 0$ does change to $d_i' = d_i + x \cdot u = d$. Also, we may spin $D_i' = D_i \# S$ once around A_i, as in [F–K], p. 87, changing $e_i' = e_i$ to $e_i' \pm 1$ and d_i' to $d_i' \pm 1$. After spinning $|d|$ times we get D_i'' with $e_i'' = e_i \pm d$ and $d_i'' = 0$. Thus $e_i'' \in 2\mathbf{Z}$ as required, but we have achieved that at the expense of introducing the intersection int $(D_i'') \cap K$. We may assume that int (D_i'') intersects K transversely. Since $d_i'' = 0$, we can arrange all the intersection points to occur in pairs with opposite ± 1 indices. For a given pair of points, we can create a Whitney circle by joining the two points by one path in int (D_i'') and another in K. Then, by taking a connected sum with $S^2 \times S^2$ (framed surgery along a nearby curve) we can find an embedded 2-disk in which to perform the Whitney trick. By iteration we can cancel all points of intersection, so that $\partial D_i'' = D_i'' \cap K = A_i$. This proves the claim for an even form λ.

In the odd case, λ can be diagonalized, that is, there is a basis $\{\gamma_i\}$ of $H_2(W_k)$ such that $\gamma_j \cdot \gamma_h = \pm\delta_{jh}$. Then $x = \Sigma\, a_j\gamma_j$, and since x is ordinary, $a_j \in 2\mathbf{Z}$ for some index j. That γ_j can be represented by an embedded 2-sphere S. As before we take $D_i' = D_i \# S$, for which we now have $e_i' = e_i + \gamma_j \cdot \gamma_j \in 2\mathbf{Z}$ and $d_i' = d_i + a_j = a_j$. This new disk D_i' can be spun $|a_j|$-times around A_i to produce D_i'' with $e_i'' = e_i' \pm a_j \in 2\mathbf{Z}$ and $d_i'' = 0$. Finally, we cancel all intersections int $(D_i'') \cap K$ to guarantee that $\partial D_i'' = D_i'' \cap K = A_i$, thereby proving the claim.

As noted before, this shows that $x \in H_2(N)$ can be stably represented. To finish the proof, we need only observe that each embedding $f: S^2 \to N_k$ can be improved to a simple embedding by surgery on embedded circles generating the commutator subgroup of $\pi_1(N_k - f(S^2))$. Since N_k is simply connected, framings can be picked so that each surgery operation replaces N_{k+r} by N_{k+r+1}. □

COROLLARY 2.3. *Suppose N is a spin 4-manifold. For a characteristic class $x \in H_2(N)$, x can be stably represented by a (simple) locally flat 2-sphere if and only if $x \cdot x \equiv 0$ mod 16.*

Proof. Since $KS(N) = \frac{1}{8}\sigma(N) \bmod 2$, we have $x \cdot x \equiv \Theta(x) \cdot 8 \bmod 16$. □

COROLLARY 2.4. *Let N, N' be homotopy equivalent, but nonhomeomorphic, simply connected 4-manifolds. Let $\psi : [H_2(N), \lambda] \to [H_2(N'), \lambda']$ be an isometry of intersection pairings. For each class $x \in H_2(N)$, either x or $\psi(x)$ can be stably represented by a simple, locally flat 2-sphere.*

Proof. $KS(N) \neq KS(N')$ implies that either $\Theta(x) = 0$ or $\Theta(\psi(x)) = 0$. □

THEOREM 2.5. *If $x \in H_2(N)$ is an odd divisibility class, then any two simple embeddings $f_1, f_2 : S^2 \to N$ representing x are stably homeomorphic.*

Proof. According to [F_2] Theorem 10, each locally flat 2-sphere in N has a topological vector bundle neighborhood. Let v_i $(i = 1, 2)$ be such a neighborhood of $f_i(S^2)$. Since v_i is an oriented 2-plane bundle classified by its euler number, there is an oriented homeomorphism $(v_1, f_1(S^2)) \cong (v_2, f_2(S^2))$. Let $X_i = N - v_i$. We wish to extend $\partial X_i \cong \partial X_2$ to an oriented homeomorphism between $X_1 \# k(S^2 \times S^2)$ and $X_2 \# k(S^2 \times S^2)$, $k \gg 0$.

Notice first that X_i admits a spin structure unless N is nonspin and x is not characteristic in which case X_i is a nonspin manifold. In the spin case, ∂X_i inherits the spin structure from X_i. In fact, ∂X_i is homeomorphic to either a lens space or $S^1 \times S^2$ (the latter occurs precisely when $x \cdot x = 0$), so ∂X_i always admits a spin structure and we can impose it arbitrarily when X_i is not spin. Form $X = X_1 \cup_\partial (-X_2)$ by identifying the boundaries via a spin reversing homeomorphism $\partial X_1 \cong \partial(-X_2)$ where $-X_2$ denotes X_2 with the orientation reversed. It follows that X is a spin manifold iff X_i is.

Let d denote as usual the divisibility of x. Since f_i is a simple embedding, $\pi_1(X_i) \cong H_1(X_i) \cong \mathbf{Z}_d$ by Poincaré duality (cf. [H–S] Lemma 3.1). Furthermore, $\pi_1(\partial X_i)$ maps onto $\pi_1(X_i)$, so by Van Kampen $\pi_1(X) \cong \mathbf{Z}_d$. The manifold X determines then a class in the oriented bordism group $\Omega_4(K(\mathbf{Z}_d, 1))$ and in the spin case in $\Omega_4^{\mathrm{Spin}}(K(\mathbf{Z}_d, 1))$.

From the Atiyah–Hirzebruch spectral sequence we immediately see that $\Omega_4^{\mathrm{spin}}(K(\mathbf{Z}_d, 1)) \cong H_0(\mathbf{Z}_d, \Omega_4^{\mathrm{Spin}}) \cong \mathbf{Z}$ is generated by the E_8 manifold. Similarly, $\Omega_4(K(\mathbf{Z}_d, 1)) \cong \mathbf{Z} \oplus \mathbf{Z}_2$ with the cyclic summands detected respectively by the signature and the Kirby–Siebenmann invariant. Now depending on whether $x \cdot x = 0$ or not we have either $\sigma(X_i) = \sigma(N)$ or $\sigma(X_i) = \sigma(N) \pm 1$. In either case $\sigma(X_i)$ depends only on x and not on i. Thus $\sigma(X) = \sigma(X_1) - \sigma(X_2) = 0$. Also $KS(X) = 0$, so $[X] = 0$ in $\Omega_4(K(\mathbf{Z}_d, 1))$ (resp. $\Omega_4^{\mathrm{Spin}}(K(\mathbf{Z}_d, 1))$). Thus there is an oriented (resp. spin) 5-manifold W^5 such that $\partial W = X$. Since ∂X_i has a bicollared neighborhood in X, W can also be viewed as a relative bordism from $(X_1, \partial X_1)$ to $(X_2, \partial X_2)$. It follows now from [Kr] §2 that there is a relative s-cobordism between

$X_1 \# k(S^2 \times S^2)$ and $X_2 \# k(S^2 \times S^2)$, $k \gg 0$. This together with Freedman's s-cobordism theorem [F_2] implies that $(N, f_1(S^2))$ and $(N, f_2(S^2))$ are stably homeomorphic, as required. □

3. Group actions on 4-manifolds.

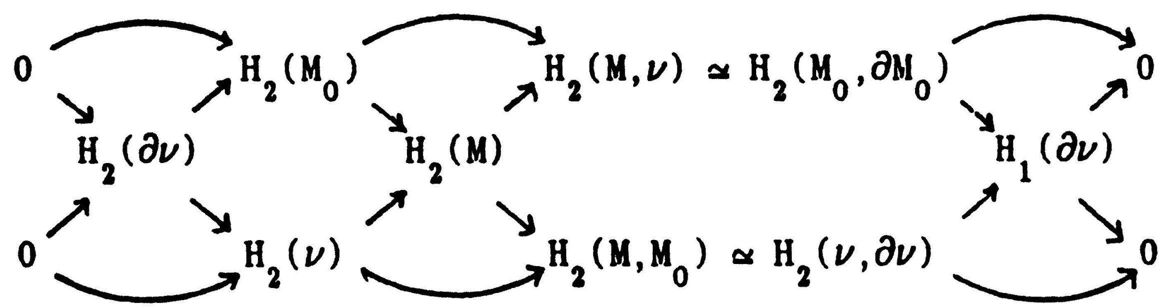
In this section, we discuss the unstable classifications of locally flat, simple embeddings $S^2 \to N$ representing the same homology class. The topological classification of such embeddings will be reduced to a problem concerning certain hermitian pairings. This algebraic problem will in turn be settled in Section 4.

