Baumslag-Solitar groups and some other groups of cohomological dimension two.

Comment. Math. Helvetici 65 (1990) 547–558 0010-2571/90/040547-12$1.50 + 0.20/0

Baumslag–Solitar groups and some other groups of cohomological dimension two

P. H. KROPHOLLER

The purpose of this paper is to establish a common generalisation of the following two results on finitely generated groups of cohomological dimension two. Let G be such a group.

THEOREM A (Bieri [2], Corollaries 8.7 and 8.9). *If G is non-abelian and has non-trivial centre then its centre is infinite cyclic and its central factor group is free-by-finite.*

THEOREM B (Gildenhuys [6]). *If G is soluble then it is isomorphic to*

$$\langle x, y; x^y = x^n \rangle$$

for some non-zero integer n.

Already common characteristics are apparent. In both cases G has an infinite cyclic subgroup H which meets all of its conjugates non-trivially. In Theorem A one takes H to be the centre, and in Theorem B, the group generated by x in the above presentation. Quite generally, if H is infinite cyclic, the assertion that it meets all of its conjugates can be expressed as

$$\mathrm{Comm}_G(H) = G, \qquad (*)$$

in the notation of [10] or [11], (cf. especially the proof of Proposition 1 of [11]). Perhaps the most famous groups satisfying this condition are the one-relator groups

$$\langle x, y; (x^m)^y = x^n \rangle;$$

the so-called Baumslag–Solitar groups.

Let $\mathfrak{X}$ denote the class of all finitely generated groups of cohomological dimension two which have an infinite cyclic subgroup H such that (*) holds. We prove here that

THEOREM C. *A non-cyclic group belongs to $\mathfrak{X}$ if and only if it is the fundamental group of a finite graph of infinite cyclic groups.*

One implication here is clear, for if G is the fundamental group of a finite graph of infinite cyclic groups then it is certainly finitely generated of cohomological dimension ≤ 2, and moreover any choice H of vertex or edge group satisfies (*). In addition to Baumslag–Solitar groups, and the groups described in Theorems A and B, some quite complicated looking presentations arise in this way. For example let G be

$$\langle u, v, w, x, y, z;\, w^2 = x^3,\, x^4 = z^7,\, u^{-1}z^2u = w^3,\, x^3 = y^3,\, v^{-1}y^6v = x^2\rangle$$

and let H be the subgroup generated by x. Plainly any such group has a presentation with one fewer relator than generator. In particular it follows from Theorem C that every $\mathfrak{X}$-group is finitely presented.

Theorem C can be rephrased. It shows that every $\mathfrak{X}$-group admits an action on a tree so that the vertex and edge stabilisers are infinite cyclic. From this it follows that every finitely generated subgroup of an $\mathfrak{X}$-group is a free product of an $\mathfrak{X}$-group and a free group. Thus we have

COROLLARY 1. *Every $\mathfrak{X}$-group is coherent, meaning that its finitely generated subgroups are finitely presented.*

Moreover, in the same spirit as Corollary 8.9 of [2] we can deduce

COROLLARY 2. *If G is an $\mathfrak{X}$-group then its second derived group is free.*

Proof. Let x generate an infinite cyclic subgroup of G for which (*) holds. Then for any g in G there are non-zero integers p, q such that $(x^p)^g = x^q$. Define $\varphi : G \to \mathbb{Q}^\times$ by $\varphi g = q/p$. As observed in lemma 0 of [11], φ is a well-defined homomorphism from G to the multiplicative group of rational numbers. Furthermore any finitely generated subgroup K of Ker φ centralises x^n for some positive n, and $\langle K, x^n\rangle$ is a group of cohomological dimension ≤ 2 with non-trivial centre. Corollary 8.9 of [2] shows that $K' \cap \langle x\rangle = 1$. Since this is true of every choice of K it follows that the derived subgroup of Ker φ does not meet $\langle x\rangle$. Since $G' \subseteq$ Ker φ we conclude that $G'' \cap \langle x\rangle = 1$. Now our proof of Theorem C actually shows that there is a G-tree such that every vertex and edge stabiliser is infinite cyclic and commensurable with $\langle x\rangle$. Therefore G'' is free for it must act freely on this tree.

