Fixed point free low dimensional real algebraic actions of A5 on contractible varieties.

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Fixed point free low dimensional real algebraic actions of A_5 on contractible varieties

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Dedicated to Professor Shôrô Araki on the occasion of his 60th birthday.

1. Introduction

Let G be a compact Lie group, and let Ω be an orthogonal representation of G. A *real algebraic G variety* is a G invariant set

$$V = \{x \in \Omega \mid p_1(x) = \cdots = p_m(x) = 0\}$$

for polynomials $p_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, m$. We also say that G acts real algebraically on the variety V. All varieties considered in this paper are non-singular. The purpose of this note is to show

THEOREM A. *The alternating group A_5 acts real algebraically with exactly one fixed point on a variety which is diffeomorphic to S^6. Furthermore, A_5 acts real algebraically without a fixed point on a variety which is diffeomorphic to $\mathbb{R}^n$* for any $n \geq 6$.

The principal motivation for this theorem is the

FIXED POINT CONJECTURE. *A compact Lie group G acts real algebraically without a fixed point on a variety diffeomorphic to $\mathbb{R}^n$ if and only if G acts smoothly without a fixed point on the disk $D^m = \{x \in \mathbb{R}^m \mid \|x\| \leq 1\}$.*

Petrie and Rendall [PR] showed the necessity part ($\Rightarrow$) of the conjecture, and here one may set $m = n$. Previous partial results in this direction are also discussed in their paper. The conjecture demonstrates a clear difference between smooth and the real algebraic actions. Conner and Floyd constructed (smooth) cyclic group actions on $\mathbb{R}^n$ without fixed point [CF]. By the Lefschetz Fixed Point Theorem, every smooth cyclic action on a disk has a fixed point. Combined with the result of Petrie and Randall this implies that there are no cyclic, fixed point free, real algebraic actions on any variety diffeomorphic to $\mathbb{R}^n$.

Our theorem deals with the sufficiency part ($\Leftarrow$) of the conjecture. Observe that A_5 acts without a fixed point on a disk [Br, page 55], and Theorem A implies

COROLLARY. *The Fixed Point Conjecture holds for* A_5.

The corollary was obtained previously by Dovermann, Masuda, and Petrie [DMP]. It was derived from a theorem similar to Theorem A, but there $n \geq 24$. Furthermore, Dovermann, Knop, and Suh showed the sufficiency part of the Fixed Point Conjecture for odd order abelian groups [DKS].

A fixed point free complex algebraic action of a reductive group (such as a finite group or $\mathbb{C}^*$) on $\mathbb{C}^n$ would be a striking counter example to the Linearity Conjecture by Kambayashi [Ka]: "*Any reductive complex algebraic action on* $\mathbb{C}^n$ *is conjugate to a linear action.*" For some results supporting this conjecture see [BH] and [Kr]. Recently G. Schwarz [Sc] has shown that Kambayashi's conjecture is false. Many groups, such as $O(n, \mathbb{C}) \times \mathbb{C}^*$, have algebraic actions on $\mathbb{C}^n$ which are not conjugate to linear actions. But, these actions have fixed points. Recently Masuda and Petrie extended the results of Schwarz [MP]. The actions in Theorem A are not conjugate to linear actions because they have no fixed points. We only know that the underlying variety of this action is diffeomorphic to $\mathbb{R}^n$, and algebraically it may not be $\mathbb{R}^n$. Generally there are infinitely many real algebraic varieties diffeomorphic to one smooth manifold (e.g., see [BK]).

The first part of Theorem A addresses a problem raised by Montgomery and Samelson [MS]. *Which groups can act on a homotopy sphere with exactly one fixed point?* Stein [St] showed that the binary icosahedral group has this property, and Petrie showed the same for several classes of groups [P2], [P3]. One such class are odd order abelian groups with at least three non-cyclic Sylow subgroups. Theorem A provides an answer to the question of Montgomery and Samelson in the real algebraic category.

The question of low-dimensional smooth one fixed point actions on spheres was raised by Morimoto [Mo1]. He constructed *smooth* actions as in Theorem A (see [Mo3] and [Mo4]). In dimensions ≤ 5 one fixed point actions on spheres do not exist. See [Mo2] and [F] if the dimension is ≤ 4 and [BKS] if the dimension is 5. We give a short proof of Morimoto's theorem which does not only provide smooth A_5 actions on S^6, but which also provides such actions in the more rigid real algebraic category.

