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A calculation of Pin^+ bordism groups

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We begin by recalling the definition of the *Pin* and *Spin*-bordism groups. For each integer $n \geq 1$ there are compact Lie groups, $Spin(n)$, $Pin^-(n)$ and $Pin^+(n)$. Atiyah, Bott and Shapiro [ABS], described the groups $Spin(n)$ and $Pin^-(n)$ in terms of the Clifford algebra associated to the negative definite form on $\mathbf{R}^n$. Lam [*L*], describes these as well as $Pin^+(n)$, the group coming from the Clifford algebra associated to the positive definite form on $\mathbf{R}^n$. Another definition is the following. The group $Spin(n)$ is the double cover of the group $SO(n)$. It is a $Z/2$ central extension of $SO(n)$ and is classified by $w_2 \in H^2(BSO(n); Z/2)$: indeed it is the unique non-trivial $Z/2$ central extension. The two groups $Pin^\pm$ are double covers of $O(n)$. They are also $Z/2$ central extensions: Pin^- is classified by $w_2 + w_1^2 \in H^2(BO(n); Z/2)$ and Pin^+ is classified by w_2.

There is a bordism theory of manifolds with *Spin*, Pin^-, or Pin^+ structure, and we use the term bordism groups for the bordism groups of a point. Anderson, Brown and Peterson calculated the *Spin*-bordism groups, [ABP1], and the Pin^--bordism groups, [ABP2]. We complete the story by calculating the Pin^+-bordism groups.

Both the $Pin^\pm$-bordism groups are 2-torsion, and they have cyclic summands of order equal to an arbitrarily high power of 2. Both bordism groups are modules over the *Spin* bordism ring. Of the real projective spaces, the RP^{4k}'s have Pin^+ structures and the RP^{4k+2}'s have Pin^- structures. The other result in this paper is that $Pin^\pm$-bordism, modulo the *Spin* bordism submodule generated by the real projective spaces, is a $Z/2$ vector space.

To describe our results in more detail, recall the 2-local decomposition of the spectrum **MSpin** from [ABP1].

$$\mathbf{MSpin} \to \bigvee_{k \geq 0} \pi(2k)\mathbf{bo}\langle 8k \rangle \bigvee_{k>0} \pi(2k+1)\mathbf{bo}\langle 8k+2 \rangle \bigvee_{k>0} \alpha(k)\mathbf{K}(Z/2, k)$$

where $\mathbf{bo}\langle r \rangle$ denotes the spectrum obtained from the usual BO spectrum by killing all the homotopy groups in dimensions less than r, and $\mathbf{K}(A, r)$ denotes the Eilenberg–MacLane spectrum with one non-zero homotopy group isomorphic to A

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in dimension r. Furthermore, $\pi(k)$ denotes the number of partitions of k with all the pieces greater than 1. If $p(k)$ denotes the usual partition function, then $\pi(k) = p(k) - p(k-1)$. The numbers $\alpha(k)$ in principle can be computed. Since the cohomologies of **MSpin**, $\mathbf{bo}\langle r\rangle$ and $\mathbf{K}(Z/2, r)$ are known, as are the $\pi(k)$, the $\alpha(k)$ are the unique numbers which give equality of cohomologies.

The decomposition above is not unique, but we choose one such decomposition and fix it for the rest of the paper.

The Anderson, Brown and Peterson calculations of *Spin* and Pin^- bordism are similar. The homotopy groups of the $\mathbf{bo}\langle r\rangle$ and the $\mathbf{K}(Z/2, r)$ are known, so once they prove the decomposition formula, they can easily calculate *Spin* bordism. For Pin^- bordism, they argue that $MPin^-$ is homotopy equivalent to the spectrum $\mathbf{MSpin} \wedge \Sigma^{-1}T(\xi)$, where $T(\xi)$ denotes the suspension spectrum of the Thom space of the canonical bundle over RP^∞. They apply the decomposition formula and compute (via Adams spectral sequence methods) $\pi_*(\mathbf{bo}\langle 8k\rangle \wedge \Sigma^{-1}T(\xi))$ and $\pi_*(\mathbf{bo}\langle 8k+2\rangle \wedge \Sigma^{-1}T(\xi))$. We describe our answer in a similar fashion. We will show that $MPin^+$ is homotopy equivalent to the spectrum $\mathbf{MSpin} \wedge \Sigma^{-3}T(3\xi)$, where $T(s\xi)$ will denote the suspension spectrum of the Whitney sum of s copies of the canonical bundle over RP^∞. We record our answer in the theorem below along with the results of the Anderson, Brown and Peterson calculation for $MPin^-$, which we will need later. Here are some well-known formulae which will simplify what follows.

(i) For any spectrum X,

$$\pi_i(X \wedge \mathbf{K}(Z/2, r)) = \pi_{i-r}(X \wedge \mathbf{K}(Z/2, 0)).$$

(ii) For any spectrum X,

$$\pi_i(X \wedge \mathbf{bo}\langle 8k\rangle) = \pi_{i-8k}(X \wedge \mathbf{bo}\langle 0\rangle)$$

$$\pi_i(X \wedge \mathbf{bo}\langle 8k+2\rangle) = \pi_{i-8k}(X \wedge \mathbf{bo}\langle 2\rangle).$$

Hence we only describe the answers below. In the sequel, we let $\mathbf{M}(r)$ denote the spectrum $\Sigma^{-r}T(r\xi)$.

THEOREM 1.

$$\pi_i(\mathbf{M}(1) \wedge \mathbf{K}(Z/2, 0)) = \pi_i(\mathbf{M}(3) \wedge \mathbf{K}(Z/2, 0)) = Z/2 \quad \textit{for all } i \geq 0.$$

$\pi_{8n+i} =$	$8n$	$8n+1$	$8n+2$	$8n+3$	$8n+4$	$8n+5$	$8n+6$	$8n+7$
$\mathbf{M}(1) \wedge \mathbf{bo}\langle 0\rangle$	$Z/2$	$Z/2$	$Z/2^{4n+3}$	0	0	0	$Z/2^{4n+4}$	0
$\mathbf{M}(3) \wedge \mathbf{bo}\langle 0\rangle$	$Z/2^{4n+1}$	0	$Z/2$	$Z/2$	$Z/2^{4n+4}$	0	0	0

for $0 \leq i < 8$ *and* $n \geq 0$.

