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Autor(en): **Mislin, Guido**

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## On group homomorphisms inducing mod- $p$ cohomology isomorphisms

GUIDO MISLIN

Let  $\rho : F_1 \rightarrow F_2$  be a homomorphism of finite groups  $F_1$  and  $F_2$  inducing an isomorphism  $H^*(F_2; \mathbb{Z}/p) \rightarrow H^*(F_1; \mathbb{Z}/p)$ ,  $p$  a fixed prime. By a result of S. Jackowski [5] it is known that then

- (i)  $\ker(\rho)$  is of order prime to  $p$ ,
- (ii)  $\text{im}(\rho)$  has index prime to  $p$ .

Simple examples show that in general (i) and (ii) alone do not suffice for  $\rho$  to induce a  $\mathbb{Z}/p$ -cohomology isomorphism. The purpose of this note is to describe necessary and sufficient conditions on  $\rho$  in group theoretic terms for  $\rho$  to induce an  $H^*\mathbb{Z}/p$ -isomorphism. It turns out to be natural to work in the more general setting of compact Lie groups. The following notations and terminology will be used throughout this note.

For  $\rho : G \rightarrow H$  a morphism of compact Lie groups we write

$$C(\rho) = \{h \in H \mid h\rho(g) = \rho(g)h \text{ for all } g \in G\}$$

for the centralizer of  $\rho$ ,

$$N(\rho) = \{h \in H \mid h\rho(G) = \rho(G)h\}$$

for the normalizer of  $\rho$ , and

$$W(\rho) = N(\rho)/C(\rho)$$

for the Weyl group of  $\rho$ . Note that  $W(\rho)$  is a compact Lie group. It is a finite group, if for instance  $\rho(G)$  is a finite subgroup of  $H$ . In case  $\rho : T \rightarrow G$  stands for the inclusion of a maximal torus into a compact connected Lie group,  $W(\rho) = W(G)$ , the classical Weyl group of  $G$ . As usual

$$\text{Rep}(G, H)$$

stands for the representations of  $G$  in  $H$ , that is, the set of  $H$ -conjugacy classes of continuous homomorphisms  $G \rightarrow H$ . For  $p$  a prime we write

$$Q_p(G)$$

for the Quillen-category of finite  $p$ -subgroups of  $G$ ; its objects are the finite  $p$ -subgroups of  $G$ , and morphisms  $P_1 \rightarrow P_2$  are homomorphisms of the form  $c^g : x \rightarrow g^{-1}xg$  for some  $g \in G$ .

Our theorem then takes the following form.

**THEOREM.** *Let  $\rho : G \rightarrow H$  be a morphism of compact Lie groups and let  $p$  be a prime. Then the following are equivalent:*

- (A)  $H^*B\rho : H^*(BH; \mathbb{Z}/p) \rightarrow H^*(BG; \mathbb{Z}/p)$  is an isomorphism.
- (B)  $\text{Rep}(\rho) : \text{Rep}(\pi, G) \rightarrow \text{Rep}(\pi, H)$  is a bijection for every finite  $p$ -group  $\pi$ .
- (C)  $Q_p(\rho) : Q_p(G) \rightarrow Q_p(H)$  is an equivalence of categories.

**REMARK.** The reader verifies easily that (B) implies

(Bi):  $\ker(\rho)$  contains no element of order  $p$ .

(Bii): every finite  $p$ -subgroup in  $H$  is conjugate to a subgroup in  $\rho(G) \subset H$ .

These statements generalize (i) and (ii) above to the case of compact Lie groups.