The first part of the proof of Theorem A will follow from the next three results. Let $\mathscr{P} = SO(3)/A_5 = \{gA_5 \mid g \in SO(3)\}$ denote the Poincaré homology sphere. Here we identify A_5 with the icosahedral group $I \subset SO(3)$. The alternating group A_5 acts on $\mathscr{P}$ by left multiplication. With this action $\mathscr{P}$ is a closed smooth A_5 manifold. The action has exactly one fixed point because the normalizer of I in $SO(3)$ is I itself.

PROPOSITION B. ([DMP, Section 2]) *The Poincaré homology sphere $\mathscr{P}$ is equivariantly diffeomorphic to a real algebraic A_5 variety.*

Let G be a compact Lie group, and let X and X' be closed (i.e., compact and without boundary) smooth G manifolds. They are called *equivariantly cobordant* if there exists a smooth G manifold W such that its boundary is the disjoint union of X and X'.

Let $\mathscr{P}_2$ be $\mathscr{P} \times \mathscr{P}$ with a diagonal action of A_5. In particular, $\mathscr{P}_2$ is equivariantly diffeomorphic to a real algebraic A_5 variety, and the action has exactly one fixed point.

THEOREM C. *The manifold $\mathscr{P}_2$ is A_5 equivariantly cobordant to S^6 with an action of A_5 which has exactly one fixed point.*

THEOREM D. ([DMP, Theorem 1.3]) *Suppose G is a compact Lie group and M is a closed smooth G manifold. Suppose M is G cobordant to a real algebraic G variety. Then M is G diffeomorphic to a real algebraic G variety.*

Theorem D is a partial generalization of Tognoli's Theorem [T] (the proof of the Nash Conjecture), and it is this result which links smooth and real algebraic transformation groups. Tognoli's Theorem is without group action. With the help of Theorem D the proof of Theorem A has been reduced to a smooth problem, and this problem is solved in Theorem C.

The proof of Theorem C is given in Section 3. There are two propositions which prepare it. In Proposition 2.1 we construct a cobordism between $\mathscr{P}_2$ and a manifold X, and the non-free orbits in X have the properties implied by an A_5 action on a six-dimensional homotopy sphere. The method of proof is an explicit low dimensional construction. In Proposition 3.1 we show that a manifold as in the conclusion of Proposition 2.1 is equivariantly cobordant to a sphere. In its proof more abstract equivariant surgery techniques are used to provide cobordism in the theorem.

The second part of Theorem A follows easily from the first part with the help of our next

LEMMA E. (See [M] and [DMP]) *Let G be a compact Lie group. Let V be a real algebraic G variety and W a G invariant subvariety. Then $V\backslash W$ is equivariantly diffeomorphic to a real algebraic G variety.*

Proof. Let $(p_1, \ldots, p_k)$ be the ideal which defines the variety V. Set $p = \Sigma\, p_i^2$. Then $p^{-1}(0) = V$. Let Ξ denote the representation of G in which V is the zero set. Find a G invariant polynomial q such that $q^{-1}(0) = W$. Here one may start with a non-equivariant polynomial with this property and average its square over G. Then

$$V \backslash W = \{x \in V \mid p(x) = 0 \quad \text{and} \quad q(x) \neq 0\}.$$

The assignment which maps x to $(x, 1/q(x))$ defines an equivariant diffeomorphism

$$V \backslash W \to A = \{(x, y) \in \Xi \oplus \mathbb{R} \mid p(x) = 0 \text{ and } yq(x) - 1 = 0\}.$$

The action of G on A is real algebraic. □

Proof of Theorem A. Proposition B implies that $\mathscr{P}_2$ is equivariantly diffeomorphic to a real algebraic A_5 variety. Theorem C implies that $\mathscr{P}_2$ is equivariantly cobordant to S^6, and the action on S^6 has exactly one fixed point. Theorem D implies that S^6 with this action of A_5 is equivariantly diffeormorphic to a real algebraic A_5 variety. This shows the first part of Theorem A.

Furthermore, Lemma E implies that $S^6 \backslash (S^6) A^5$ is equivariantly diffeomorphic to a real algebraic A_5 variety, and this variety is diffeomorphic to $\mathbb{R}^6$. This implies the second claim in Theorem A for $n = 6$. For $n > 6$ the result follows then trivially. □

2. Low dimensional surgeries

The topic of this section is the construction of the manifold X described in Proposition 2.1. The manifold will be used later. We denote the tangent bundle of a smooth manifold X by TX. Let Ω be a representation of a group G and B a G space. The product bundle $p_1 : B \times \Omega \to B$, whose projection map is projection on this first factor, is denoted by $\underline{\Omega}$. The base space will be understood from context. We say that a smooth G manifold X has an equivariant stable framing if there are representations $\Omega_\pm$ and a G vector bundle isomorphism $TX \oplus \underline{\Omega}_- \to \underline{\Omega}_+$.