$\pi_{8n+i} =$	$8n$	$8n+1$	$8n+2$	$8n+3$	$8n+4$	$8n+5$	$8n+6$	$8n+7$
$\mathbf{M}(1) \wedge \mathbf{bo}\langle 2\rangle$	$Z/2 \oplus Z/2$	$Z/2$	$Z/2^{4n+1}$	$Z/2$	$Z/2$	0	$Z/2^{4n+2}$	$Z/2$
$\mathbf{M}(3) \wedge \mathbf{bo}\langle 2\rangle$	$Z/2^{4n-1}$	$Z/2$	$Z/2 \oplus Z/2$	$Z/2$	$Z/2^{4n+2}$	$Z/2$	$Z/2$	0

for $0 \le i < 8$, $n \ge 0$ *and* $8n + i \ge 3$. *In the case* $n = 0$, $i = 0$ *or* 1,

$$\pi_i(\mathbf{M}(1) \wedge \mathbf{bo}\langle 2\rangle) = \pi_i(\mathbf{M}(3) \wedge \mathbf{bo}\langle 2\rangle) = 0.$$

In the case $n = 0$, $i = 2$,

$$\pi_2(\mathbf{M}(1) \wedge \mathbf{bo}\langle 2\rangle) = \pi_2(\mathbf{M}(3) \wedge \mathbf{bo}\langle 2\rangle) = Z/2.$$

COROLLARY 2. *The top line of the first table, with* $n = 0$, *gives the* Pin^- *bordism groups through dimension* 7; *the second line of the first table, with* $n = 0$, *gives the* Pin^+ *bordism groups through dimension* 7.

An alternate calculation of these bordism groups through dimension 4 is given in [KT]. While trying to understand these low-dimensional calculations, we were led to the general results presented here. The proofs will be given in the second section and a short table of the bordism groups is included at the end of the paper.

Notice that Pin^- bordism is a $Z/2$ vector space except in dimensions congruent to 2 mod 4. Moreover, RP^n has a Pin^- structure if n is congruent to 2 mod 4. Likewise, Pin^+ bordism is a $Z/2$ vector space except in dimensions congruent to 0 mod 4 and RP^n has a Pin^+ structure if n is congruent to 0 mod 4.

Recall some facts about the structure of the *Spin* bordism ring. The $\mathbf{bo}\langle\ \rangle$ factors are indexed by partitions. For a fixed $n = 8k$ we have a different $\mathbf{bo}\langle 8k\rangle$ for each partition, J, of $2k$ such that J has no 1's in it. For any partition, let $n(J)$ denote the sum of the elements of J, or in other words, $n(J)$ is the integer for which J is a partition. The $\mathbf{bo}\langle 8k+2\rangle$'s are indexed by the partitions, J, with no 1's for which $n(J) = 2k+1$. In the sequel, let $\mathbf{bo}\langle J\rangle$ denote $\mathbf{bo}\langle 4n(J)\rangle$ if $n(J)$ is even or $\mathbf{bo}(4n(J) - 2\rangle$ if $n(J)$ is odd. There is also a copy of $\mathbf{bo}\langle 0\rangle$. There are elements M_J in dimensions $4n(J)$, where J is a partition of $n(J)$ with no 1's. These manifolds satisfy the condition that in our fixed decomposition of **MSpin**, the bordism class of M_J is a generator of $\pi_{4n(J)}\mathbf{bo}\langle J\rangle$ and maps to zero in $\pi_{4n(J)}$ of all the other summands.

Let $X(J, n) = RP^n \times M_J$ if $n(J)$ is even. If n is even, fix a $Pin^\pm$ structure on RP^n and consider $X(J, n)$ as an element of $Pin^\pm$ bordism. If $n(J)$ is odd, $RP^n \times M_J$ will be divisible by 2 in the corresponding *Pin* bordism group, so let $X(J, n)$ denote an element in $Pin^\pm$ bordism such that $2X(J, n) = RP^n \times M_J$. Note that for Pin^+ bordism we are asserting that $M_J = M_J \times RP^0$ is divisible by 2. Let $C(J, 2n)$ denote a cyclic group whose order is the order of the element $X(J, 2n)$ in the appropriate *Pin* bordism group. There are natural maps $C(J, 4n) \to MPin^+_{4n(J)+4n}$ and $C(J, 4n+2) \to MPin^-_{4n(J)+4n+2}$.

THEOREM 3. *The order of* $X(J, 2n)$ *is given as follows*:

	$2n = 8k$	$2n = 8k+2$	$2n = 8k+4$	$2n = 8k+6$
$n(J)$ even	2^{4k+1}	2^{4k+3}	2^{4k+4}	2^{4k+4}
$n(J)$ odd	2^{4k+2}	2^{4k+2}	2^{4k+3}	2^{4k+5}

The sum of the natural maps

$$\bigoplus_{J,n} C(J, 4n) \to MPin^+_*$$

is injective with image a summand: the complementary summand is a $Z/2$ *vector space. The sum of the natural maps*

$$\bigoplus_{J,n} C(J, 4n+2) \to MPin^-_*$$

is injective with image a summand: the complementary summand is a $Z/2$ *vector space. In both sums,* $n \geq 0$ *and* J *runs over all partitions with no* 1*'s.*

COROLLARY 4. *The* Pin^+ *bordism groups, modulo the Spin bordism submodule generated by the* RP^{4n}*, are* $Z/2$ *vector spaces. The* Pin^- *bordism groups, modulo the Spin bordism submodule generated by the* RP^{4n+2}*, are* $Z/2$ *vector spaces.*

Finally, we pause to consider the standard question of the image of Pin^+ bordism in unoriented bordism, denoted $\mathcal{N}_*$. Using the techniques of Anderson, Brown and Peterson [ABP2], we show