Before proving the Theorem, we want to recall some basic facts on homotopy fixed-points. All spaces considered are supposed to be of the homotopy type of CW-complexes. If  $X$  denotes a  $\pi$ -space,  $\pi$  a group, one writes  $X^{h\pi}$  for the homotopy fixed-point space of the  $\pi$  action on  $X$ . It is by definition equal to  $\text{map}_\pi(E\pi, X)$ , the space of  $\pi$ -maps from the universal  $\pi$ -space  $E\pi$  to  $X$ . The space of fixed-points  $X^\pi$  maps naturally to  $X^{h\pi}$  and the induced map

$$(Z/p)_\infty(X^\pi) \rightarrow ((Z/p)_\infty X)^{h\pi}, \tag{1}$$

is known to be an equivalence, if  $\pi$  is a finite  $p$ -group and  $X$  a finite dimensional  $\pi$ -space (Theorem of Carlsson, Miller and Lannes, cf. [2]). The functor  $(Z/p)_\infty(-)$  denotes the Bousfield–Kan  $Z/p$ -completion functor [1]. It has the basic property that it turns  $H_*( ; Z/p)$ -isomorphisms into homotopy equivalences. The following lemma is implicit in Carlsson’s paper [2].

**LEMMA 1.** *Let  $X$  be a finite dimensional  $\pi$ -space,  $\pi$  a finite  $p$ -group. Then the natural map*

$$X^{h\pi} \rightarrow ((Z/p)_\infty X)^{h\pi}$$

*induces a bijection of connected components.*

*Proof.* From the equivalence (1) we see that

$$\pi_0((Z/p)_\infty X)^{h\pi} \cong \pi_0(X^\pi).$$

By [2, VI.12],

$$\pi_0(X^{h\pi}) \cong \pi_0(((Z/p)_\infty^{\text{tot}} X)^\pi),$$

where the space  $((Z/p)_\infty^{\text{tot}} X)^\pi$  consists of a disjoint union of certain partial completions of the components of  $X^\pi$  (cf. [2, IV.3]). Therefore,

$$\pi_0(((Z/p)_\infty^{\text{tot}} X)^\pi) \cong \pi_0(X^\pi)$$

and the lemma follows.

We will also need the following result which applies to arbitrary (not necessarily finite dimensional) spaces  $X$ .

**LEMMA 2.** *Suppose  $X$  is an  $i$ -connected  $\pi$ -space,  $\pi$  a finite  $p$ -group and  $i \geq 2$ . Then the canonical map*

$$\Theta : X^{h\pi} \rightarrow ((Z/p)_\infty X)^{h\pi}$$

*induces a  $\pi_0$ -bijection, and isomorphisms*

$$\pi_j(X^{h\pi}, x) \rightarrow \pi_j(((Z/p)_\infty X)^{h\pi}, \Theta x)$$

*for  $j < i$  and all  $x \in X^{h\pi}$ .*

*Proof.* Since  $X$  is  $i$ -connected,  $(Z/p)_\infty X$  is  $i$ -connected too and it follows that the fibre  $F$  of  $X \rightarrow (Z/p)_\infty X$  is  $(i - 1)$ -connected, with uniquely  $p$ -divisible homotopy groups. The (homotopy) fibre  $F_y$  of  $\Theta$  over a point  $y \in ((Z/p)_\infty X)^{h\pi}$  may be identified with  $F^{h\pi}$  for some action of  $\pi$  on  $F$ . Since  $F$  is 1-connected mod- $p$  acyclic,  $F^{h\pi}$  is  $p$ -acyclic too [3, 2.3]. Thus  $F_y$  is non-empty and connected, which implies that  $\Theta$  is a  $\pi_0$ -bijection. The obstruction theory spectral sequence

$$H^*(\pi, \underline{\pi}_* F) \Rightarrow \pi_*(F^{h\pi})$$

then collapses, because the groups  $\pi_k F$  are all uniquely  $p$ -divisible, and it follows that

$$\pi_k(F^{h\pi}) \cong (\pi_k F)^\pi$$

for all  $k$ . In particular,  $\pi_k(F^{h\pi}) = \pi_k(F_y) = 0$  for  $k < i$  since  $F$  is  $(i - 1)$ -connected. It follows then that  $\Theta$  is a  $\pi_j$ -isomorphism for  $j < i$ .