Let $f : S^2 \to N$ be a locally flat, simple embedding representing a homology class $x \in H_2(N)$ of divisibility d. As noted in Section 2, $\pi_1(N - f(S^2)) \cong H_1(N - f(S^2))$ is a cyclic group of order d. Following [H–S] and [R], we can form a d-fold ramified covering $\pi : M \to N$ branched over $f(S^2)$. By construction then M supports a continuous action by the group $C_d = \pi_1(N - f(S^2))$. The fixed point set of this action is a locally flat 2-sphere $S = \pi^{-1}(f(S^2))$, and π maps S homeomorphically onto $f(S^2)$. Since $f : S^2 \to N$ is a simple embedding, so is $S \to M$. In fact, $\pi_1(M - S) = 0$, so the homology class $z = [S] \in H_2(M)$, represented by S, is primitive. Furthermore, it follows from [F_2] Theorem 10 that $S \subseteq M$ has a neighborhood homeomorphic to a C_d-vector bundle over S, i.e. C_d acts locally linearly on M. By choosing correctly the generator g of C_d, we may assume that g acts on the normal fiber to S via multiplication by $e^{2\pi i/d}$. Also, it follows from the van Kampen theorem that M is simply connected. Conversely, a semifree, locally linear C_d-action on a simply connected 4-manifold M, whose fixed point set is a 2-sphere representing a primitive class $z \in H_2(M)$, corresponds to the ramified covering $M \to M/C_d$.

The above construction allows us to reformulate the classification problem for embeddings as a similar problem for the corresponding group actions.

PROPOSITION 3.1. *There is a one-to-one correspondence between the isomorphism classes of locally flat, simple embeddings $f : S^2 \to N^4$, $\pi_1(N) = 0$ representing homology classes of divisibility d and the isomorphism classes of semifree, locally linear, cyclic group actions (C_d, M), $\pi_1(M) = 0$, for which the fixed point set* Fix (C_d, M) *is a simply embedded* 2*-sphere representing a primitive homology class.* □

Let M be a closed, oriented, simply connected 4-manifold, and let $G = C_d$ be a cyclic group acting locally linearly on M. Assume that the action is semifree, and that the fixed point set $M^G = \text{Fix}(G, M)$ is a simply embedded 2-sphere representing a primitive class $z \in H_2(M)$. It follows that G preserves both the orientation of M and its intersection pairing $\lambda : H_2(M) \times H_2(M) \to \mathbf{Z}$. Let ν denote the equivariant tubular neighborhood of $M^G \subseteq M$, and let $M_0 = M - \operatorname{int} \nu$ be the complement. There is a commutative diagram of exact sequences



From the diagram we see that the triple $[H_2(M), \lambda, z]$ completely determines the homology group $H_2(M_0)$ and its intersection pairing.

Our first goal is to determine the G-isovariant homotopy type of (G, M). Following [Wil], we introduce two auxiliary G-spaces X and B. The first G-space X is a $G - CW$ complex obtained by attaching free n-cells ($n \geq 4$) of the form $G \times D^n$ to M so that (i) each closed cell is disjoint from ν, and (ii) $\pi_i(X - \nu) = 0$ for $i \neq 2$. Similarly, the space B is obtained by attaching free n-cells ($n \geq 4$) to X so that $\pi_i(B) = 0$ for $i \neq 2$.

Now suppose we are given another G-manifold (G, M') satisfying the same conditions as those required from (G, M). In addition, suppose there exists a $\mathbf{Z}G$-module isomorphism $\theta : H_2(M) \to H_2(M')$ which preserves the intersection pairings, $\theta^*\lambda' = \lambda$, and sends $z = [M^G]$ to $z' = [M'^G]$. In such a case we shall say that θ is a $\mathbf{Z}G$-isometry, and write $\theta : [H_2(M), \lambda, z] \cong [H_2(M'), \lambda', z']$. We wish to show that θ can be realized by an isovariant homotopy equivalence $h : M \to M'$.

Since $z \cdot z = z' \cdot z'$ and the action of G near the fixed 2-spheres M^G and M'^G is completely determined by these intersection numbers, there exists an equivariant homeomorphism $\nu \to \nu'$ between the tubular neighborhoods. In fact, to simplify our notation, we shall identify ν and ν' via this homeomorphism and write $\nu = \nu'$ from now on. We wish to extend this identification to all of M and M'.

The first step towards the required extension is to find an isovariant homotopy equivalence $\varphi : X \to X'$ rel ν which induces θ on $\pi_2(X) \cong H_2(M)$. It follows from our braid diagram that θ induces as a $\mathbf{Z}G$-isomorphism $\theta_0 : H_2(M_0) \to H_2(M'_0)$ which respects the induced intersection pairings. Now from obstruction theory, the map $\varphi : X \to X'$ can be constructed provided $(\theta_0)_* k = k'$, where $k \in \operatorname{Ext}^2_{ZG}(\tilde{H}_0(\partial\nu), \pi_2(M_0))$ is the relative first k-invariant of (G, M) defined in [Wil]. Since $\partial\nu$ is connected, $k = k' = 0$ and the condition is trivially satisfied.

Therefore there exists a diagram

$$(3.2)\qquad \begin{array}{ccccc} \partial v \subseteq M_0 \subseteq M & & \searrow^{i} & & \\ {\scriptstyle \mathrm{id}}\downarrow \qquad \quad \vdots {\scriptstyle h} & & & X & \\ \partial v \subseteq M'_0 \subseteq M' & & \nearrow_{i'} & & \end{array}$$

where i is the inclusion of M in X, i' is the composite map $M' \hookrightarrow X' \xrightarrow{\varphi^{-1}} X$, and the dotted arrow indicates the map h to be constructed.

Since B is an Eilenberg–MacLane space, it follows from [Mac] that $H_4(B)$ can be identified with the module of symmetric 2-tensors $\Gamma\pi_2(M) \subseteq \pi_2(M) \otimes_{\mathbf{Z}} \pi_2(M)$. Under this identification, the image of the fundamental class $j_* i_*[M] \in H_4(B)$, $j : X \hookrightarrow B$, corresponds to the 2-tensor given by the intersection pairing on $\pi_2(M) \cong H_2(M)$. The fact that θ is an isometry can now be interpreted as saying that $j_* i_*[M] = j_* i'_*[M']$.

We assert that $j_* : H_4(X) \to H_4(B)$ is a monomorphism, and so the above equality implies that $i_*[M] = i'_*[M']$. From a Mayer–Vietoris sequence argument, we see that $H_4(X) = H_4(X_0) \oplus \mathbf{Z}$ where $X_0 = X - v$. Since X_0 is also an Eilenberg–MacLane space, $H_4(X_0)$ can be identified with $\Gamma\pi_2(X_0)$. Under this identification, the map j_* restricted to $H_4(X_0)$ corresponds to the induced map $\Gamma\pi_2(X_0) \to \Gamma\pi_2(X)$ which is clearly injective. On the other hand, the $\mathbf{Z}$-summand in $H_4(X)$ is mapped under j_* to the subspace $\mathbf{Z}\langle z \otimes z'\rangle$, where $z' \in \pi_2(X)$ is the element λ-dual to z. This proves that $H_4(X)$ and $j_* H_4(X)$ have the same rank over $\mathbf{Z}$, and the assertion follows.

We also claim that once the required isovariant map $h : (M, M_0) \to (M', M'_0)$ is constructed so that diagram (3.2) commutes up to isovariant homotopy, then h is automatically an isovariant homotopy equivalence. For any such h induces the isomorphism θ on the second homotopy group, and

$$i_*[M] = i'_* h_*[M] = (\deg h) i'_*[M'] = (\deg h) i_*[M].$$

Thus $\deg h = 1$, and the claim follows from Poincaré duality and the Whitehead theorem.