COROLLARY 5. *The image of the natural map* $MPin^+_* \to \mathcal{N}_*$ *equals all bordism classes all of whose Stiefel–Whitney numbers involving* $w_2(\tau)$ *vanish, where* τ *denotes the tangent bundle.*

After this paper was submitted, we learned of the paper of Giambalvo [G], which also calculates $MPin^+$ bordism. Giambalvo does the calculation via the Adams' spectral sequence and arrives at the same answer we do. He also attempted to analyse the role of the RP^{2n}'s in Pin^+ and Pin^- bordism, using the map ψ described below, but his results differ considerably from ours. Specifically, we claim that the order of RP^{8n+4} in Pin^+ bordism is 2^{8n+4} and that his Corollary 3.5 is wrong (see the discussion preceding Theorem 3). The table on page 399 is also incorrect: the factor corresponding to $\mathbf{M}(2) \wedge \mathbf{bo}\langle 8\rangle$ is missing and the Z_2^8 should be $Z/2^8$.

We would like to thank S. Stolz for numerous conversations on the subject of *Pin* bordism.

Proofs

We begin with two lemmas to reduce the calculation to a diagram chase.

LEMMA 6. *The* ith *Pin*$^+$ *bordism group is isomorphic to*

$\pi_i(\mathbf{MSpin} \wedge \mathbf{M}(4k+3))$ *for any* $k \geq 0$.

The ith Pin$^-$ *bordism group is isomorphic to*

$\pi_i(\mathbf{MSpin} \wedge \mathbf{M}(4k+1))$ *for any* $k \geq 0$.

In both cases, the usual transversality construction gives the isomorphism.

Proof. Let us begin with the *Pin*$^+$ case. Standard transversality constructions identify $\pi_i(\mathbf{MSpin} \wedge \mathbf{M}(4k+3))$ with the bordism theory of i-dimensional manifolds with a *Spin* structure on the bundle $\tau \oplus (4k+3)\det(\tau)$, where τ is the tangent bundle to the manifold and $\det(\tau)$ is the determinant line bundle. It is easy to check that for any bundle η, 4η has a canonical *Spin* structure, so the above bordism theory is equivalent to the bordism theory of i-dimensional manifolds with a *Spin* structure on the bundle $\tau \oplus 3\det(\tau)$. Next one can compute that any bundle η has a *Pin*$^+$ structure iff $\eta \oplus 3\det(\eta)$ has a *Spin* structure, and, since this is a universal relation, one can set up a one-to-one correspondence between *Spin* structures on $\eta \oplus 3\det(\eta)$ and *Pin*$^+$ structures on η. Hence our bordism theory is equivalent to the bordism theory of i-dimensional manifolds with a *Pin*$^+$ structure on the tangent bundle.

The *Pin*$^-$ case is entirely similar. □

Let $\mathbf{M}(Z/2, 0) = \mathrm{e}^0 \cup \mathrm{e}^1$ with attaching map of degree 2 and denote the homotopy ith group of $\mathbf{MSpin} \wedge \mathbf{M}(Z/2,0)$ by $(\mathbf{MSpin} \wedge Z/2)_i$. These groups can largely be calculated by applying *Spin* bordism to the cofibration sequence $S^0 \xrightarrow{\times 2} S^0 \to \mathbf{M}(Z/2, 0)$, since the degree 2 map on S^0 induces multiplication by 2 on the *Spin* bordism groups.

These groups have an interpretation as $Z/2$-*Spin* bordism. This is the bordism theory consisting of a manifold M with a codimension-one submanifold N; an orientation on $M - N$ which does not extend across any component of N; an orientation of the normal bundle of N in M; a *Spin* structure on $M - N$; a *Spin* structure on N; and diffeomorphisms which preserve the *Spin* structures from N to the boundary components of $M - N$. We do not need this interpretation in the sequel.

LEMMA 7. *There exists a cofibration sequence*

$$\mathbf{M}(Z/2, 0) \to \mathbf{M}(2r - 1) \to \Sigma^2\mathbf{M}(2r + 1) \tag{8}$$

Hence we get long exact sequences

$$\cdots \to (\mathbf{MSpin} \wedge Z/2)_i \to MPin_i^+ \xrightarrow{\psi} MPin_{i-2}^- \to \cdots$$

$$\cdots \to (\mathbf{MSpin} \wedge Z/2)_i \to MPin_i^- \xrightarrow{\psi} MPin_{i-2}^+ \to \cdots$$

In both cases, the map ψ is defined by starting with a manifold M, finding a submanifold $N \subset M$ dual to ω_1, and then forming the transverse intersection, $N \cap N$. Notice that ψ can also be described by taking the natural map $\psi\colon \mathbf{M}(r) \to \Sigma^2\mathbf{M}(r + 2)$ and smashing it with **MSpin**. *In particular, the two exact sequences above decompose in the same way that* **MSpin** *does.*

Proof. Recall that $T(r\xi) = RP^\infty/RP^{r-1}$. Indeed, $RP^n \subset RP^{n+r}$ with normal bundle $r\xi|_{RP^n}$. Hence we have a map $RP^{n+r} \to T(r\xi|_{RP^n})$ and the composite $RP^n \subset RP^{n+r} \to T(r\xi|_{RP^n})$ is the zero-section. Hence a copy of RP^{r-1} disjoint from RP^n in RP^{n+r} is null-homotopic in $T(r\xi|_{RP^n})$, so we get a map $RP^{n+r}/RP^{r-1} \to T(r\xi|_{RP^n})$ which is easily checked to be a homotopy equivalence.

The cofibration sequence is now clear since RP^{2r}/RP^{2r-2} is homotopy equivalent to $T((2r - 1)\xi|_{RP^1})$ and this is $\Sigma^{2r-1}\mathbf{M}(Z/2, 0)$.

