Smooth algebras and vanishing of Hochschild homology.

Comment. Math. Helvetici 65 (1990) 474–477 0010-2571/90/030474-04$1.50 + 0.20/0

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For a given homomorphism of commutative noetherian rings $A \to B$, we consider the $B \otimes_A B$-module structure in B induced by the canonical surjective homomorphism $B \otimes_A B \to B$, $b \otimes b' \mapsto bb'$. We denote by $fd_{B \otimes_A B}(B)$ the flat dimension of this module.

Let K be a field; then a noetherian K-algebra C is called geometrically regular if for any finite field extension $L \mid K$, the ring $C \otimes_K L$ is regular. A homomorphism $A \to B$ of noetherian rings is called regular if it is flat and its fibers are geometrically regular.

In this paper we prove the following result

THEOREM 1. *Let A be a noetherian ring and let B be a flat and noetherian A-algebra. If $fd_{B \otimes_A B}(B) < \infty$, then the homomorphism $A \to B$ is regular.*

This result has been obtained in [8] with two additional hypotheses: B has characteristic zero and $B \otimes_A B$ is a noetherian ring. Moreover, in [8] has been proved that if $A \to B$ is regular and $B \otimes_A B$ noetherian, then $fd_{B \otimes_A B}(B) < \infty$. Nevertheless, it is not true for an arbitrary regular homomorphism. Indeed, for a regular homomorphism $A \to B$, we have isomorphisms [2, Proposition C]

$$\bigwedge^n \Omega_{B \mid A} \xrightarrow{\sim} \operatorname{Tor}_n^{B \otimes_A B}(B, B),$$

where $\Omega_{B \mid A}$ is the Kahler differentials module of B over A.

The proof of the version of Theorem 1 given in [8] uses the Hodge decomposition of Hochschild cohomology in characteristic zero, as well as a strong result of L. Avramov [3], which characterizes the local complete intersection rings by the finiteness of flat dimension of its cotangent complex. Here we give a more elementary proof based on the properties of Hochschild homology exposed in [5]. We also use a result of J.-L. Brylinski [4] concerning to the localization of this homology.

Observe that the condition $fd_{B \otimes_A B}(B) < \infty$ is equivalent to the existence of an integer n such that the Hochschild homology

$$H_n(B, M) = \mathrm{Tor}_n^{B \otimes_A B}(B, M) = 0$$

for all $B \otimes_A B$-modules M.

The results we will be using are the following

R.1. Let K be a commutative ring and let Λ, L be commutative K-algebras such that Λ is K-flat. Then for any $(L \otimes_K \Lambda) \otimes_L (L \otimes_K \Lambda)$-module M there are isomorphisms $H_n(\Lambda, M) \simeq H_n(L \otimes_K \Lambda, M)$ (on the left hand side we have homology of Λ over K, and on the right one, homology of $L \otimes_K \Lambda$ over L).

R.2. Let K be a field and let Λ be a commutative K-algebra. Then for any Λ-modules M and N we have isomorphisms $H_n(\Lambda, M \otimes_K N) \simeq \mathrm{Tor}_n^{\Lambda}(M, N)$.

R.3. Let $A \to B$ be a flat homomorphism of commutative rings. Let q be a prime ideal of B and let p be its contraction in A. Then for any $B \otimes_A B$-module M we have isomorphisms

$$\mathrm{Tor}_n^{B \otimes_A B}(B, M)_q \simeq \mathrm{Tor}_n^{B_q \otimes_{A_p} B_q}(B_q, M_q).$$

R.1 and R.2 are in [5]. R.3 is in [4].

Proof of Theorem 1. Using R.3 it is sufficient to show that if $A \to B$ is a flat homomorphism of noetherian local rings, K the residue field of A, and $fd_{B \otimes_A B}(B) < \infty$, then $B \otimes_A K$ is a geometrically regular K-algebra. Let $C = B \otimes_A K$. It follows from R.1 that $fd_{C \otimes_K C}(C) < \infty$. Since C is noetherian, R.2 implies that the global dimension of C is finite. Then C is a regular local ring (by the classical result of J. P. Serre [10]). Moreover, a new application of R.1 and R.2, shows that C remains regular after a finite extension of K. Therefore C is geometrically regular over K, and Theorem 1 is proved.