PROPOSITION 2.1. *There is an A_5 equivariant cobordism between $\mathscr{P}_2$ and an A_5 manifold X such that*

(1) *X^{A_5} consists of exactly one point.*
(2) *$X^H = S^0$ if H is a dihedral or tetrahedral subgroup of A_5.*
(3) *$X^C = S^2$ for any non-trivial cyclic subgroup C of A_5,*
(4) *X has an equivariant stable framing.*
(5) *X is orientable, and the A_5 action on X preserves the orientation.*

We prepare the proof of Proposition 2.1. As before we identify A_5 with the icosahedral group $I \subset SO(3)$. Let $\mathbb{H}$ denote the quaternions and $\pi : S(\mathbb{H}) \to SO(3)$ the double cover. The binary icosahedral group is $\tilde{I} = \pi^{-1}(I)$. We will reserve the letter Ξ for the representation of I whose underlying space is $\mathbb{H}$. The action $I \times \Xi \to \Xi$ is given by $(\pi(\gamma), v) \mapsto \gamma v \gamma^{-1}$.

We describe $T\mathscr{P} \oplus \underline{\mathbb{R}}$. Set $S(\mathbb{H}) \times \mathbb{H} \doteq T\mathbb{H}_{|S(\mathbb{H})} = TS(\mathbb{H}) \oplus \underline{\mathbb{R}}$. The binary icosahedral group acts by left and right multiplication on $S(\mathbb{H}) \times \mathbb{H}$. Taking quotients with respect to the right multiplication we get $T\mathscr{P} \oplus \underline{\mathbb{R}}$.

Consider $\eta : S(\mathbb{H}) \times \mathbb{H} \to \mathscr{P} \times \Xi$ defined by $\eta(g, v) = ([g], vg^{-1})$. This map factors through $T\mathscr{P} \oplus \mathbb{R}$ and identifies $T\mathscr{P} \oplus \underline{\mathbb{R}}$ with $\mathscr{P} \times \Xi$. The action of $\tilde{I}$ on $\mathscr{P}$ is not effective and induces an action of I. The map η is equivariant with respect to the left $\tilde{I}$ actions on $S(\mathbb{H}) \times \mathbb{H}$ and the I action on $\mathscr{P} \times \Xi$.

Let $x \in \mathscr{P}$ be the fixed point. Then $T_x\mathscr{P} \oplus \mathbb{R} = \Xi$. The Thom–Pontrjagin map $c : \mathscr{P} \to S(\Xi)$ collapses the complement of a small invariant disk around x to the point at infinity in the one point compactification $T_x\mathscr{P}^+$ of $T_x\mathscr{P}$. It is an equivariant map. We identify $T_x\mathscr{P}^+$ with $S(\Xi)$ such that $0 \in T_x\mathscr{P}$ maps to $1 \in \Xi$, and such that $-1 \in \Xi$ corresponds to the compactification point. We define the bundle isomorphism

$$B : T\mathscr{P} \oplus \underline{\mathbb{R}} = \mathscr{P} \times \Xi \to \mathscr{P} \times \Xi \quad \text{with} \quad B([g], v) = ([g], vc([g])).$$

Let H be a non-trivial proper subgroup of I. Then $\mathscr{P}^H$ is diffeomorphic ($\cong$) to the boundary of a disk D and the normal bundle $\nu(\mathscr{P}^H, \mathscr{P})$ is trivial. Let ν be its fibre, $\mathbb{D} := D \times D(\nu)$, and $\mathbb{D}_0 := \partial D \times D(\nu)$. Then $\mathbb{D}_0 \subset \partial \mathbb{D}$. Let $B_H : \nu(\mathscr{P}^H, \mathscr{P}) \to \partial D \times \nu$ be the trivialization induced by B, and let $t : D(\nu(\mathscr{P}^H, \mathscr{P})) \to \mathscr{P}$ be the map which identifies the normal disk bundle with a tubular neighbourhood of $\mathscr{P}^H$ in $\mathscr{P}$. Define $\varphi : \mathbb{D}_0 \to \mathscr{P}$ as the composition of $(B_H)^{-1}$ (restricted to the disk bundle) and t. Set $W_H = \mathscr{P} \times [0, 1] \cup_\varphi \mathbb{D}$ where $\mathbb{D}$ is attached along $\mathscr{P} \times \{1\}$.