Finally, there is a secondary obstruction to construction of h, lying in the equivariant cohomology group

$$\begin{aligned} H^4_G(M_0, \partial v; \pi_4(X_0, M'_0)) &\cong H_0(M_0/G; \pi_4(X_0, M'_0)) \\ &\cong \pi_4(X_0, M'_0) \otimes_{\mathbf{Z}G} \mathbf{Z} \\ &\cong H_4(X_0, M'_0) \otimes_{\mathbf{Z}G} \mathbf{Z} \\ &\cong H_4(X_0/G, M'_0/G). \end{aligned}$$

From the following commutative diagram

$$\begin{array}{ccccccc} H_4(M/G) & \longrightarrow & H_4(M/G, \nu/G) & \xleftarrow{\cong} & H_4(M_0/G, \partial\nu/G) & & \\ {\scriptstyle (i/G)_*}\downarrow & & \downarrow & & \downarrow & & \\ H_4(X/G) & \longrightarrow & H_4(X/G, \nu/G) & \xleftarrow{\cong} & H_4(X_0/G, \partial\nu/G) & \longrightarrow & H_4(X_0/G, M_0'/G) \end{array}$$

we see that this obstruction corresponds to the class $\alpha = (i/G)_*[M/G] - (i'/G)_*[M'/G]$ in $H_4(X_0/G, M_0'/G)$. Furthermore, if we consider the transfer homomorphism

$$tr_* : H_4(X_0/G, M_0'/G) \to H_4(X_0, M_0'),$$

then $tr_*(\alpha)$ is the obstruction to a nonequivariant extension $\bar{h} : (M, M_0) \to (M', M_0')$ such that $i' \circ \bar{h} \simeq i$ rel ν. However, this nonequivariant obstruction

$$tr_*(\alpha) = i_*[M] - i'_*[M'] = 0,$$

and so $\alpha \in \operatorname{Ker}(tr_*) \cong \hat{H}_0(G; H_4(X_0, M_0'))$.

To compute this last group, we note that $H_3(M_0') \cong H^1(M_0', \partial\nu) = 0$, and consequently,

$$H_4(X_0, M_0') \cong H_4(X_0) \cong \Gamma\pi_2(X_0).$$

Also, it follows from [Wil] Proposition 2.3 that $\pi_2(X_0) \cong \pi_2(M_0)$ fits into the exact sequence of $\mathbf{Z}G$-modules

$$0 \to \Omega\mathbf{Z} \to \pi_2(X_0) \oplus r\mathbf{Z}G \to H^1(M, M^G) \to 0.$$

Since $H^1(M, M^G) = 0$ and $\Omega\mathbf{Z}$ is represented by the augmentation ideal I of $\mathbf{Z}G$, we conclude that $\pi_2(X_0)$ and I are stably isomorphic $\mathbf{Z}G$-modules, i.e.,

$$\pi_2(X_0) \oplus r\mathbf{Z}G \cong I \oplus s\mathbf{Z}G$$

for some integers r and s. Now in the case of an odd order d, the results of [H–K] §3 imply that $\Gamma\pi_2(X_0)$ is a stably free module, whereas for d even $\Gamma\pi_2(X_0)$ is stably isomorphic to a certain permutation module. In either case $\hat{H}_0(G; \Gamma\pi_2(X_0)) = 0$, and consequently the obstruction α vanishes.

To replace the resulting G-isovariant homotopy equivalence $h : (M, M_0) \to (M', M_0')$ by an equivariant homeomorphism, we must also assume that M and M'

have the same Kirby–Siebenmann invariant. A calculation with the topological surgery exact sequence shows that this condition is also sufficient when d is odd. Thus we have the following

THEOREM 3.3. *Let* (G, M) *and* (G, M') *be group actions satisfying all the requirements specified in Proposition* 3.1. *Given a* $\mathbf{Z}G$-isometry $\theta : [H_2(M), \lambda, z] \cong [H_2(M'), \lambda', z']$, *there exists an isovariant homotopy equivalence* $h : M \to M'$ rel v *which extends the identity map on the tubular neighborhood* v *and induces* θ *on* $H_2(M)$. *Furthermore, if* M *and* M' *have the same Kirby–Siebenmann invariant and* d *is odd, then* h *can be required to be an equivariant homeomorphism.* □

Our next result describes the $\mathbf{Z}G$-module structure of $H_2(M)$.

THEOREM 3.4. $H_2(M)$ *is a stably free* $\mathbf{Z}G$*-module.*

Proof. We begin by showing that $H_2(M)$ is a projective $\mathbf{Z}G$-module. We have already seen that $H_2(M_0)$ is stably isomorphic to the augmentation ideal I. The two modules are related to each other via the following exact sequence

$$(3.5) \quad 0 \to H_2(v) \to H_2(M) \to H_2(M, v) \to 0.$$

Notice that $H_2(M, v) \cong H_2(M_0, \partial v)$ is isomorphic via Poincaré duality to $H^2(M_0) = \mathrm{Hom}_{\mathbf{Z}}(H_2(M_0), \mathbf{Z})$ and as such it is also stably isomorphic to I.

We shall show that $H^i(G; H_2(M)) = 0$ for each $i \geqslant 0$ ($i = 2, 3$ is enough) by examining the long exact sequence in cohomology

$$H^{i-1}(G; H_2(M, v)) \xrightarrow{\delta} H^i(G; H_2(v)) \longrightarrow H^i(G; H_2(M)) \longrightarrow H^i(G; H_2(M, v)).$$

Without loss of generality, we can assume that G has prime power order, say $d = p^r$. Let $K \subseteq G$ denote the unique subgroup of order p. For i even, consider the following commutative diagram

$$\begin{array}{ccc} \mathbf{Z}_{p^r} = H^{i-1}(G; H_2(M, v)) & \xrightarrow{\delta} & H^i(G; H_2(v)) = \mathbf{Z}_{p^r} \\ {\scriptstyle \mathrm{res}} \downarrow & & \downarrow {\scriptstyle \mathrm{res}} \\ \mathbf{Z}_p = H^{i-1}(K; H_2(M, v)) & \xrightarrow{\delta} & H^i(K; H_2(v)) = \mathbf{Z}_p \end{array}$$

It follows that for each $i > 0$, $H^i(G; H_2(M)) = 0$ iff $H^i(K; H_2(M)) = 0$. (Notice that $H^i(G; \mathbb{Z}) = H^{i+1}(G; I) = 0$ for i odd.) But $H^i(K; H_2(M)) = 0$ for $i > 0$ by [E], so $H_2(M)$ is a projective $\mathbf{Z}G$-module, as required.

Finally, to prove that $H_2(M; \mathbb{Z})$ is stably free, we invoke the pull-back diagram of commutative rings (Rim's square)

$$\begin{array}{ccc} \Lambda & \xrightarrow{\bmod \Sigma} & \Lambda_1 \\ {\scriptstyle \bmod I}\downarrow & & \downarrow{\scriptstyle \bmod I} \\ \mathbf{Z} & \xrightarrow[\bmod d]{} & \mathbf{Z}_d \end{array} \tag{3.6}$$

Here $\Lambda = \mathbf{Z}G$, $\Sigma = 1 + g + \cdots + g^{d-1}$ is the sum of all groups elements in G, and $\Lambda_1 = \Lambda/(\Sigma)$. Associated with this diagram there is a Mayer–Vietoris sequence of K-groups

$$\to K_1(\Lambda) \to K_1(\Lambda_1) \oplus K_1(\mathbf{Z}) \to K_1(\mathbf{Z}_d) \to K_0(\Lambda) \to K_0(\Lambda_1) \oplus K_0(\mathbf{Z}) \to K_0(\mathbf{Z}_d)$$

(cf. [M_2]). In the present situation, $K_1(\mathbf{Z}_d)$ is the group of units in $\mathbf{Z}_d$ and the induced map $K_1(\Lambda_1) \to K_1(\mathbf{Z}_d)$ is a surjection. Hence

$$0 \to K_0(\Lambda) \to K_0(\Lambda_1) \oplus K_0(\mathbf{Z}) \to K_0(\mathbf{Z}_d) \to 0,$$

and we can determine the structure of a projective module over Λ by taking the tensor product with Λ_1, and examining the resulting element in $K_0(\Lambda_1)$. We make use again of the exact sequence in (3.5)

$$0 \to \mathbf{Z} \to H_2(M) \to H_2(M, v) \to 0.$$

Tensoring with Λ_1,

$$\mathbf{Z} \otimes_\Lambda \Lambda_1 \to H_2(M) \otimes_\Lambda \Lambda_1 \to H_2(M, v) \otimes_\Lambda \Lambda_1 \to 0.$$