*Proof of the Theorem.* (A)  $\Rightarrow$  (B). By Dwyer–Zabrodsky [3] one has a natural bijection

$$\text{Rep}(\pi, G) \rightarrow \pi_0 \text{map}(B\pi, BG), \tag{2}$$

associating with a homomorphism  $\varphi : \pi \rightarrow G$  the component of  $\text{map}(B\pi, BG)$  containing  $B\varphi$ ; we denote that component by  $\text{map}(B\pi, BG)_\varphi$ . As  $B\rho : BG \rightarrow BH$  is an  $H_*( ; \mathbb{Z}/p)$ -isomorphism, the induced map

$$\text{map}(B\pi, (\mathbb{Z}/p)_\infty BG) \rightarrow \text{map}(B\pi, (\mathbb{Z}/p)_\infty BH), \tag{3}$$

is an equivalence. Thus, to prove (B) it suffices to show that for a general compact Lie group  $G$

$$\pi_0(\text{map}(B\pi, BG)) \rightarrow \pi_0(\text{map}(B\pi, (\mathbb{Z}/p)_\infty BG)), \tag{4}$$

is a bijection. This is certainly so for  $G = SU(n)$  as we see from Lemma 2 (trivial  $\pi$ -action on  $BSU(n)$ ). In the general case we choose an embedding  $\epsilon : G \rightarrow SU(n)$  for some  $n$ , and we look at the fibration

$$SU(n)/G \rightarrow BG \rightarrow BSU(n), \tag{5}$$

If we fix a map  $\sigma : \pi \rightarrow SU(n)$  which factors through  $G \subset SU(n)$ , then we obtain a fibration sequence

$$Z \rightarrow \coprod_{\alpha} \text{map}(B\pi, BG)_{\sigma_\alpha} \rightarrow \text{map}(B\pi, BSU(n))_{\sigma}, \tag{6}$$

where  $\sigma_\alpha : \pi \rightarrow G$  runs over all  $G$ -conjugacy classes for which  $\epsilon\sigma_\alpha$  is  $SU(n)$ -conjugate to  $\sigma$ . We can identify  $Z$  with the space of sections of the fibration

$$SU(n)/G \rightarrow E\pi \times_{\pi} (SU(n)/G) \rightarrow B\pi$$

which is obtained by pulling back (5) along  $B\sigma : B\pi \rightarrow BSU(n)$ . As a result

$$Z \cong \text{map}_{\pi}(E\pi, SU(n)/G) = (SU(n)/G)^{h\pi}$$

where  $\pi$  acts on  $SU(n)/G$  via  $\sigma$ . Since  $BSU(n)$  is simply connected,  $(\mathbb{Z}/p)_\infty(-)$  turns

(5) into a fibration sequence

$$(Z/p)_\infty(SU(n)/G) \rightarrow (Z/p)_\infty BG \rightarrow (Z/p)_\infty BSU(n), \tag{7}$$

which will give rise, as before, to a fibration

$$(((Z/p)_\infty(SU(n)/G))^{h\pi} \rightarrow \text{map}(B\pi, (Z/p)_\infty BG)_{R(\sigma)} \rightarrow \text{map}(B\pi, (Z/p)_\infty BSU(n))_\sigma, \tag{8}$$

where  $\text{map}(B\pi, (Z/p)_\infty BG)_{R(\sigma)}$  denotes the disjoint union of those connected components of  $\text{map}(B\pi, (Z/p)_\infty BG)$  which map to  $\text{map}(B\pi, (Z/p)_\infty BSU(n))_\sigma$ , the component of  $(Z/p)_\infty B\sigma$  of  $\text{map}(B\pi, (Z/p)_\infty BSU(n))$ . To ensure that the map in (4) is bijective it obviously suffices to check that