The description of the map ψ also follows. Consider a *Spin* boundary M^{m+2r-1} and a map $f\colon M \to T((2r - 1)\xi)$. The map ψ sends f to the composite $M \to T((2r + 3)\xi)$ of f and the map $g\colon T((2r + 1)\xi) \to T((2r + 3)\xi)$. To see what happens to the underlying *Pin* manifolds, we can assume that f lands in $T((2r - 1)\xi|_{RP^N})$ for some large N, and we get a cofibration sequence like (8) but taking place inside of RP^{N+2r+1} instead of RP^∞. We make the new map transverse to the zero-section to get out *Pin* manifold, P. The map g becomes a map $g\colon T((2r + 1)\xi|_{RP^N}) \to T((2r + 3)\xi|_{RP^{N-2}})$, so to get $\psi(P)$ we make the map

$P \to RP^N$ transverse to RP^{N-2}. But this is the same as making it transverse to RP^{N-1}, which gives a dual to ω_1 in P and then intersecting this dual with itself. □

PROPOSITION 9. *The two tables below are obtained by smashing* **bo**$\langle 0\rangle$ *with the cofibration sequence* (8) *and taking homotopy groups. The first table takes* $2r - 1 = 4k + 1$ *and the second takes* $2r - 1 = 4k + 3$.

π_{8n+i}	$\mathbf{M}(Z/2, 0) \wedge \mathbf{bo}\langle 0\rangle$	$\mathbf{M}(4k+1) \wedge \mathbf{bo}\langle 0\rangle$	$\Sigma^2\mathbf{M}(4k+3) \wedge \mathbf{bo}\langle 0\rangle$
0	$Z/2$	$Z/2$	0
1	$Z/2$	$Z/2$	0
2	$Z/4$	$Z/2^{4n+3}$	$Z/2^{4n+1}$
3	$Z/2$	0	0
4	$Z/2$	0	$Z/2$
5	0	0	$Z/2$
6	0	$Z/2^{4n+4}$	$Z/2^{4n+4}$
7	0	0	0

π_{8n+i}	$\mathbf{M}(Z/2, 0) \wedge \mathbf{bo}\langle 0\rangle$	$\mathbf{M}(4k+3) \wedge \mathbf{bo}\langle 0\rangle$	$\Sigma^2\mathbf{M}(4k+5) \wedge \mathbf{bo}\langle 0\rangle$
0	$Z/2$	$Z/2^{4n+1}$	$Z/2^{4n}$
1	$Z/2$	0	0
2	$Z/4$	$Z/2$	$Z/2$
3	$Z/2$	$Z/2$	$Z/2$
4	$Z/2$	$Z/2^{4n+4}$	$Z/2^{4n+3}$
5	0	0	0
6	0	0	0
7	0	0	0

Proof. The groups in the first columns follow from Bott periodicity and the work on coefficients of Araki and Toda, [AT1] [AT2]. Note that all the groups follow easily from the cofibration sequence $S^0 \xrightarrow{\times 2} S^0 \to \mathbf{M}(Z/2,0)$ except the extension for π_2. The groups in the column for $\mathbf{M}(4k+1)$ are taken from Anderson, Brown and Peterson, [ABP2]. From the second table $\mathbf{bo}\langle 0\rangle \wedge \mathbf{M}(3)_{8n+7} = \mathbf{bo}\langle 0\rangle \wedge \mathbf{M}(3)_{8n+6} = \mathbf{bo}\langle 0\rangle \wedge \mathbf{M}(3)_{8n+5} = 0$. From the first table $\mathbf{bo}\langle 0\rangle \wedge \mathbf{M}(3)_{8n+4} = Z/2^{4n+4}$ and $\mathbf{bo}\langle 0\rangle \wedge \mathbf{M}(3)_{8n+3} = \mathbf{bo}\langle 0\rangle \wedge \mathbf{M}(3)_{8n+2} = Z/2$. An easy diagram chase using the second table shows that $\mathbf{bo}\langle 0\rangle \wedge \mathbf{M}(3)_{4n+1} = 0$ and that $0 \to Z/2 \to \mathbf{bo}\langle 0\rangle \wedge \mathbf{M}(3)_{8n} \to Z/2^{4n} \to 0$ is exact. An easy diagram chase using the first table shows that $\mathbf{bo}\langle 0\rangle \wedge \mathbf{M}(3)_{8n} = Z/2^{4n+1}$. □

PROPOSITION 10. *The two tables below are obtained by smashing* $\mathbf{bo}\langle 2\rangle$ *with the cofibration sequence (8) and taking homotopy groups. The first table takes* $2r-1=4k+1$ *and the second takes* $2r-1=4k+3$.

π_{8n+i}	$\mathbf{M}(Z/2,0)\wedge\mathbf{bo}\langle 2\rangle$	$\mathbf{M}(4k+1)\wedge\mathbf{bo}\langle 2\rangle$	$\Sigma^2\mathbf{M}(4k+3)\wedge\mathbf{bo}\langle 2\rangle$
4	$Z/2$	$Z/2$	$Z/2\oplus Z/2$
5	0	0	$Z/2$
6	0	$Z/2^{4n+2}$	$Z/2^{4n+2}$
7	0	$Z/2$	$Z/2$
8	$Z/2$	$Z/2\oplus Z/2$	$Z/2$
9	$Z/2$	$Z/2$	0
10	$Z/4$	$Z/2^{4n+5}$	$Z/2^{4n+3}$
11	$Z/2$	$Z/2$	$Z/2$

π_{8n+i}	$\mathbf{M}(Z/2,0)\wedge\mathbf{bo}\langle 2\rangle$	$\mathbf{M}(4k+3)\wedge\mathbf{bo}\langle 2\rangle$	$\Sigma^2\mathbf{M}(4k+5)\wedge\mathbf{bo}\langle 2\rangle$
4	$Z/2$	$Z/2^{4n+2}$	$Z/2^{4n+1}$
5	0	$Z/2$	$Z/2$
6	0	$Z/2$	$Z/2$
7	0	0	0
8	$Z/2$	$Z/2^{4n+3}$	$Z/2^{4n+2}$
9	$Z/2$	$Z/2$	$Z/2$
10	$Z/4$	$Z/2\oplus Z/2$	$Z/2\oplus Z/2$
11	$Z/2$	$Z/2$	$Z/2$

The groups π_{8n+i} *for* $n=0$ *and* $i=0$ *or* 1 *vanish for dimensional reasons. For* $n=0$ *and* $i=2$ *or* 3, $\pi_{8n+i}(\mathbf{M}(4k+3)\wedge\mathbf{bo}\langle 2\rangle)=\pi_{8n+i}(\mathbf{M}(Z/2,0)\wedge\mathbf{bo}\langle 2\rangle)=\pi_{8n+i}(\mathbf{M}(4k+1)\wedge\mathbf{bo}\langle 2\rangle)=Z/2$.