Since $H_2(M) \otimes_\Lambda \Lambda_1$ is a projective Λ_1-module which has no $\mathbf{Z}$-torsion and $\mathbf{Z} \otimes_\Lambda \Lambda_1 \cong \mathbf{Z}_d$ we obtain an isomorphism $H_2(M) \otimes_\Lambda \Lambda_1 \cong H_2(M, v) \otimes_\Lambda \Lambda_1$ of Λ_1-modules. But $H_2(M, v)$ is stably isomorphic to I which is a free Λ_1-module. It follows that the tensor product $H_2(M, v) \otimes_\Lambda \Lambda_1$ is a stably free Λ_1-module, and hence it represents the trivial element in the reduced K-group $\tilde{K}_0(\Lambda_1)$. This proves the theorem. □

The last two results suggest that we study "pointed" integral Λ-lattices $[L, \lambda, z]$, where L is finitely generated, stably free Λ-module ($\Lambda = \mathbf{Z}G$) equipped with a Λ-invariant unimodular pairing $\lambda : L \times L \to \mathbf{Z}$ and a base point, a distinguished primitive class $z \in L^G$. However, for technical reasons, it will be more convenient to replace λ by the corresponding hermitian pairing $h : L \times L \to \Lambda$ defined as follows

$$h(x, y) = \sum_{g \in G} \lambda(g^{-1}x, y)g.$$

The term "hermitian" refers to the property $h(x, y) = \overline{h(y, x)}$, where $- : \Lambda \to \Lambda$ is the natural involution on the group ring given by $g \mapsto g^{-1}$.

Thus, associated with each group action (G, M) of the type considered there is a pointed Λ-isometry class of $[H_2(M), h_M, z]$, where h_M is the nonsingular hermitian pairing corresponding to the intersection pairing λ and $z = [M^G]$. By Theorem 3.3, group actions (G, M) are classified by the corresponding pointed Λ-isometry classes.

We conclude this section with an observation based on the proof of Theorem 3.4. Our proof that $H_2(M)$ is stably free Λ-module also shows that the extension in (3.5) is a generator of the group $\mathrm{Ext}^1_\Lambda(\Lambda_1, \mathbf{Z}) = \mathbf{Z}_d$. Thus up to Galois automorphism of Λ_1, (3.5) is (stably) isomorphic to

$$0 \to \mathbf{Z} \xrightarrow{(\Sigma, 0)} \Lambda \oplus k\Lambda \xrightarrow{\pi \oplus 1} \Lambda_1 \oplus k\Lambda \to 0.$$

An easy computation with transfer shows that the corresponding sequence for the orbit manifold M/G,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2(\nu/G) & \xrightarrow{(d,0)} & H_2(M/G) & \longrightarrow & H_2(M/G, \nu/G) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & \mathbf{Z} & & \mathbf{Z} \oplus k\mathbf{Z} & & \mathbf{Z}_d \oplus k\mathbf{Z} & & \end{array}$$

results from tensoring (3.5) over Λ with $\mathbf{Z}$.

A similar relationship exists between the intersection pairings of M and M/G. By a straightforward geometric argument, we find that the intersection pairing on $H_2(M/G) = H_2(M) \otimes_\Lambda \mathbf{Z}$ is given as

$$h \otimes_\Lambda 1 : (H_2(M) \otimes_\Lambda \mathbf{Z}) \times (H_2(M) \otimes_\Lambda \mathbf{Z}) \to \mathbf{Z}$$

$$(h \otimes_\Lambda 1)(x, y) = \sum_{g \in G} \lambda(g^{-1}x, y).$$

This we record as

PROPOSITION 3.7. *There is a* $\mathbf{Z}$*-isometry of pointed* $\mathbf{Z}$*-lattices*

$$\pi_* \otimes_\Lambda 1 : [H_2(M), h_M, z] \otimes_\Lambda \mathbf{Z} \cong [H_2(M/G), \lambda_{M/G}, x]$$

where $\pi_* : H_2(M) \to H_2(M/G)$ *is the projection,* $z = [M^G]$, *and* $x = \pi_*(z)$. □

4. Pointed hermitian pairings

Let Λ be a ring with involution; for simplicity assume Λ is commutative with 1. A *pointed hermitian pairing* over Λ is a triple $[P, h, z]$, where P is a finitely generated, projective Λ-module, $h : P \times P \to \Lambda$ is a hermitian pairing and $z \in P$ is a base point. An *isometry* of pointed hermitian pairings over Λ is an isomorphism of Λ-modules which respects the pairings and the base points. We define addition of pointed hermitian pairings by the following formula

$$[P, h, z] \oplus [P', h', z'] = [P \oplus P', h \oplus h', z \oplus z'].$$

For a homomorphism of rings with involution $\varphi : \Lambda \to \Lambda'$, we have a change-of-rings operation:

$$[P, h, z] \otimes_\Lambda \Lambda' = [P \otimes_\Lambda \Lambda', h \otimes_\Lambda 1, z \otimes_\Lambda 1].$$

(A special case of this operation appeared already in Proposition 3.7.) $[P, h, z]$ and $[P', h', z']$ are said to be *stably equivalent* if for some integers r and s

$$[P, h, z] \oplus H(\Lambda^r) \cong [P', h', z'] \oplus H(\Lambda^s),$$

where $H(\Lambda^r)$ stands for the hyperbolic pairing $[\Lambda^{2r}, H(\Lambda^r), 0] = r[\Lambda^2, H(\Lambda), 0]$. When $[P, h, z]$ and $[P', h', z']$ are stably equivalent, we write $[P, h, z] \cong_s [P', h', z']$. $[P, h, 0]$ will often be abbreviated to $[P, h]$.

If $G = C_d$ is a cyclic group and $\Lambda = \mathbf{Z}G$ with the involution given by $g \mapsto g^{-1}$, then a pointed hermitian pairing $[P, h, z]$ over Λ is said to be *realizable* whenever there exists an action (G, M) of the type considered in Proposition 3.1 such that $[P, h, z] \cong [H_2(M), h_M, [M^G]]$. By Theorem 3.4, the underlying module of a realizable pairing is stable free.

PROPOSITION 4.1. *A pointed hermitian pairing* $[P, h, z]$ *over* $\mathbf{Z}G$ *is realizable if and only if it is stably realizable.*

Proof. Suppose

$$[P, h, z] \oplus H(\Lambda^r) \cong [H_2(M), h_M, [M^G]] \oplus H(\Lambda^s).$$

Now $[H_2(M), h_M, [M^G]] \oplus H(\Lambda^s)$ can be realized by the equivariant connected sum

$$(G, M_1) = (G, M) \# s(G, G \times S^2 \times S^2)$$

of (G, M) with copies of $S^2 \times S^2$. In the exact sequence

$$0 \to H_2(M_1 - M_1^G) \to H_2(M_1) \to H_2(M_1, M_1 - M_1^G) \to 0$$

the third nonzero group is isomorphic to $H^2(M_1^G)$ by Poincaré duality and the hyperbolic summand $H(\Lambda^r)$ corresponds to a subspace in $H_2(M_1 - M_1^G)$. Therefore by Freedman's disk theorem [F_2] with $\pi_1 = G$, there exist framed embedded 2-spheres $S^2 \times D^2 \to M_1 - M_1^G$ representing the hyperbolic generators of $H(\Lambda^r)$. Using these framed 2-spheres we can perform surgery on $M_1 - M_1^G$ to kill $H(\Lambda^r)$. The result of this surgery realizes $[P, h, z]$. □

We now proceed to formulate the cancellation law for pointed hermitian pairings over $\mathbf{Z}G$. We shall assume for the rest of this section that $G = C_d$ is a cyclic group of odd order.