PROPOSITION 2.2. *The I equivariant stable framing $B : T\mathscr{P} \oplus \underline{\mathbb{R}} \to \mathscr{P} \times \Xi$ extends to an H equivariant stable framing $\beta_H : TW_H \oplus \underline{\mathbb{R}} \to W_H \times (\Xi \oplus \mathbb{R})$.*

Proof. Let $W_0 = \mathscr{P} \times [0, 1]$. We use B to define $\beta_0 : TW_0 \oplus \underline{\mathbb{R}} \to W_0 \times (\Xi \oplus \mathbb{R})$. In the direction of $\mathscr{P}$ we use B, and the summand $\mathbb{R}$ which we added to Ξ is used to frame the direction $[0, 1]$ in W_0. We show that β_0 extends over W_H. Set $W' = W_0 \cup (D \times \{0\})$. Because W_H contracts to W' it suffices to extend β_0 to $\gamma : (TW_H)_{|W'} \oplus \underline{\mathbb{R}} \to W' \times [\Xi \oplus \mathbb{R})$.

Restricted over W_0^H, β_0 decomposes as a direct sum $(\beta_0)^H \oplus (\beta_0)_H$, where $(\beta_0)^H$ is the isomorphism on the H fixed point set, and $(\beta_0)_H$ is the isomorphism on its orthogonal complement. Because we used φ to identify $\mathbb{D}_0$ with a tubular neighbourhood of $\mathscr{P}^H$ in $\mathscr{P}$, it follows that $(\beta_0)_H$ extends over D and W'. (The same argument was used in [DP, Lemma 4.28].)

We extend β_0^H. Consider the case where H is dihedral or tetrahedral. Then $\mathscr{P}^H \cong S^0$. Denote the points in $\mathscr{P}^H$ by x and y, where $\{x\} = \mathscr{P}^I$. In particular, $c(x) = 1$, $c(y) = -1$ and

$$B(x, v) = (x, v) \quad \text{and} \quad B(y, v) = (y, -v) \tag{2.3}$$

Because of the specific construction of φ using the framing B and the fact that β_0 is defined by B, it follows that β_0^H extends over the disk $D^1 = [0, 1]$ bounded by $\mathscr{P}^H$.

Let H be a non-trivial cyclic subgroup of I. In this case $\mathscr{P}^H \cong S^1$. The stable framing of $T\mathscr{P}$, induced by η and restricted to $\mathscr{P}^H$, gives the stabilized Lie framing on $\mathscr{P}^H$. This framing is twisted, and c untwists it. For this reason B extends H equivariantly over the disk bounded by $\mathscr{P}^H$. This completes the proof of the proposition also in this case. □

Proof of Proposition 2.1. We construct an A_5 equivariant cobordism N between $\mathscr{P}_2 = \mathscr{P} \times \mathscr{P}$ (with diagonal action) and a manifold X, and an equivariant stable framing c of N. The manifold X will have all of the desired properties.

Let $N_0 := \mathscr{P}_2 \times [0, 1]$. A stable framing $c_0 : \underline{\mathbb{R}}^2 \oplus TN_0 \to \underline{\Xi} \oplus \underline{\Xi} \oplus \underline{\mathbb{R}}$ of N_0 is given as follows. Use the product

$$B \times B : \underline{\mathbb{R}}^2 \oplus T\mathscr{P}_2 \cong (\underline{\mathbb{R}} \oplus T\mathscr{P})^2 \cong (\mathscr{P} \times \Xi)^2 \to (\mathscr{P} \times \Xi)^2 \cong \mathscr{P}_2 \times (\Xi \oplus \Xi)$$

as stable framing of $\mathscr{P}_2$. The summand $\underline{\mathbb{R}}$ added to $\underline{\Xi} \oplus \underline{\Xi}$ accounts for the $[0, 1]$ direction in N_0.