Proof. The first columns follow just as above. Indeed, $\pi_i(\mathbf{M}(Z/2,0)\wedge\mathbf{bo}\langle 2\rangle)=\pi_i(\mathbf{M}(Z/2,0)\wedge\mathbf{bo}\langle 0\rangle)$ unless $i=0$, 1 or 2, in which case $\pi_i(\mathbf{M}(Z/2,0)\wedge\mathbf{bo}\langle 2\rangle)=0$ if $i=0$, or 1, and $\pi_2(\mathbf{M}(Z/2,0)\wedge\mathbf{bo}\langle 2\rangle)=Z/2$. (This is why we have started our rows with 4 and gone to 11.) From the second table $\mathbf{bo}\langle 2\rangle\wedge\mathbf{M}(3)_{8n+7}=0$ and $\mathbf{bo}\langle 2\rangle\wedge\mathbf{M}(3)_{8n+6}=Z/2$. From the first table $\mathbf{bo}\langle 2\rangle\wedge\mathbf{M}(3)_{8n+4}=Z/2^{4n+2}$ and $\mathbf{bo}\langle 2\rangle\wedge\mathbf{M}(3)_{8n+5}=Z/2$. Feeding these values back into the second table, we see that $\mathbf{bo}\langle 2\rangle\wedge\mathbf{M}(3)_{8n+9}$ has order at least 2 and at most 4. From the first table, it is a subgroup of $Z/2$, and so $\mathbf{bo}\langle 2\rangle\wedge\mathbf{M}(3)_{8n+9}=Z/2$. From the first table, $\mathbf{bo}\langle 2\rangle\wedge\mathbf{M}(3)_{8n+8}$ is cyclic of order at least 2^{4n+3} and at most 2^{4n+5}. Feeding this into the second table, $\mathbf{bo}\langle 2\rangle\wedge\mathbf{M}(3)_{8n+8}=Z/2^{4n+3}$ and $\mathbf{bo}\langle 2\rangle\wedge\mathbf{M}(3)_{8n+10}$ has order at least 2 and at most 8. From the first table, $\mathbf{bo}\langle 2\rangle\wedge\mathbf{M}(3)_{8n+11}$ is a subgroup of $Z/2$, whereas from the second table it has order at least 2. Hence $\mathbf{bo}\langle 2\rangle \wedge \mathbf{M}(3)_{8n+11} = Z/2$, and from either table $\mathbf{bo}\langle 2\rangle \wedge \mathbf{M}(3)_{8n+10}$ has order 4. Indeed, the tenth row of the second table is $Z/4 \to \mathbf{bo}\langle 2\rangle \wedge \mathbf{M}(3)_{8n+10} \to Z/2 \oplus Z/2$. These sequences are modules over $\pi_*(\mathbf{bo}\langle 0\rangle)$, and the product with the non-zero class in $\pi_1(\mathbf{bo}\langle 0\rangle)$ induces an epimorphism $\pi_{8n+10}(\mathbf{M}(Z/2,0) \wedge \mathbf{bo}\langle 2\rangle) \to \pi_{8n+11}(\mathbf{M}(Z/2,0) \wedge \mathbf{bo}\langle 2\rangle)$. This shows that $\mathbf{bo}\langle 2\rangle \wedge \mathbf{M}(3)_{8n+10} = Z/2 \oplus Z/2$ since the map $\pi_{8n+11}(\mathbf{M}(Z/2,0) \wedge \mathbf{bo}\langle 2\rangle) \to \pi_{8n+11}(\mathbf{M}(4k+3) \wedge \mathbf{bo}\langle 2\rangle)$ is an isomorphism. To see the required product relation, first of all observe that it is an equally valid relation in $\pi_*(\mathbf{M}(Z/2,0) \wedge \mathbf{bo}\langle 0\rangle)$. In $\pi_*(\mathbf{bo}\langle 0\rangle)$ it is a well-known relation that $\pi_{8n+1}(\mathbf{bo}\langle 0\rangle) \to \pi_{8n+2}(\mathbf{bo}\langle 0\rangle)$ is an isomorphism. The required relation is an easy diagram chase.

The case $8n + i = 2$ or 3 can be dealt with similarly, and it is not hard to see that $\mathbf{bo}\langle 2\rangle \wedge \mathbf{M}(3)_3 = \mathbf{bo}\langle 2\rangle \wedge \mathbf{M}(3)_2 = Z/2$. Note that $\pi_2(\mathbf{bo}\langle 2\rangle \wedge \mathbf{M}(Z/2,0)) = Z/2$, not the $Z/4$ one might have expected, and that $\pi_3(\Sigma^2\mathbf{M}(r) \wedge \mathbf{bo}\langle 2\rangle) = 0$, as does π_2 since $\Sigma^2\mathbf{M}(r) \wedge \mathbf{bo}\langle 2\rangle$ is 3-connected. □

The reader can easily deduce Theorem 1 from Propositions 9 and 10.