THEOREM 4.2. *Let* $[P, h, z]$, $[P', h', z']$ *be nonsingular pointed hermitian pairings over* $\Lambda = \mathbf{Z}G$ *with P stably free of rank* ≥ 3. *Assume that* $z \in P^G$ *and* $[P, h, z]$ *has a hyperbolic summand equivalent to* $H(\Lambda)$. *If* $[P, h, z] \cong_s [P', h', z']$ *and* $[P, h, z] \otimes_\Lambda \mathbf{Z} \cong [P', h', z'] \otimes_\Lambda \mathbf{Z}$, *then* $[P, h, z] \cong [P', h', z']$. *Furthermore, each isometry between* $[P, h, z] \otimes_\Lambda \mathbf{Z}$ *and* $[P', h', z'] \otimes_\Lambda \mathbf{Z}$ *is induced by one between* $[P, h, z]$ *and* $[P', h', z']$.

Proof. Let Γ be the usual maximal order in $\mathbf{Q}G$ containing Λ. Consider the cartesian square

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \hat{\Lambda} & \longrightarrow & \hat{\Gamma} \end{array}$$

where $\hat{\Lambda} = \Pi \hat{\Lambda}_p$, $\hat{\Gamma} = \Pi \hat{\Gamma}_p$, and the products are taken over the primes dividing d. We shall show that (i) $[P, h, z] \otimes_\Lambda \Gamma \cong [P', h', z'] \otimes_\Lambda \Gamma$ and $[P, h, z] \otimes_\Lambda \hat{\Lambda} \cong [P', h', z'] \otimes_\Lambda \hat{\Lambda}$, and (ii) that (i) implies $[P, h, z] \cong [P', h', z']$.

We first note that since $\Gamma = \Pi_{n \mid d} \mathbf{Z}[\zeta_n]$ and d is odd, the hermitian pairings $[P, h] \otimes_\Lambda \mathbf{Z}[\zeta_n]$, $n \neq 1$, and $[P, h] \otimes_\Lambda \hat{\Lambda}_p$ have quadratic refinements. It follows now from [W_5] Theorem 10 that $[P, h] \otimes_\Lambda \mathbf{Z}[\zeta_n] \cong [P', h'] \otimes_\Lambda \mathbf{Z}[\zeta_n]$. This together with the hypothesis over $\mathbf{Z}$ implies $[P, h, z] \otimes_\Lambda \Gamma \cong [P', h', z'] \otimes_\Lambda \Gamma$. (Notice that $[P, h, z] \otimes_\Lambda \mathbf{Z}[\zeta_n] \cong [P, h, 0] \otimes_\Lambda \mathbf{Z}[\zeta_n]$ for $z \in P^G$ and $n \neq 1$.) By Lemma 1 and

Theorem 2 of [W_5], the statement over $\hat{\Lambda}_p$ can be reduced to one over $\mathbf{F}_p C_r$ where $C_d = C_{p^i} \times C_r$. Since $\mathbf{F}_p C_r$ is semisimple, cancellation is possible over $\hat{\Lambda}_p$, and so $[P, h, z] \otimes_\Lambda \hat{\Lambda} \cong [P', h', z'] \otimes_\Lambda \hat{\Lambda}$, as required.

The isometry classes of pointed hermitian pairings over Λ that are equivalent to $[P, h, z]$ over Γ and $\hat{\Lambda}$ are in one-to-one correspondence with elements of the double coset space

$$\mathrm{Aut}\,([P, h, z] \otimes_\Lambda \hat{\Lambda})\backslash \mathrm{Aut}\,([P, h, z] \otimes_\Lambda \hat{\Gamma})/\mathrm{Aut}\,([P, h, z] \otimes_\Lambda \Gamma).$$

This follows from [Ba_2] Theorem 7.30 and the general discussion in [W_6]. Let $[\alpha]$ be the double coset corresponding to $[P', h', z']$.

We claim that $[\alpha]$ has a representative $\alpha' \in \mathrm{Aut}([P, h, z] \otimes_\Lambda \hat{\Gamma})$ of determinant 1. Since $[P', h', z'] \oplus H(\Lambda^r) \cong [P, h, z] \oplus H(\Lambda^r)$ for some r, $[P', h', z'] \oplus H(\Lambda^r)$ corresponds to the trivial double coset in

$$\mathrm{Aut}\,([P, h, z] \otimes_\Lambda \hat{\Lambda} \oplus H(\hat{\Lambda}^r))\backslash \mathrm{Aut}\,([P, h, z] \otimes_\Lambda \hat{\Gamma} \oplus H(\hat{\Gamma}^r))/\mathrm{Aut}\,([P, h, z] \otimes_\Lambda \Gamma \oplus H(\Gamma^r)).$$

Therefore for some representative α, $\det(\alpha) = \hat{a} \cdot a_\Gamma$ where $\hat{a} \in \hat{\Lambda}$, $a_\Gamma = \Pi_{n|d}\, a_n \in \Gamma$, $a_1 = 1$, $\hat{a}\bar{\hat{a}} = a_n \bar{a}_n = 1$. Since $\hat{\Lambda}$ is a complete semilocal ring and $H(\hat{\Lambda}/\mathrm{rad})$ is diagonalizable, there is

$$\hat{\beta} \in \mathrm{Aut}\,(H(\hat{\Lambda})) \times 1 \subseteq \mathrm{Aut}\,([P, h, z] \otimes_\Lambda \hat{\Lambda})$$

with $\det(\hat{\beta}) = \hat{a}^{-1}$. By the Dirichlet unit theorem, $a_n = (\pm \zeta_n^i) \cdot u_n$ where $u_n \in \mathbf{Z}[\zeta_n]^\times$ is of infinite order (unless $u_n = 1$) and has the property that $\bar{u}_n = u_n$ modulo finite units. From $a_n \bar{a}_n = 1$, it follows that $u_n = 1$, and consequently $a_n^{-1} = \det(\beta_n)$ for some

$$\beta_n \in \mathrm{Aut}\,(H(\mathbf{Z}[\zeta_n])) \times 1 \subseteq \mathrm{Aut}\,([P, h, z] \otimes_\Lambda \mathbf{Z}[\zeta_n]).$$

If we now let $\beta_1 = 1$ and $\beta_\Gamma = \Pi_{n|d}\, \beta_n$, then $\alpha' = \hat{\beta}\alpha\beta_\Gamma$ is the required representative.

Finally, by the strong approximation theorem [Sh, 5.12], applied to the special unitary group $\mathrm{SAut}\,([P, h, z] \otimes_\Lambda \mathbf{Z}[\zeta_n]$, the latter is dense in

$$\mathrm{SAut}\,([P, h, z] \otimes_\Lambda \hat{\mathbf{Z}}[\zeta_n]).$$

Since the space of left cosets

$$\mathrm{Aut}\,([P, h, z] \otimes_\Lambda \hat{\Lambda})\backslash \mathrm{Aut}([P, h, z] \otimes_\Lambda \hat{\Gamma})$$

is finite, this shows that $[\alpha]$ has a representative in $\mathrm{SAut}\,([P, h, z] \otimes_\Lambda \Gamma)$. Thus $[\alpha]$ is the trivial double coset and $[P', h', z'] \cong [P, h, z]$, as required.

To prove the second statement of the theorem, it suffices to show that the natural map

$$\mathrm{Aut}\,([P, h, z]) \to \mathrm{Aut}\,([P, h, z] \otimes_\Lambda \mathbf{Z})$$

is surjective. By virtue of Rim's square (3.6), this is equivalent to showing that the image of $\mathrm{Aut}\,([P, h] \otimes_\Lambda \mathbf{Z})$ in $\mathrm{Aut}\,([P, h] \otimes_\Lambda \mathbf{Z}_d)$ is contained in the image of

$$\mathrm{Aut}\,([P, h] \otimes_\Lambda \Lambda_1) \to \mathrm{Aut}\,([P, h] \otimes_\Lambda \mathbf{Z}_d).$$

Notice that this last statement is true stably, when $[P, h, z]$ is replaced by $[P, h, z] \oplus H(\Lambda^r)$, $r \gg 0$, since in the stable range the lifting problem for isometries over $\mathbf{Z}_d$ reduces to a determinant question. This shows that given

$$\alpha_{\mathbf{Z}} \in \mathrm{Aut}\,([P, h, z] \otimes_\Lambda \mathbf{Z}) \times 1 \subseteq \mathrm{Aut}\,([P, h, z] \otimes_\Lambda \mathbf{Z} \oplus H(\mathbf{Z}^r)),$$

there exists $\alpha_r \in \mathrm{Aut}\,([P, h, z] \oplus H(\Lambda^r))$ such that $\alpha_{\mathbf{Z}} \oplus 1 = \alpha_r \otimes_\Lambda \mathbf{Z}$.