We attach handles to N_0 along $\mathscr{P}_2 \times \{1\}$. Let H be a non-trivial proper subgroup of A_5 which is an isotropy group. In the set-up of Proposition 2.2 we assigned to H a handle $\mathbb{D}$ with boundary piece $\mathbb{D}_0$. Let Ξ_0 be the nontrivial summand of Ξ, so Ξ_0 is the tangent representation of $\mathscr{P}$ at the fixed point. Set

$$\mathbb{E}_0(H) = \begin{cases} A_5 \times (\mathbb{D}_0 \times D(\Xi_0)) & \text{if } \dim \mathscr{P}^H = 0 \\ A_5 \times (D(\Xi_0) \times \mathbb{D}_0) & \text{if } \dim \mathscr{P}^H = 1 \end{cases}.$$

This is an A_5 space with diagonal action. Let z be a point in $\mathscr{P}$ with isotropy group H, and let $\eta : D(\Xi_0) \to \mathscr{P}$ be an H equivariant embedding which identifies $D(\Xi_0)$ with a tubular neighbourhood of z. Let φ be as in the set-up of 2.2. We define an A_5 map

$$\Psi(H) : \mathbb{E}_0(H) \to \mathscr{P} \times \mathscr{P} = \mathscr{P}_2 \times \{1\}$$

by

$$\Psi(H)(g, u, v) = \begin{cases} (g\varphi(u), g\eta(v)) & \text{if } \dim \mathscr{P}^H = 0 \\ (g\eta(u), g\varphi(v)) & \text{if } \dim \mathscr{P}^H = 1 \end{cases}.$$

Let $\tilde{\mathbb{D}}_0(H)$ be the following quotient of $\mathbb{E}_0(H)$.

$$\tilde{\mathbb{D}}_0(H) = \begin{cases} A_5 \times_H (\mathbb{D}_0 \times D(\Xi_0)) & \text{if } \dim \mathscr{P}^H = 0 \\ A_5 \times_H (D(\Xi_0) \times \mathbb{D}_0) & \text{if } \dim P^H = 1 \end{cases}.$$

Then $\Psi(H)$ factors through $\tilde{\mathbb{D}}_0(H)$ and defines an A_5 equivariant embedding

$$\psi(H) : \tilde{\mathbb{D}}_0(H) \to \mathscr{P} \times \mathscr{P} = \mathscr{P}_2 \times \{1\}.$$

Finally, we define

$$\tilde{\mathbb{D}}(H) = \begin{cases} A_5 \times_H (\mathbb{D} \times D(\Xi_0)) & \text{if } \dim \mathscr{P}^H = 0 \\ A_5 \times_H (D(\Xi_0) \times \mathbb{D}) & \text{if } \dim \mathscr{P}^H = 1 \end{cases}.$$

Observe that $\tilde{\mathbb{D}}_0(H)$ is part of the boundary of $\tilde{\mathbb{D}}(H)$. Using $\psi(H)$ as attaching map we form $N_0 \cup_{\tilde{\mathbb{D}}_0(H)} \tilde{\mathbb{D}}(H)$.

We extend the stable framing c_0 of N_0 over the attached handles $\tilde{\mathbb{D}}(H)$. The stable framing c_0 restricts to an H equivariant stable framing of $\mathbb{D}_0$, which extends to an H equivariant stable framing of $\mathbb{D}$ (see Proposition 2.2). A stable framing of $D(\Xi_0)$ is obtained easily because this space is contractible. Their product provides a stable framing of $\mathbb{D} \times D(\Xi_0)$ and $D(\Xi_0) \times \mathbb{D}$. These stable framings extend A_5 equivariantly over $\tilde{\mathbb{D}}(H)$ because the handles are attached equivariantly. This provides an A_5 equivariant extension of c_0.

Consider the set of conjugacy classes of non-trivial proper subgroups of A_5 which are isotropy groups of the action on $\mathscr{P}$. In each of these classes choose one representative H, and attach the associated handles $\tilde{\mathbb{D}}(H)$ to N_0 in the way described above. These handles are disjoint from each other. The resulting manifold N is the cobordism which we set out to construct. Because we have extensions of the stable framing c_0 over each of the handles, we get a stable framing c of N. The manifold X is defined by $\partial N = \mathscr{P}_2 \cup X$ and a stable framing of X is defined as restriction of c over X.

We need to verify that X satisfies (1)–(5). Property (4) and (5) are obvious. By assumption, $\mathscr{P}_2^{A_5}$ consists of exactly one point, and this point stays untouched in the process of attaching handles. This point is also contained in X, and (1) is clear. We check (2). Let H be an isotropy group of the action on $\mathscr{P}$, such that $H \neq G$, and dim $\mathscr{P}^H = 0$. Then $\mathscr{P}_2^H \cong S^0 \times S^0$. We attached one handle cancelling two points in $\mathscr{P}_2^H$. Thus $X^H \cong S^0$.