LEMMA 11. *There exists a positive integer valued function, $\phi(r)$, such that, for any integer s such that $s > r$ and $s \equiv -(r+1) \bmod 2^{\phi(r)}$, there exists a map $c : S^{r+s} \to T(s\xi)$ which is transverse to the zero section with inverse image RP^r.*

Proof. The reduced K-theory of RP^r is cyclic of order a power of 2 and generated by ξ, the canonical line bundle. The power of 2 is the number $\phi(r)$, where $\phi(r)$ is the number of integers, t, with $0 < t \le r$ and $t \equiv 0, 1, 2, 4 \bmod 8$. Since the tangent bundle of RP^r is well-known to be $(r+1)\xi$, if we choose s as above, $s\xi$ is a stable bundle which is a normal bundle for an embedding of RP^r into S^{r+s}. Apply the Pontrjagin–Thom construction to this embedding to produce the map c. □

Remark. The map c, or even its homotopy class, is not unique. Indeed, since the $r+s$ sphere above has a unique *Spin* structure, c endows RP^{4n} with a Pin^+ structure and RP^{4n+2} with a Pin^- structure. It is not hard to see that c may be chosen to get either of the two $Pin^\pm$ structures that exist on an RP^{2m}.

We conclude this section with the proof of Theorem 3.

It will be convenient in what follows to fix a $Pin^\pm$ structure on RP^{2n}. Begin by noticing that $MPin_2^- = Z/8$ and RP^2 is a generator. Finally, if we switch the Pin^- structure on RP^2, we get the negative of our previous element. Fix a Pin^- structure on RP^2. It is easy to calculate our map ψ from above in this case: $\psi(RP^{2n}) = RP^{2n-2}$ and if we switch *Pin* structure on RP^{2n} we also switch *Pin* structure on RP^{2n-2}. Hence, having fixed a Pin^- structure on RP^2, we can use iterations of the ψ's to pick out a *Pin* structure on all the RP^{2n}'s.

Recall another result on the decomposition of **MSpin** from [ABP1]. The $\mathbf{bo}\langle\ \rangle$ factors are index by partitions J with no 1's in them. If we set $\mathbf{bo}(J) = \mathbf{bo}(4n(J))\rangle$ if $n(J)$ is even and $\mathbf{bo}(J) = \mathbf{bo}\langle 4n(J) - 2\rangle$ if $n(J)$ is odd, then, localized at 2, **MSpin** is a wedge of some $\mathbf{K}(Z/2,)$'s and $\mathbf{bo}\langle J\rangle$'s where we have one factor for each partition J with no 1's. For each partition J with no 1's select a *Spin* manifold, M_J. The class represented by M_J is a generator of $\pi_{4n(J)}(\mathbf{bo}\langle J\rangle)$ and maps to zero in the other factors in our fixed decomposition of **MSpin** localized at 2.

Define $Y(J, 4n) = M_J \times RP^{4n}$ as a Pin^+ manifold using our fixed Pin^+ structure on RP^{4n}. Define $Y(J, 4n+2) = M_J \times RP^{4n+2}$ as a Pin^- manifold using our fixed Pin^- structure on RP^{4n+2}. Note $\psi(Y(J, 2n)) = Y(J, 2n-2)$. By Lemma 11 we can identify the image of $Y(J, 2n)$ rather well in $\pi_*(\mathbf{MSpin} \wedge \mathbf{M}(s))$ where s is any integer so that RP^{2n} comes from $\pi_{2n}(\mathbf{M}(s))$. With our fixed decomposition of **MSpin**, $Y(J, 2n)$ vanishes in all components of the decomposition except for the Jth.

To proceed further, we need to analyze cases. Begin with the $8k$ case. If $k = 0$, then we just have M_J which is a generator of $\pi_{4n(J)}(\mathbf{bo}(4n(J))\rangle)$ localized at 2, and hence a generator of $\pi_{4n(J)}(\mathbf{bo}\langle 4n(J)\rangle \wedge \mathbf{M}(Z/2, 0))$. From the second table in Proposition 9, we see that this element continues to have order 2 in Pin^+ bordism. Fix any s as in Lemma 11 for $r = 8k$, and note that this s also satsifies the hypotheses of Lemma 11 for $r = 8k - 8$. Consider Pin^+ bordism as $\pi_*(\mathbf{MSpin} \wedge \mathbf{M}(s))$. Under our decomposition of **MSpin**, $Y(J, 8k)$ goes to 0 in all the pieces except for $\pi_*(\mathbf{bo}(J) \wedge \mathbf{M}(s))$. We claim that in this summand it is a generator of order 2^{4k+1}. Notice first that by Proposition 9 the element lives in a cyclic group of order 2^{4k+1}. The four-fold iterate of ψ defines a homomorphism from this group to the corresponding one for $8k - 8$, and $Y(J, 8k)$ goes to $Y(J, 8k - 8)$. If we assume by induction that $Y(J, 8k - 8)$ is a generator, then it follows that $Y(J, 8k)$ is also a generator and has the desired order.

Now suppose that $n(J)$ is odd. The manifold M_J still represents a generator of $\pi_{4n(J)}(\mathbf{bo}\langle 4n(J) - 2\rangle \wedge \mathbf{M}(Z/2, 0))$, but this time, consulting table two in Proposition 10, we see that M_J is divisible by 2. Let $s \equiv 3 \bmod 4$ and identify Pin^+ bordism with $\pi_*(\mathbf{MSpin} \wedge \mathbf{M}(s))$. Let $X(s, J, 0)$ denote a choice of element in Pin^+ bordism which lives in the $\mathbf{bo}\langle J\rangle \wedge \mathbf{M}(s)$ summand so that $2X(s, J, 0) = Y(J, 0) = M_J$. Fix any s as in Lemma 11 for $r = 8k$, and suppose that we defined $X(s, J, 8k - 8)$ which lives in the $\mathbf{bo}\langle J\rangle \wedge \mathbf{M}(s)$ summand of $\mathbf{MSpin} \wedge \mathbf{M}(s)$ and which satisfies $2X(s, J, 8k - 8) = Y(J, 8k - 8)$ and $X(s, J, 8k - 8)$ is a generator. From Proposition 10 this means that $X(s, J, 8k - 8)$ has order 2^{4k-2}. As above, consider the four-fold iteration of ψ. Restricted to the $\mathbf{bo}\langle J\rangle \wedge \mathbf{M}(s)$ factor, the map is just the epimorphism $Z/2^{4k+2} \to Z/2^{4k-2}$ and $Y(J, 8k)$ goes to $Y(J, 8k - 8)$. It is now easy to select $X(s, J, 8k)$ satisfying the required conditions. (Note that there are always two choices.)