As a particular case of Theorem 1 we obtain that a field extension $L \mid K$ is separable if $fd_{L \otimes_K L}(L) < \infty$. It is a generalization of Theorem 9 in [8].

COROLLARY 2. *Let A be a noetherian ring and let B be an A-algebra of finite type. The following conditions are equivalent*

(i) *B is a smooth A-algebra*

(ii) *B is a flat A-module and* $fd_{B \otimes_A B}(B) < \infty$.

Proof. Let us recall the characterizations of regularity and smoothness using the André-Quillen 1-dimensional (co) homology functors $H(A, B, -)$ (see [1]). This functors can be described as follows. Let $B \simeq R/J$ be a presentation of B where R is a polynomial A-algebra, and consider the canonical homomorphism $J/J^2 \to \Omega_{R|A} \otimes_R B$. Then, for a B-module M we have

$$H_1(A, B, M) \simeq \operatorname{Ker}(J/J^2 \otimes_B M \to \Omega_{R|A} \otimes_R M)$$

$$H^1(A, B, M) \simeq \operatorname{Coker}(\operatorname{Hom}_B(\Omega_{R|A} \otimes_R B, M) \to \operatorname{Hom}_B(J/J^2, M)).$$

The above mentioned characterizations are the following:

(1) $A \to B$ is regular if and only if $H_1(A, B, -) = 0$, i.e., $H_1(A, B, B) = 0$ and $\Omega_{B|A}$ is a flat B-module;

(2) B is a smooth A-algebra if and only if $H^1(A, B, -) = 0$, i.e., $H_1(A, B, B) = 0$ and $\Omega_{B|A}$ is a projective B-module. Since B is noetherian and $\Omega_{B|A}$ is a finitely generated B-module, these two conditions on $\Omega_{B|A}$ are equivalent. Then the result is a consequence of [8] and Theorem 1.

REMARK 3. The proof of $H_1(A, B, -) = 0$ for a regular homomorphism is easy if B is an A-algebra of finite type. In the general case, the proof is difficult since it involves the homology functors of higher dimensions (see [1], pp. 330–331).

The homomorphism of rings $B \otimes_A B \to B$ enable us to consider a B-module M as a $B \otimes_A B$-module. If $B \otimes_A B$ is a noetherian ring, then the vanishing of $H_n(B, M)$ for an integer n and all B-modules M, is sufficient for $fd_{B \otimes_A B}(B) < \infty$. It is an easy consequence of the following well known result: if R is a local noetherian ring with residue field K, and I is an ideal of R, then $fd_R(R/I) < \infty$ if and only if there exists an integer n such that $\operatorname{Tor}_n^R(R/I, K) = 0$. On the other hand, the calculation of Hochschild homology of complete intersections in [11] shows that there exist locally complete intersection algebras A, of finite type over a field of characteristic zero such that $H_n(A, A) \neq 0$ for infinitely many n. We consider this as an evidence for the following conjecture:

CONJECTURE. *Let K be a field of characteristic zero and let A be a K-algebra of finite type. If $H_n(A, A) = 0$ for n sufficiently large, then A is a smooth K-algebra.*

A conjecture of J. Herzog implies that such an algebra must be at least a locally complete intersection. In fact, since K has characteristic zero, $H_n(K, A, A)$ is a summand of $H_{n+1}(A, A)$ (see [6], where $H_n(K, A, A)$ is called $H_{1,n}(A, A)$). Then $H_n(K, A, A) = 0$ for n sufficiently large. Let R be a polynomial K-algebra of finite type such that there exists a surjective K-homomorphism $R \to A$. Since the André–Quillen homology of R vanishes in dimensions >0, the Jacobi–Zriski sequence associated to $K \to R \to A$ yields $H_n(R, A, A) = 0$ for n sufficiently large. In this situation, Herzog conjectures [7] (p. 62) that A must be a locally complete intersection.

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Received October 30, 1989