We check (3). Let C be a non-trivial cyclic subgroup of A_5 with normalizer NC and $W(C) := NC/C$. Observe that $W(C)$ is of order 2, and $\mathscr{P}^C \cong S^1$. Identify $\mathscr{P}^C$ with the unit complex numbers such that $\{1\}$ corresponds to the A_5 fixed point in $\mathscr{P}$, and $\{-1\}$ corresponds to the other point left fixed by the action of NC. Then $\mathscr{P}_2^C \cong S^1 \times S^1$ and $(\mathscr{P}_2^C)^{W(C)} \cong S^0 \times S^0$. We describe the spheres in $\mathscr{P}_2^C$ on which we did surgery when we attached handles to N_0. We used $S^0 = \{\pm 1\} \times \{-1\}$ as the sphere on which to do surgery when we eliminated two NC fixed points. We also did surgery on the spheres $S^1 \times \{\pm i\}$ when we attached the handles associated with C. So, with above notation we choose $z = i \in \mathscr{P}^C \subset \mathscr{P}$. The reader is invited to draw a picture of the torus and carry out the surgeries which we just described. The result of these surgeries is S^2, as it was claimed for the C fixed point set in X. This completes the proof. □

3. Surgery on free orbits

The next step in the proof of Theorem C provides an equivariant cobordism between X as in the conclusion of Proposition 2.1 and S^6 with an A_5 action. Restricted to all non-free orbits the corbordism is a product. We discuss an obstruction for finding such a cobordism, and we show that the group in which it lives is trivial, hence the obstruction vanishes.

The obstruction lies in a Witt group $W_2(A_5, \Gamma)$. More generally, Bak introduced groups $W_*(G, \Gamma)$ where G is a group and Γ is a form parameter [B, Section 1]. Let $v = \Sigma\, a_g g \in \mathbb{Z}[G]$. Then $\bar{v}$ is defined as $\Sigma\, a_g g^{-1}$. In our situation $G = A_5$ and

$$\Gamma = \left\{ v + \bar{v} + \sum \alpha_g g \mid v \in \mathbb{Z}[A_5],\quad \alpha_g \in \mathbb{Z} \text{ and } 1 \neq g \in A_5 \text{ with } g^2 = 1 \right\}$$

In comparison to the elements in Wall's surgery obstruction group $L_2^h(\mathbb{Z}[A_5], 1)$ where the self intersection form takes values in $\mathbb{Z}[A_5]/\{v + \bar{v} \mid v \in \mathbb{Z}[A_5]\}$, the self intersection form now takes values in $\mathbb{Z}[A_5]/\Gamma$. Morimoto noticed the importance of these Witt groups for equivariant surgery.

DEFINITION. Let X be a closed smooth A_5 manifold. We call it adjusted if X^P is a homotopy sphere for all non-trivial subgroups P of A_5 of prime power order.

THEOREM 3.1. *Let X be an adjusted A_5 manifold of dimension $4k + 2$ such that*

(1) $\dim X^g \leq 2k$ *for* $1 \neq g \in A_5$ *and equality holds if* $g^2 = 1$.
(2) $X^{A_5} \neq \emptyset$.
(3) *X has a stable equivariant framing.*
(4) *X is orientable, and then A_5 action on X preserves the orientation.*

Then X is A_5 equivariantly cobordant to a homotopy sphere Σ such that Σ^{A_5} is diffeomorphic to X^{A_5}.

Proof of Theorem C. It is the conclusion of Proposition 2.1 that $\mathcal{P}_2 = \mathcal{P} \times \mathcal{P}$ is A_5 equivariantly cobordant to a manifold X which satisfies the assumptions of Theorem 3.1 and which has exactly one fixed point. It is the conclusion of Theorem 3.1 that X is equivariantly cobordant to a homotopy sphere Σ such that Σ^{A_5} consists of exactly one point. Then $\mathcal{P}_2$ and Σ are equivariantly cobordant. In particular, Σ is diffeomorphic to S^6, and Theorem C is proved. □

We need two algebraic computations in the proof of Theorem 3.1.