The other cases are similar so we only discuss the key points. Begin with the next case, $8k+2$ and start with $k=0$. This means we are trying to identify RP^2 in $MPin_2^- = Z/8$. Applying ψ and consulting the first table from Proposition 9, we see that it is a generator. We can now use induction and the four-fold iterate of ψ to handle the case $n(J)$ even. In the case $n(J)$ is odd, we need to identify $M_J \times RP^2$. It lives in a group of order 4, and table one of Proposition 10, shows that ψ is an isomorphism, so $M_J \times RP^2$ is of order 2 in Pin^- bordism, since M_J has order 2 in Pin^+ bordism. Hence we define $X(s, J, 8k+2)$ as above using the four-fold iterate of ψ. The cases $8k+4$ and $8k+6$ are done in the same way.

Now let us define $X(J, 2n) = Y(J, 2n)$ if $n(J)$ is even; for $n(J)$ odd, define $X(J, 2n) = X(2^{\phi(2n)+1} - (2n+1), J, 2n)$. From the above discussion, we know the orders of each of the $X(J, 2n)$'s: let $C(J, 2n)$ denote a cyclic group of this order with a fixed generator and map $C(J, 2n)$ to $MPin^{\pm}$ by sending the fixed generator to $X(J, 2n)$. We get maps

$$\oplus_{J,n} C(J, 4n) \to MPin_*^+ \quad \text{and} \quad \oplus_{J,n} C(J, 4n+2) \to MPin_*^- .$$

For n fixed we see from above that $\oplus_{J,n} C(J, 4n) \to MPin_*^+$ and $\oplus_{J,n} C(J, 4n+2) \to MPin_*^-$ are split injective. Theorem 3 asserts that these maps are still split injective when we also sum over the n.

We do the Pin^+ case. Fix a dimension $r = 8k$. Note that $C(J, 4n)$ lands in dimension r iff $r = 4n(J) + 4n$. If $n(J)$ is even, then $C(J, 4n)$ has order 2^{2n+1} and if $n(J)$ is odd, $C(J, 4n)$ has order 2^{2n+2}. In particular, two $C(J, 4n)$'s which land in the same dimension and have the same order have the same n and the same $n(J)$. If $r = 8k+4$ we get different numbers but the same conclusion. Finally note that both $\oplus_{r=4n(J)+4n} C(J, 4n)$ and $Mpin_r^+$ have the same number of $Z/2^k$ summands for all $k > 1$, and if we restrict the map $\oplus_{r=4n(J)+4n} C(J, 4n) \to MPin_r^+$ to the summands of order 2^k we get a split injection. It is an elementary algebra exercise to verify that this means that the map is a split injection and the complementary summand is a $Z/2$ vector space.

The Pin^- case is entirely similar.

The proof of Corollary 5

We begin with a general discussion of characteristic numbers. Let BG be a space such as BSO, $BPin^+$, etc. equipped with a map to BO. Let M be a manifold with a G structure; i.e. the tangent bundle map $M \to BO$ has a fixed lift to a map $\tau : M \to BG$. Then M^n determines a homomorphism $H^n(BG; Z/2) \to Z/2$ given by sending $x \in H^n(BG; Z/2)$ to $\tau^*(x)$ evaluated on the fundamental class of M. This defines a homomorphism $T : \Omega_n^G \to \mathrm{Hom}\,(H^n(BG; Z/2), Z/2)$. If we let $M(G)$ denote the Thom spectrum for the inverse to the universal bundle over BO pulled-back to

BG, the Thom isomorphism shows that we can equally regard T as a homomorphism $T: \Omega_n^G \to \mathrm{Hom}(H^n(M(G); Z/2), Z/2)$. If a homomorphism $b: H^n(M(G); Z/2) \to Z/2$ is to be in the image of T, then $b(ax) = 0$ for any a in the mod 2 Steenrod algebra of dimension at least 1 and any $x \in H^*(M(G); Z/2)$. If we let $\mathscr{A}$ denote the mod 2 Steenrod algebra, we can turn $Z/2$ into an $\mathscr{A}$ module by letting all the Sq^i act trivially. Then $\mathrm{Hom}_{\mathscr{A}}(H^n(M(G); Z/2), Z/2) \subset \mathrm{Hom}(H^n(M(G); Z/2), Z/2)$ is precisely the set of homomorphisms satisfying our condition and Condition P of [ABP2] merely says that the image of T is precisely $\mathrm{Hom}_{\mathscr{A}}(H^n(M(G); Z/2), Z/2)$. (It is also true that $\mathrm{Hom}_{\mathscr{A}}(H^n(M(G); Z/2), Z/2) = E_2^{0,n}(M(G))$ in the Adams spectral sequence for $\pi_*(M(G))$. Moreover, $E_\infty^{0,n}(M(G)) \subset E_2^{0,n}(M(G))$ is precisely the image of T. Hence the collapse of the Adams spectral sequence is sufficient for $M(G)$ to have Property P.)