$$\pi_0\left(\coprod_x \text{map}(B\pi, (Z/p)_\infty BG)_{\sigma_x}\right) \rightarrow \pi_0(\text{map}(B\pi, (Z/p)_\infty BG)_{R(\sigma)}), \tag{9}$$

is bijective for every  $\sigma : \pi \rightarrow SU(n)$  which factors through  $G \subset SU(n)$ . For this, consider the natural map of the fibration (6) to that of (8). Because

$$\pi_1 \text{map}(B\pi, BSU(n))_\sigma \rightarrow \pi_1 \text{map}(B\pi, (Z/p)_\infty BSU(n))_\sigma$$

is an isomorphism (Lemma 2) we see that (9) is a bijection, if the map on fibres

$$\pi_0(SU(n)/G)^{h\pi} \rightarrow \pi_0((Z/p)_\infty(SU(n)/G))^{h\pi}$$

is a bijection. But this is the case by Lemma 1.

(B)  $\Rightarrow$  (C). We first check that  $Q_p(\rho)$  induces a bijection on isomorphism classes of objects. Let  $A, B$  be finite  $p$ -subgroups of  $G$  with  $\rho(A)$  and  $\rho(B)$  isomorphic as objects of  $Q_p(H)$  so that there exists an  $h \in H$  with  $c^h : \rho(A) \rightarrow \rho(B)$  a group isomorphism. Note that  $A \rightarrow \rho(A)$  is injective in view of (B). Thus, there is a group isomorphism  $\Theta : A \rightarrow B$  rendering the diagram

$$\begin{array}{ccc} A & \xrightarrow{\rho} & \rho(A) \\ \Theta \downarrow & & \downarrow c^h \\ B & \xrightarrow{\rho} & \rho(B) \end{array}$$

commutative; we will show that  $\Theta = c^g$  for some  $g \in G$ , proving that  $A$  is isomor-

phic to  $B$  in  $Q_p(G)$ . Namely, because the bijection

$$\text{Rep}(\rho) : \text{Rep}(A, G) \rightarrow \text{Rep}(A, H)$$

maps the class of  $\tilde{\Theta} : A \rightarrow G, (x \rightarrow \Theta x)$ , to  $\rho\tilde{\Theta} = c^h\rho : A \rightarrow H$ , which is the same as the image under  $\text{Rep}(\rho)$  of the inclusion  $A \subset G$ , we infer that  $\tilde{\Theta}$  is  $G$ -conjugate to this inclusion; thus  $\Theta = c^g : A \rightarrow B$  for some  $g \in G$ . This shows that  $Q_p(G) \rightarrow Q_p(H)$  is one-one on isomorphism classes of objects. Actually, the same argument shows that  $Q_p(\rho)$  is full: for any objects  $A, B \in Q_p(G)$ , the induced map of  $Q_p$ -morphisms

$$\text{Mor}(A, B) \rightarrow \text{Mor}(\rho(A), \rho(B))$$

is surjective.

If  $P$  is any finite  $p$ -subgroup of  $H$ , we apply (B) with  $\pi = P$  to infer a commutative diagram

$$\begin{array}{ccc} & & G \\ & \nearrow f & \downarrow \rho \\ P & & H \\ & \searrow c^h & \end{array}$$

Thus  $P \in Q_p(H)$  is isomorphic, as object of  $Q_p(H)$ , to  $\rho(fP)$ , showing that  $Q_p(G) \rightarrow Q_p(H)$  is onto on isomorphism classes of objects.

It remains to check that  $Q_p(\rho)$  is faithful, i.e., that for any  $A, B \in Q_p(G)$

$$\text{Mor}(A, B) \rightarrow \text{Mor}(\rho A, \rho B)$$

is injective. But this is obvious because  $\text{Mor}(A, B) \subset \text{Hom}(A, B)$ ,  $\text{Mor}(\rho A, \rho B) \subset \text{Hom}(\rho A, \rho B)$  and  $\rho : B \rightarrow \rho(B)$  is a group isomorphism as observed earlier.