LEMMA 4.3. *There exists an isometry* $\sigma \in \mathrm{Aut}\,([P, h, z] \oplus H(\Lambda^r))$ *such that* $\sigma \cdot \alpha_r$ *preserves the hyperbolic summand of* $[P, h, z] \oplus H(\Lambda^r)$ *and* $\sigma \otimes_\Lambda \mathbf{Z} = 1$.

It follows from Lemma 4.3 that $\sigma \cdot \alpha_r = \alpha_1 \oplus \alpha_2$ where $\alpha_1 \in \mathrm{Aut}\,([P, h, z])$ and $\alpha_2 \in \mathrm{Aut}\,(H(\Lambda^r))$. Thus $\alpha_{\mathbf{Z}} \oplus 1 = \alpha_r \otimes_\Lambda \mathbf{Z} = (\alpha_1 \otimes_\Lambda \mathbf{Z}) \oplus (\alpha_2 \otimes_\Lambda \mathbf{Z})$. □

LEMMA 4.4. *There exists a special isometry* $\hat{\sigma} \in \mathrm{SAut}\,([P, h, z] \otimes_\Lambda \hat{\Lambda} \oplus H(\hat{\Lambda}^r))$ *such that* $\hat{\sigma} \cdot (\alpha_r \otimes_\Lambda \hat{\Lambda})$ *preserves the hyperbolic summand of* $[P, h, z] \otimes_\Lambda \hat{\Lambda} \oplus H(\hat{\Lambda}^r))$ *and* $\hat{\sigma} \otimes_{\hat{\Lambda}} \hat{\mathbf{Z}} = 1$.

Proof of Lemma 4.3. Each isometry σ over Λ can be thought of as a pair of isometries $\hat{\sigma}$ and σ_Γ over $\hat{\Lambda}$ and Γ, respectively, which are compatible over $\hat{\Gamma}$. Let $\hat{\sigma} \in \mathrm{SAut}\,([P, h, z] \otimes_\Lambda \hat{\Lambda} \oplus H(\hat{\Lambda}^r))$ be the isometry provided by Lemma 4.4. By the strong approximation theorem applied to $\mathrm{SAut}\,([P, h, z] \otimes_\Lambda \mathbf{Z}[\zeta_n])$ for each $n|d$, $n > 1$, $\hat{\sigma} \otimes_{\hat{\Lambda}} \hat{\Gamma}$ can be lifted to an isometry $\sigma_\Gamma \in \mathrm{SAut}\,([P, h, z] \otimes_\Lambda \Gamma \oplus H(\Gamma^r))$ such that $\sigma_\Gamma \otimes_\Gamma \mathbf{Z} = 1$. By construction, $\hat{\sigma}$ and σ_Γ are compatible over $\hat{\Gamma}$ and the corresponding $\sigma \in \mathrm{Aut}\,([P, h, z] \oplus H(\Lambda^r))$ has the required properties. □

Proof of Lemma 4.4. *Consider the cartesian square*

$$\begin{array}{ccc} \hat{\Lambda} & \xrightarrow{\bmod \Sigma} & \hat{\Lambda}_1 \\ \downarrow & & \downarrow \\ \hat{\mathbf{Z}} & \longrightarrow & \hat{\mathbf{Z}}_d \end{array}$$

where $\hat{\Lambda}_1 = \hat{\Lambda}/(\Sigma)$, $\Sigma = 1 + g + \cdots + g^{d-1}$, $\hat{\mathbf{Z}} = \Pi_{p \mid d} \hat{\mathbf{Z}}_p$, and $\hat{\mathbf{Z}}_d = \hat{\mathbf{Z}} \otimes \mathbf{Z}_d$. We wish to lift the identity $1 \in \mathrm{SAut}([P, h] \otimes_\Lambda \hat{\mathbf{Z}}_d \oplus H(\hat{\mathbf{Z}}_d^r))$ to a special isometry $\hat{\sigma}_1 \in \mathrm{SAut}([P, h) \otimes_\Lambda \hat{\Lambda}_1 \oplus H(\hat{\Lambda}_1^r))$ such that $\hat{\sigma}_1 \cdot (\alpha_r \otimes_\Lambda \hat{\Lambda}_1)$ preserves the hyperbolic summand of $[P, h] \otimes_\Lambda \hat{\Lambda}_1 \oplus H(\hat{\Lambda}_1^r))$. By Lemma 1 and Theorem 2 of [W_5], this is equivalent to the corresponding lifting problem modulo the radical. The construction of a lifting having all the required properties modulo the radical is clearly possible as follows from the following diagram

$$\begin{array}{ccccc} \hat{\Lambda}_1 & \to & \hat{\Lambda}_1/\mathrm{rad} & \xrightarrow{\cong} & \prod_{p \mid d} \left(\prod \mathbf{F}_{p^i} \right) \\ \downarrow & & \downarrow & & \downarrow \\ \hat{\mathbf{Z}}_d & \to & \hat{\mathbf{Z}}_d/\mathrm{rad} & \xrightarrow{\cong} & \prod_{p \mid d} \mathbf{F}_p. \end{array}$$

□

REMARK 4.5. It follows from the proof of Theorem 4.2 that the hypothesis of a hyperbolic summand in $[P, h, z]$ can be replaced by the weaker condition that such summands exist only over $\hat{\Lambda}$ and $\mathbf{Z}[\zeta_n]$ for each $n \mid d$, $n > 1$. The point here is of course that we no longer require a hyperbolic summand in $[P, h, z] \otimes_\Lambda \mathbf{Z}$ which would be too restrictive for our purposes. Notice that if $[A, \lambda, x]$ is a pointed $\mathbf{Z}$-lattice, then it is possible that $[A, \lambda]$ may split off a copy of $H(\mathbf{Z})$ but $[A, \lambda, x]$ may not. For example, take any $[A, \lambda, x]$ for which $[A, \lambda] = [A_0, \lambda_0] \oplus H(\mathbf{Z})$ where $[A_0, \lambda_0]$ is a positive definite lattice, $x \neq 0$, and $\lambda(x, x) = 0$.

For any hermitian pairing $[P, h]$ over $\mathbf{Z}G$, the hermitian pairing $[P, h] \otimes_{\mathbf{Z}} \mathbf{C}$ decomposes over $\mathbf{C}G$ as $\oplus_{j=0}^{d-1} [P(j), h_j]$ where $P(j) \subseteq P \otimes_{\mathbf{Z}} \mathbf{C}$ is the subspace of $P \otimes_{\mathbf{Z}} \mathbf{C}$ on which the generator of G acts by multiplication by $e^{2\pi i j/d}$ and h_j is the component of $h \otimes_{\mathbf{Z}} \mathbf{C}$ corresponding to $P(j)$. Let σ_j denote the signature of $[P(j), h_j]$.

THEOREM 4.6. *Let $[P, h, z]$ be a nonsingular pointed hermitian pairing over $\Lambda = \mathbf{Z}G$ (d odd) with $z \in P^G$. Assume that P is a stably free Λ-module of rank $m \geq 3$ and that $[P, h, z] \otimes_\Lambda \mathbf{Z}$ has a hyperbolic summand $H(\mathbf{Z}^k)$ for some $0 < k < m/2$. If $m \geq \max_{0 \leq j < d} |\sigma_j| + 2k$, then there exists a pointed hermitian pairing $[P_0, h_0, z_0]$ over Λ such that $[P, h, z] \cong [P_0, h_0, z_0] \oplus H(\Lambda^k)$.*

Proof. By assumption, there exists a pointed $\mathbf{Z}$-lattice $[A, \lambda, x]$ and an isometry

(4.7) $\alpha_{\mathbf{Z}} : [P, h, z] \otimes_\Lambda \mathbf{Z} \cong [A, \lambda, x] \oplus H(\mathbf{Z}^k)$.

Consider now $[P, h, z] \otimes_\Lambda \Gamma$. Since $k > 0$, the hermitian pairing $[P, h] \otimes_\Lambda \mathbf{Z}[\zeta_n]$, $n \mid d$, $n > 1$ is indefinite at each archimidean place, and since d is odd it also has a quadratic refinement. Therefore by [W_5] Theorem 11, there exists a pointed hermitian pairing $[P_\Gamma, h_\Gamma, z_\Gamma]$ over Γ and an isometry

$$\alpha_\Gamma : [P, h, z] \otimes_\Lambda \Gamma \cong [P_\Gamma, h_\Gamma, z_\Gamma] \oplus H(\Gamma^k)$$

such that $\alpha_\Gamma \otimes_\Gamma \mathbf{Z} = \alpha_{\mathbf{Z}}$.