Now $\mathrm{Hom}_{\mathscr{A}}(H^n(M(G); Z/2), Z/2)$ behaves like any other Hom, so we can apply it to the short exact sequences of cohomology groups coming from (8). It is not hard to see directly that $E_2^{0,r}(\mathbf{M}(Z/2, 0) \wedge \mathbf{bo}\langle 0\rangle) = Z/2$ if $r = 0$; $E_2^{0,r}(\mathbf{M}(Z/2, 0) \wedge \mathbf{bo}\langle 2\rangle) = Z/2$ if $r = 2$ and both groups are 0 otherwise. Theorem 4.4 of [ABP2] says that $E_2^{0,r}(\mathbf{M}(1) \wedge \mathbf{bo}\langle 0\rangle) = Z/2$ if $r = 0$ or $r \equiv 2 \pmod 4$; $E_2^{0,r}(\mathbf{M}(1) \wedge \mathbf{bo}\langle 2\rangle) = Z/2$ if $r = 2$ or $r \equiv 0 \pmod 4$ and both groups are 0 otherwise. One can also check by hand that $E_2^{0,r}(\mathbf{M}(3) \wedge \mathbf{bo}\langle 0\rangle) = Z/2$ if $r = 0$ and is 0 for $r < 3$ and that $E_2^{0,r}(\mathbf{M}(3) \wedge \mathbf{bo}\langle 2\rangle) = Z/2$ if $r = 2$ and is 0 otherwise for $r < 5$. By comparing the two exact sequences coming from (8) we can compute $E_2^{0,r}(\mathbf{M}(3) \wedge \mathbf{bo}\langle 0\rangle)$ and $E_2^{0,r}(\mathbf{M}(3) \wedge \mathbf{bo}\langle 2\rangle)$. More importantly, we can see that $\psi: E_2^{0,r}(\mathbf{M}(1) \wedge \mathbf{bo}\langle 0\rangle) \to E_2^{0,r-2}(\mathbf{M}(3) \wedge \mathbf{bo}\langle 0\rangle)$ and $\psi: E_2^{0,r}(\mathbf{M}(1) \wedge \mathbf{bo}\langle 2\rangle) \to E_2^{0,r-2}(\mathbf{M}(3) \wedge \mathbf{bo}\langle 2\rangle)$ are both epic. Since $\mathbf{M}(1) \wedge \mathbf{bo}\langle 0\rangle)$ and $\psi: E_2^{0,r}(\mathbf{M}(1) \wedge \mathbf{bo}\langle 2\rangle) \to E_2^{0,r-2}(\mathbf{M}(3) \wedge \mathbf{bo}\langle 2\rangle)$ are both epic. Since $\mathbf{M}(1) \wedge \mathbf{bo}\langle 0\rangle$ and $\mathbf{M}(1) \wedge \mathbf{bo}\langle 2\rangle$ satisfy Property P by [ABP2], this shows that $\mathbf{M}(3) \wedge \mathbf{bo}\langle 0\rangle$ and $\mathbf{M}(3) \wedge \mathbf{bo}\langle 2\rangle$ also satisfy Property P. The Eilenberg–MacLane summands also satisfy Property P, hence so does $MPin^+$.

Since $H^*(BO; Z/2) \to H^*(BPin^+; Z/2)$ is onto, it follows formally that a manifold, M^n, is unoriented bordant to a Pin^+ manifold iff all the characteristic numbers in the kernel of $H^n(BO; Z/2) \to H^n(BPin^+; Z/2)$ vanish on M. This kernel is the ideal in $H^*(BO; Z/2)$ generated by w_2 and its images under the Steenrod algebra: e.g. w_3 is in the kernel. It is always the case however that, if all the characteristic BO-numbers of a manifold which involve $x \in H^r(BO; Z/2)$ vanish, then all the numbers involving $a(x)$ for any $a \in \mathscr{A}$ also vanish. Hence M is bordant to a Pin^+ manifold iff all tangential characteristic numbers involving w_2 vanish.

We may as well finish by remarking that $\mathbf{MSpin} \wedge Z/2$ satisfies Property P and that a manifold is unoriented bordant to an element in $\mathbf{MSpin} \wedge Z/2$ iff all the numbers involving ω_2 and ω_1^2 vanish.

The tables

Here are the promised Pin^+ bordism groups through dimension 95, arranged in two tables. The second table gives $A(n)$, the number of $Z/2$ summands in $MPin_n^+$. The first table gives numbers $\pi(n)$ which enable us to find the other summands in dimensions congruent to 0 mod 4. For $MPin_{8n+4}^+$, the summands of order greater than 2 are $\oplus \pi(i) Z/2^{4n+4-2i}$ beginning with $i = 0$ and continuing until $4n + 4 - 2i = 2$. For $MPin_{8n+8}^+$, the summands of order greater than 2 are $\oplus \pi(i) Z/2^{4n+5-2i}$ beginning with $i = 0$ and continuing until $4n + 4 - 2i = 3$. As an example, $28 = 8 \cdot 3 + 4$ so $MPin_{28}^+ = 4Z/2 \oplus (1Z/2^{16} \oplus 0Z/2^{14} \oplus 1Z/2^{12} \oplus 1Z/2^{10} \oplus 2Z/2^8 \oplus 2Z/2^6 \oplus 4Z/2^4 \oplus 4Z/2^2)$

n	$\pi(n)$	n	$\pi(n)$	n	$\pi(n)$	n	$\pi(n)$	n	$\pi(n)$	n	$\pi(n)$
0	1	4	2	8	7	12	21	16	55	20	137
1	0	5	2	9	8	13	24	17	66	21	165
2	1	6	4	10	12	14	34	18	88	22	210
3	1	7	4	11	14	15	41	19	105	23	253

n	$A(n)$	n	$A(n)$	n	$A(n)$	n	$A(n)$	n	$A(n)$	n	$A(n)$	n	$A(n)$	n	$A(n)$
0	1	12	0	24	6	36	17	48	113	60	394	72	1556	84	4965
1	0	13	1	25	5	37	34	49	130	61	526	73	1764	85	5843
2	1	14	1	26	20	38	41	50	244	62	606	74	2440	86	6541
3	1	15	0	27	17	39	27	51	222	63	548	75	2423	87	6605
4	0	16	2	28	4	40	43	52	152	64	673	76	2224	88	7536
5	0	17	1	29	12	41	49	53	220	65	771	77	2694	89	8412
6	0	18	8	30	15	42	109	54	258	66	1150	78	3041	90	10515
7	0	19	7	31	8	43	96	55	218	67	1114	79	2995	91	10814
8	1	20	1	32	16	44	54	56	281	68	959	80	3475	92	10730
9	0	21	4	33	17	45	89	57	324	69	1209	81	3907	93	12365
10	3	22	5	34	48	46	106	58	534	70	1378	82	5103	94	13750
11	3	23	2	35	41	47	81	59	503	71	1310	83	5168	95	14135

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