(C)  $\Rightarrow$  (A). Define a cofunctor  $F : Q_p(G) \rightarrow Ab$  by mapping  $P$  to  $H^*(BP; Z/p)$ . The natural map

$$\text{Res} : H^*(BG; Z/p) \rightarrow \lim_{\leftarrow} F$$

is then an isomorphism. In the case of a finite group  $G$  this follows from the classical result describing  $H^*(BG; Z/p)$  in terms of the stable elements in the cohomology of a  $p$ -Sylow subgroup of  $G$ ; the general case was dealt with in [4, Theorem 2.3]. The implication (C)  $\Rightarrow$  (A) is then plain.

The next result is an immediate consequence of the Theorem. It relates Weyl-groups of maps with group cohomology.

**COROLLARY 1.** *Let  $\rho : G \rightarrow H$  be a map of compact Lie groups inducing an isomorphism  $H^*(BH; \mathbb{Z}/p) \rightarrow H^*(BG, \mathbb{Z}/p)$ . Then for every homomorphism  $\varphi : \pi \rightarrow G$  with  $\pi$  a finite  $p$ -group, the induced map of Weyl-groups*

$$\rho_* : W(\varphi) \rightarrow W(\rho\varphi)$$

*is a group isomorphism.*

*Proof.* Note that  $W(\varphi)$  is the automorphism group of the object  $\varphi(\pi) \in Q_p(G)$ ; similarly for  $W(\rho\varphi)$ . Thus part (C) of the theorem shows that the natural map  $W(\varphi) \rightarrow W(\rho\varphi)$  is an isomorphism.

It seems surprising that  $\mathbb{Z}/p$ -cohomology information can contain such precise information on Weyl-groups, which are in general not  $p$ -groups. The following application shall illustrate this; as a variation of the theme we use rational cohomology information as input.

**COROLLARY 2.** *Let  $\rho : G \rightarrow H$  be a map of connected compact Lie groups inducing an isomorphism*

$$H^*(BH; \mathbb{Q}) \rightarrow H^*(BG; \mathbb{Q}).$$

*Then  $\rho$  induces an isomorphism of Weyl-groups  $W(G) \rightarrow W(H)$ .*

*Proof.* Choose a prime  $p$  large enough such that  $H^*B\rho : H^*(BH; \mathbb{Z}/p) \rightarrow H^*(BG; \mathbb{Z}/p)$  is an isomorphism (any prime which does not divide the order of the kernel and cokernel of the map  $H_*(G; \mathbb{Z}) \rightarrow H_*(H; \mathbb{Z})$  will do). Clearly,  $G$  and  $H$  have the same rank and, because in addition there is no element of order  $p$  in the kernel of  $\rho$ ,  $\rho$  maps a maximal torus  $T(G) \subset G$  onto a maximal torus  $\rho T(G) = T(H) \subset H$ . The union of the finite  $p$ -subgroups is dense in  $T(G)$  and  $T(H)$ . As a result, we can find a finite  $p$ -subgroup  $\pi \subset T(G)$  with centralizer  $C(\pi) = C(T(G)) = T(G)$ , and  $C(\rho\pi) = C(T(H)) = T(H)$ ; here we used the fact that in a compact Lie group closed subgroups satisfy the descending chain condition and that in a connected compact Lie group, a maximal torus is its own centralizer. Similarly, we may assume that the normalizer of  $\pi$  satisfies  $N(\pi) = N(T(G))$ , and  $N(\rho\pi) = N(T(H))$ . Then it follows that the induced map of Weyl-groups  $W(G) \rightarrow W(H)$  is an isomorphism as one sees by applying the previous Corollary to the given map  $\rho : G \rightarrow H$  and the inclusion map  $\varphi : \pi \rightarrow G$ .

Of course, this corollary could also be proved in a more conventional way by observing that the hypothesis implies that  $\rho : G \rightarrow H$  induces an isomorphism of associated Lie algebras.



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