THEOREM 3.2. ([RU]) *Every finitely generated projective module over $\mathbb{Z}[A_5]$ is $\mathbb{Z}[A_5]$ stably free.*

THEOREM 3.3. *For Γ as above, $W_2(A_5, \Gamma) = 0$.*

The proof of Theorem 3.3 was provided to us by A. Bak, and we like to thank him. It simplifies a proof given by Morimoto in [Mo4].

Proof: Since $\tilde{K}_0(\mathbb{Z}[A_5]) = 0$ (see 3.2), it follows from [B, 8.17] that $W_2(A_5, \Gamma) = W_2^P(A_5, \Gamma)$ where P signifies that the underlying modules of our non-singular forms are allowed to be finitely generated and projective over $\mathbb{Z}[A_5]$. The maximal 2-hyperelementary subgroups of A_5 are the dihedral groups D_4, D_6, and D_{10}. Thus, by induction [B, Section 12], it suffices to show that $W_2(H, \Gamma_H) = 0$ for any of these dihedral groups, and

$$\Gamma_H = \left\{ v + \bar{v} + \sum \alpha_g g \mid v \in \mathbb{Z}[H], \quad \alpha_g \in \mathbb{Z} \text{ and } 1 \neq g \in H \text{ with } g^2 = 1 \right\}.$$

Consider the maximal and minimal form parameters

$$\Gamma_{\max} = \{a \in \mathbb{Z}[H] \mid \bar{a} = a\} \quad \text{and} \quad \Gamma_{\min} = \{v + \bar{v} \mid v \in \mathbb{Z}[H]\}.$$

We first show that $W_2(H, \Gamma_{\max}) = 0$. Consider the split exact sequence [B, 11.4]

$$0 \to S(\Gamma_{\max}/\Gamma_{\min}) \to W_2^P(H, \Gamma_{\min}) \to W_2^P(H, \Gamma_{\max}) \to 0.$$

The splitting of the first map is given by the Arf invariant. By [K, §6 (2)], $W_2^p(H, \Gamma_{\min})$ is accounted for solely by the Arf invariant. Thus $W_2^p(H, \Gamma_{\max}) = 0$.

Consider once more Bak's exact sequence

$$0 \to S(\Gamma_{\max}/\Gamma_H) \to W_2^p(H, \Gamma_H) \to W_2^p(H, \Gamma_{\max}) \to 0.$$

We compute $S(\Gamma_{\max}/\Gamma_H)$. It is easy to see that $\Gamma_{\max}/\Gamma_H$ is isomorphic to $\mathbb{Z}_2$ and a generator is given by e, the trivial element in H. Consider $\Gamma_{\max}$ and Γ_H as $\mathbb{Z}[H]$-modules, where the action (right and left) is given by conjugation. By definition, S denotes the symmetric tensor product

$$S(\Gamma_{\max}/\Gamma_H) = ((\Gamma_{\max}/\Gamma_H) \otimes_{\mathbb{Z}[H]} (\Gamma_{\max}/\Gamma_H))/\{a \otimes b - b \otimes a, a \otimes b - a \otimes ba\bar{b}\}$$

The tensor product is generated by $e \otimes e$. Let t denote an element of order 2 in H, $t \neq 1$. Then $t \in \Gamma_H$, $2e + 2t \in \Gamma_H$, and in $S(\Gamma_{\max}/\Gamma_H)$

$$e \otimes e = e \otimes (e + t) = e \otimes (e + t)e\overline{(e + t)} = e \otimes (2e + 2t) = 0.$$

Thus $S(\Gamma_{\max}/\Gamma_H) = 0$ and $W_2^p(H, \Gamma_H) = W_2^p(H, \Gamma_{\max}) = 0$ as claimed. □

We prepare the proof of Theorem 3.1. Let G be a finite group. We give the definition of an adjusted G normal map as it is appropriate in our context.

DEFINITION 3.4. A G normal map consists of two closed smooth oriented G manifolds X and Y, an equivariant map $f: X \to Y$ of degree 1, and an equivariant stable framing $b : TX \oplus \underline{\Delta} \to \underline{\Omega}$ of X. Here Δ and Ω are appropriate representations of G, Y is assumed to be simply connected, and it is assumed that the actions of G on X and Y preserve orientations. The data of a normal map will be abbreviated as (X, f, b). *A* G normal map is called adjusted if $f^P : X^P \to Y^P$ is a homotopy equivalence whenever P is a non-trivial subgroup of G of prime power order.