Since $\hat{\Gamma} = \Pi_{n|d} \Pi_{p|d} \hat{\mathbf{Z}}_p[\zeta_n]$, the classification of unpointed hermitian pairings over $\hat{\Gamma}$ reduces, by [W_5] Theorem 2, to a problem over $\Pi_{n|d} \Pi_{p|d} \mathbf{F}_p[\zeta_n]$. Over $\mathbf{F}_p[\zeta_n]$, hermitian pairings are classified by the rank and the discriminant invariant in $\mathbf{F}_p^\times/\mathbf{F}_p^{\times 2}$. These are also the invariants needed for the classification over $\hat{\Lambda}$. Furthermore, a collection of invariants over $\hat{\Gamma}$ is realized by a hermitian pairing over $\hat{\Lambda}$ if it is so stably realized. Thus there is a hermitian pairing $[\hat{P}, \hat{h}]$ over $\hat{\Lambda}$ and an isometry

$$\hat{\alpha} : [P, h] \otimes_\Lambda \hat{\Lambda} \cong [\hat{P}, \hat{h}] \oplus H(\hat{\Lambda}^k).$$

We also wish to find a base point $\hat{z}$ for $[\hat{P}, \hat{h}]$. (The obvious candidate would be $\hat{\alpha}(z \otimes 1)$ but unfortunately it need not lie in the summand $\hat{P}$). Now $\hat{\alpha}$ and $\alpha_{\mathbf{Z}}$ induce isometries of $\hat{\mathbf{Z}}$-lattices

$$[P, h] \otimes_\Lambda \hat{\mathbf{Z}} \cong [\hat{P}, \hat{h}] \otimes_{\hat{\Lambda}} \hat{\mathbf{Z}} \oplus H(\hat{\mathbf{Z}}^k),$$
$$[P, h, z] \otimes_\Lambda \hat{\mathbf{Z}} \cong [A, \lambda, x] \otimes_{\mathbf{Z}} \hat{\mathbf{Z}} \oplus H(\hat{\mathbf{Z}}^k).$$

Since over $\hat{\mathbf{Z}}$ cancellation is possible, $[A, \lambda] \otimes_{\mathbf{Z}} \hat{\mathbf{Z}} \cong [\hat{P}, \hat{h}] \otimes_{\hat{\Lambda}} \hat{\mathbf{Z}}$. Let $y \in \hat{P} \otimes_{\hat{\Lambda}} \hat{\mathbf{Z}}$ be the image of $x \otimes 1$ under this isometry. We now define $\hat{z} = tr_*(y) \in \hat{P}^G$, so that by construction

$$\hat{h}(\hat{z}) = \lambda(x) = h(z) = \hat{h}(\hat{\alpha}(z \otimes 1))$$

as elements in $\hat{\Lambda}$.

If $k > 2$ then according to [Ba_1] §4, Aut $([P, h] \otimes_\Lambda \hat{\Lambda})$ acts transitively on primitive elements in $P \otimes_\Lambda \hat{\Lambda}$ of the same "length". In particular, there is in that case $\hat{\sigma} \in$ Aut $[P, h] \otimes_\Lambda \hat{\Lambda})$ such that $\hat{\sigma}(\hat{\alpha}(z \otimes 1)) = \hat{z}$. Furthermore, if $k > 3$, such a $\hat{\sigma}$ can be found with determinant 1.

Assume temporarily that $k > 3$. Let I denote the set of all isometries from $[P_\Gamma, h_\Gamma, z_\Gamma] \otimes_\Gamma \hat{\Gamma} \oplus H(\hat{\Gamma}^k)$ to $[\hat{P}, \hat{h}, \hat{z}] \otimes_{\hat{\Lambda}} \hat{\Gamma} \oplus H(\hat{\Gamma}^k)$. Define

$$\alpha = (\hat{\sigma} \otimes_{\hat{\Lambda}} \hat{\Gamma}) \circ (\hat{\alpha} \otimes_{\hat{\Lambda}} \hat{\Gamma}) \circ (\hat{\alpha}_\Gamma \otimes_\Gamma \hat{\Gamma})^{-1}.$$

Then $\alpha \in I$. Let $[\alpha]$ denote the double coset of α in

$$\mathrm{Aut}\,([\hat{P}, \hat{h}, \hat{z}] \oplus H(\hat{\Lambda}^k))\backslash I/\mathrm{Aut}\,([P_\Gamma, h_\Gamma, z_\Gamma] \oplus H(\Gamma^k)).$$

LEMMA 4.8. *There exist isometries* $\beta_1 : [P_\Gamma, h_\Gamma, z_\Gamma] \otimes_\Gamma \hat{\Gamma} \cong [\hat{P}, \hat{h}, \hat{z}] \otimes_{\hat{\Lambda}} \hat{\Gamma}$ *and* $\beta_2 \in \mathrm{Aut}\,(H(\hat{\Gamma}^k))$ *such that* $[\beta_1 \oplus \beta_2] = [\alpha]$ *and* $\det(\beta_2) = 1$.

By [Ba$_2$] Theorem 7.30, there exists a pointed hermitian pairing $[P_1, h_1, z_1]$ over Λ which over Γ (resp. $\hat{\Lambda}$) is equivalent to $[P_\Gamma, h_\Gamma, z_\Gamma]$ (resp. $[\hat{P}, \hat{h}, \hat{z}]$) and such that the composite isometry

$$[P_\Gamma, h_\Gamma, z_\Gamma] \otimes_\Gamma \hat{\Gamma} \cong [P_1, h_1, z_1] \otimes_\Lambda \hat{\Gamma} \cong [\hat{P}, \hat{h}, \hat{z}] \otimes_{\hat{\Lambda}} \hat{\Gamma}$$

is equal to β_1. Likewise, there is a Λ-projective module P_2 for which $H(P_2) \otimes_\Lambda \Gamma \cong H(\Gamma^k)$, $H(P_2) \otimes_\Lambda \hat{\Lambda} \cong H(\hat{\Lambda}^k)$, and such that the composite isometry

$$H(\hat{\Gamma}^k) = H(\Gamma^k) \otimes_\Gamma \hat{\Gamma} \cong H(P_2) \otimes_\Lambda \hat{\Gamma} \cong H(\hat{\Lambda}^k) \otimes_{\hat{\Lambda}} \hat{\Gamma} = H(\hat{\Gamma}^k)$$

coincides with β_2. Since $[\beta_1 \oplus \beta_2] = [\alpha]$, there is an isometry

$$[P, h, z] \cong [P_1, h_1, z_1] \oplus H(P_2).$$

Since $\det(\beta_2) = 1$, P_2 is a stably free module (see [M$_2$] Lemma 2.4), and so

$$(4.9) \quad [P, h, z] \oplus H(\Lambda^r) \cong [P_1, h_1, z_1] \oplus H(\Lambda^{k+r}), \quad r \gg 0.$$

It is clear now that the assumption on k is not needed if we are only interested in a stable statement of the form (4.9). (Just apply the previous argument to $[P, h, z] \oplus H(\Lambda^3)$ in place of $[P, h, z]$.) Thus (4.9) holds for any $k > 0$.

The following lemma follows easily from Rim's square (3.6).

LEMMA 4.10. *Let* $[A, \lambda, x]$ *be a pointed* **Z**-*lattice. Suppose there is a pointed hermitian pairing* $[P_1, h_1, z_1]$ *over* $\Lambda = \mathbf{Z}G$ *such that* $[P_1, h_1, z_1] \otimes_\Lambda \mathbf{Z} \oplus H(\mathbf{Z}^r) \cong [A, \lambda, x] \oplus H(\mathbf{Z}^r)$, $r \gg 0$. *There exists a pointed hermitian pairing* $[P_0, h_0, z_0]$ *over* Λ *such that* $[P_0, h_0, z_0] \otimes_\Lambda \mathbf{Z} \cong [A, \lambda, x]$ *and* $[P_0, h_0, z_0] \cong_s [P_1, h_1, z_1]$.