For G normal maps we have a concept of G normal cobordism. Let $\mathscr{W} = (X, f, b)$ and $\mathscr{W}' = (X', f', b')$ be G normal maps, with the same target space Y. Let W be a G cobordism between X and X'. Let $F : W \to Y \times I$ be an equivariant map such that F restricts to f and f', so $f = F_{|X} : X \to Y \times 0 = Y$ and $f' = F_{|X'} : X' \to Y \times 1 = Y$. Let $B : TW \oplus \underline{\Delta} \to \underline{\Omega}$ be a stable G vector bundle isomorphism which restricts to b over X and to b' and X'. Then (W, F, B) is called a G normal cobordism between $\mathscr{W}$ and $\mathscr{W}'$. The cobordism is relative to the L fixed point set, $L \subset G$, if all data restricts over the L fixed point set to a product with the unit interval. We may also add the same representation to both Δ and Ω in above definition and stabilze b with the identity map. This generates an equivalence relation for stable framings and for normal maps. We abuse language slightly and call the equivalence relation generated by both of the equivalence relations again equivariant normal cobordism.

To an orientable G manifold X of dimension $4k+2$ with orientation preserving action we assign a form parameter $\Gamma(X)$. Set

$$\gamma(X)=\{g\in G\mid 1\neq g,\, g^2=1,\, X^g \text{ has a component of dimension } 2k\}$$

$$\Gamma(X)=\left\{v+\bar{v}+\sum\alpha_g g\mid v\in\mathbb{Z}[G],\ \alpha_g\in\mathbb{Z} \text{ and } g\in\gamma(X)\right\}.$$

We say that a G manifold X satisfies the dimension assumptions for surgery, if $\dim X\geq 5$ and whenever $1\neq g\in G$, then we have for each component F of X^g that $2\dim F<\dim X$.

THEOREM 3.5. *Let (X,b,f) be an adjusted G normal map. Suppose that X is of dimension $4k+2$, X satisfies the dimension assumptions for surgery, and f is $(2k+1)$-connected (i.e., for the mapping cylinder M_f of f, $\pi_j(M_f,X)=0$ for $j\leq 2k+1$). Then*

(1) *$K(X):=\ker(H_{2k+1}(X)\to H_{2k+1}(Y))$ is a projective $\mathbb{Z}[G]$ module.*
(2) *Suppose $K(X)$ is $\mathbb{Z}[G]$ stably free. There exists an element $\sigma(f,b)\in W_2(G,\Gamma(X))$ such that the vanishing of $\sigma(f,b)$ implies that (X,f,b) is G normally cobordant to a G normal map (X',f',b') for which f' is a homotopy equivalence.*
(3) *The normal cobordism in (2) may be chosen relative to the L fixed point sets for all non-trivial subgroups L of G.*

The first part of this theorem was proved by Petrie [P]. The other two parts were shown by Morimoto [Mo5]. Compare [D, Theorem 3.11] for a different point of view.

Proof of Theorem 3.1. Let $x\in X$ be a fixed point. Let $\Upsilon:=T_xX$ be the tangent representation at x. Let $Y:=S(\Upsilon\oplus\mathbb{R})$ and $f:X\to Y$ the map obtained by collapsing the complement of an A_5 invariant disk around x to one point, the compactification point in $\Upsilon^+=S(\Upsilon\oplus\mathbb{R})$. With appropriately chosen orientations f is of degree 1. Let b be the stable framing of X. Then (X,f,b) is an adjusted normal map.

All cobordisms in this proof are relative to the L fixed point sets for all nontrivial subgroups L of A_5. Using standard equivariant surgery techniques [DP, Sec. 4] we make (X, f, b) connected up to the middle dimension and find a $(2k+1)$-connected A_5 normal map (X_0, f_0, b_0) which is A_5 normally cobordant to (X, f, b).

The normal map (X_0, f_0, b_0) satisfies the assumptions in Theorem 3.5. In particular, $K(X_0)$ is $\mathbb{Z}[A_5]$ stably free because of Theorem 3.2 and the obstruction $\sigma(f_0, b_0)$ in 3.5 (2) vanishes due to Theorem 3.3. It is the conclusion of Theorem 3.5 that (X_0, f_0, b_0) (and hence also (X, f, b)) is A_5 normally cobordant to an A_5 normal map (X_1, f_1, b_1) such that f_1 is a homotopy equivalence. So, X_1 is a homotopy sphere because Y is a sphere, and $\Sigma := X_1$ is equivariantly cobordant to X with $\Sigma^{A_5} = X^{A_5}$. This completes the proof. □

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