Apply Lemma 4.10 to the nonhyperbolic summand of $[P, h, z] \otimes_\Lambda \mathbf{Z}$ in (4.7). Thus there is a pointed hermitian pairing $[P_0, h_0, h_0]$ such that

$$(4.11) \quad \begin{aligned} &[P, h, z] \oplus H(\Lambda^r) \cong [P_0, h_0, z_0] \oplus H(\Lambda^{k+r}), \quad r \gg 0, \\ &[P, h, z] \otimes_\Lambda \mathbf{Z} \cong [P_0, h_0, z_0] \otimes_\Lambda \mathbf{Z} \oplus H(\mathbf{Z}^k). \end{aligned}$$

Notice that if the first equivalence in (4.11) is tensored over Λ with either $\hat{\Lambda}$ or $\mathbf{Z}[\zeta_n]$, $n \mid d$, $n > 1$ then a hyperbolic form of rank r can be cancelled from both sides. Therefore by Remark 4.5, Theorem 4.2 can be applied to (4.11) so that $[P, h, z] \cong [P_0, h_0, z_0] \oplus H(\Lambda^k)$, as claimed. □

REMARK 4.12. Let us record for future reference the following observation based on the previous proof. If $[P, h, z] \oplus H(\Lambda^r)$, $r \gg 0$ has a hyperbolic summand equivalent to $H(\Lambda^{r+1})$, then $[P, h, z] \otimes_\Lambda \hat{\Lambda}$ splits off a copy of $H(\hat{\Lambda})$. This statement is definitely not true over Λ as it fails already over $\mathbf{Z}$ (even when $z = 0$).

Proof of Lemma 4.8. We appeal again to the strong approximation theorem of [Sh]. It follows that modulo the action of SAut $([P, h, z] \otimes_\Lambda \hat{\Lambda})$ any element of the special unitary group SAut $([P, h, z] \otimes_\Lambda \hat{\Gamma})$ can be lifted to SAut $([P, h, z] \otimes_\Lambda \Gamma)$. That is, given $\delta \in$ SAut $([P_\Gamma, h_\Gamma, z_\Gamma] \otimes_\Gamma \hat{\Gamma} \oplus H(\hat{\Gamma}^k))$, there exist

$$\delta_\Gamma \in \text{SAut}\,([P_\Gamma, h_\Gamma, z_\Gamma] \oplus H(\Gamma^k)) \quad \text{and} \quad \hat{\delta} \in \text{SAut}\,([\hat{P}, \hat{h}, \hat{z}] \oplus H(\hat{\Lambda}^k))$$

such that $\alpha\delta = \hat{\delta}\alpha\,\delta_\Gamma$. Hence $[\alpha\delta] = [\alpha]$ for each such δ.

Since SAut $([P, h, z] \otimes_\Lambda \hat{\Gamma})$ acts transitively on hyperbolic summands $H(\hat{\Gamma}^k)$ in $[P, h, z] \otimes_\Lambda \hat{\Gamma}$, there is $\delta_1 \in$ SAut $([P_\Gamma, h_\Gamma, z_\Gamma] \otimes_\Lambda \hat{\Gamma} \oplus H(\hat{\Gamma}^k))$ and isometries $\beta_0' : [P_\Gamma, h_\Gamma, z_\Gamma] \otimes_\Gamma \hat{\Gamma} \cong [\hat{P}, \hat{h}, \hat{z}] \otimes_{\hat{\Lambda}} \hat{\Gamma}$, $\beta_1' \in$ Aut $(H(\hat{\Gamma}^k))$ such that $\alpha\delta_1 = \beta_0' \oplus \beta_1'$. Let $b = \det(\beta_1')$. Since d is odd, we can find isometries $\delta_3 \in$ Aut $([P_\Gamma, h_\Gamma, z_\Gamma] \otimes_\Lambda \hat{\Gamma})$, $\delta_4 \in$ Aut $(H(\hat{\Gamma}))$ with respective determinants b and b^{-1}. Thus if we let

$$\delta_2 = \delta_3 \oplus \delta_4 \in \text{SAut}\,([P_\Gamma, h_\Gamma, z_\Gamma] \otimes_\Gamma \hat{\Gamma} \oplus H(\hat{\Gamma}^k)),$$

then $\alpha\delta_1\,\delta_2 = \beta_1 \oplus \beta_2$ has the required properties. □

Proof of Theorem 1.1. Assume conditions (i) and (ii). By Theorem 2.1, x can be represented by a locally flat, simple embedding $S^2 \to N_k = N \# k(S^2 \times S^2)$ for some k. If $k > 0$, let $\pi : M \to N_k$ be the ramified covering corresponding to this embedding via Proposition 3.1 and let $[P, h, z] = [H_2(M), h_M, [M^G]]$. It follows from Proposition 3.7 that

$$[P, h, z] \otimes_\Lambda \mathbf{Z} \cong [H_2(N), \lambda_N, x] \oplus H(\mathbf{Z}^k).$$

Furthermore, according to [R], $\sigma_j = \sigma(M) - 2j(d-j)(1/d^2)x \cdot x$ for each $0 \leq j < d$. Since by assumption

$$\text{rank}\,(P) = b_2(N) + 2k \geq \max_{0 \leq j < d} |\sigma(N) - 2j(d-j)(1/d^2)x \cdot x| + 2k,$$

Theorem 4.6 applies to $[P, h, z]$. Thus there is a pointed hermitian pairing $[P_0, h_0, z_0]$ such that $[P, h, z] \cong [P_0, h_0, z_0] \oplus H(\Lambda^k)$ and by Lemma 4.10 we can also assume that $[P_0, h_0, z_0] \otimes_\Lambda \mathbf{Z} \cong [H_2(N), \lambda_N, x]$.

Since $[P_0, h_0, z_0] \cong_s [P, h, z]$, Proposition 4.1 shows that $[P_0, h_0, z_0]$ can be realized by an action (G, X). Since X/G results from surgery on M/G, the Kirby–Siebenmann invariants of X/G and N are the same. Therefore the isometry of intersection pairings

$$[H_2(X/G), \lambda_{X/G}, [X^G]] \cong [P_0, h_0, z_0] \otimes_\Lambda \mathbf{Z} \cong [H_2(N), \lambda, x]$$

can be realized by a homeomorphism $f: X/G \to N$. $f(X^G) \subseteq N$ is then the required 2-sphere in N. □

Proof of Theorem 1.2. Let $f, f' : S^2 \to N$ be two locally flat, simple embeddings representing $x \in H_2(N)$. Let M, $M' \to N$ be the associated ramified coverings, and let $[P, h, z]$, $[P', h', z']$ be the corresponding pointed hermitian pairings over Λ. Proposition 3.7 and Theorem 2.5 imply

$$(4.13) \qquad \begin{gathered} [P, h, z] \otimes_\Lambda \mathbf{Z} \cong [P', h', z'] \otimes_\Lambda \mathbf{Z} \cong [H_2(N), \lambda, x], \\ [P, h, z] \cong_s [P', h', z']. \end{gathered}$$

We claim that $[P, h, z] \cong [P', h', z']$. This is trivial when $d = 1$, so assume $d > 1$. By assumption,

$$\operatorname{rank}(P) = b_2(N) \geq \max_{0 \leq j < d} |\sigma(N) - 2j(d-j)(1/d^2)x \cdot x| + 2.$$

In particular, $[H_2(N), \lambda]$ is indefinite and since $b_2(N) > 2$, $[H_2(N), \lambda, x] \oplus H(\mathbf{Z}^r)$ splits off a copy of $H(\mathbf{Z}^{r+1})$, $r \gg 0$. Therefore by Theorem 4.6, $[P, h, z] \oplus H(\Lambda^r)$ has a hyperbolic summand equivalent to $H(\Lambda^{r+1})$. By Remarks 4.12 and 4.5, Theorem 4.2 can be applied to (4.13). Hence $[P, h, z] \cong [P', h', z']$, as claimed.

It follows now from Theorem 3.3 and the second part of Theorem 4.2 that there is an equivariant homeomorphism $k : M \to M'$ such that $k/G : N \to N$ induces the identity on homology. Then $(k/G) \circ f = f'$ and according to the isotopy theorem of [Q_2] and [P], k/G is isotopic to the identity on N. Thus f and f' are ambient isotopic. □

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