There are two $\mathfrak{X}$-groups which often have to be treated as special cases, the free abelian group of rank 2 and the Klein bottle group, and for convenience we shall call them *flat* $\mathfrak{X}$*-groups*. I would like to thank Martin Roller for suggesting Corollary 3(ii) here, which is analogous to the well-known theorem of Stallings and Swan that every torsion-free free-by-finite group is free.

COROLLARY 3. (i) *The infinite cyclic subgroups H of a non-flat $\mathfrak{X}$-group G which satisfy* $\mathrm{Comm}_G(H) = G$ *are all commensurable.*

(ii) *Every torsion-free finite extension of an $\mathfrak{X}$-group is again an $\mathfrak{X}$-group.*

Proof. (i) Suppose that H_1 and H_2 are two such subgroups which are not commensurable, and let φ_1 and φ_2 be the corresponding homomorphisms from G to $\mathbb{Q}^\times$, as in the proof of Corollary 2. Let K be the subgroup generated by H_1 and H_2. Suppose first that under φ_1 the image of K in $\mathbb{Q}^\times$ is infinite. Then H_1 meets K', for if x is a generator of H_1 and p, q are distinct non-zero integers such that $y^{-1}x^p y = x^q$ for some $y \in K$ then $[x^p, y] = x^{q-p}$ is a non-trivial element of $K' \cap H_1$. Clearly $H_1 \subseteq \mathrm{Ker}\ \varphi_1$, and since $K/\mathrm{Ker}\ \varphi_1$ is infinite, it follows that $K' \cap H_2$ is trivial.

If, on the other hand, φ_2 carries K to an infinite subgroup of $\mathbb{Q}^\times$ then $K' \cap H_2$ is non-trivial while $K' \cap H_1$ is trivial. The conclusion is that at least one of the φ_i carries K into the subgroup $\{\pm 1\}$ of $\mathbb{Q}^\times$. Consequently one of the H_i is normal in K and K is either abelian or isomorphic to the Klein bottle group. If K is infinite cyclic then H_1 and H_2 are commensurable, contrary to assumption. Therefore K has a free abelian subgroup of rank two of finite index. Let A be a maximal abelian subgroup of G meeting K in a subgroup of finite index. Since $\mathrm{Comm}_G(H_i) = G$ for $i = 1, 2$, we have $\mathrm{Comm}_G(A) = G$. Now for any $g \in G$, $A \cap A^g$ is central in $\langle A, A^g \rangle$, both groups have cohomological dimension two, and by Bieri's Theorem A, $\langle A, A^g \rangle$ must be abelian. By choice of A, it contains A^g for all g. Thus A is normal and by Theorem 8.2 of [2], A has finite index in G. Therefore G is flat.

(ii) Let G be a torsion-free group with a normal $\mathfrak{X}$-subgroup N of finite index. Certainly G is finitely generated and of cohomological dimension two. Let H be an infinite cyclic subgroup of N with $\mathrm{Comm}_N(H) = N$. For any g in G, $\mathrm{Comm}_N(H^g) = N$. Therefore by (i), either N is abelian, or H and H^g are commensurable. Since this holds for all g, either G belongs to $\mathfrak{X}$, or it is abelian-by-finite. In the latter case, G is either free abelian of rank two or isomorphic to the Klein bottle group. In any case, G belongs to $\mathfrak{X}$.

The proof of Theorem C itself proceeds in a series of steps. In the first, it is shown that every $\mathfrak{X}$-group is of type $(FP)_2$. For this we use Strebel's clever finiteness criterion [13]. It leads at once to the observation that every G in $\mathfrak{X}$ is a two-dimensional duality group. The next step is to show that if H is an infinite cyclic subgroup of G satisfying (*) then $\tilde{e}(G, H) \geq 2$. Here, $\tilde{e}$ is the end invariant for pairs of groups introduced in [10]. There are convenient methods for computing $\tilde{e}(G, H)$ when G and H are duality groups which are described in [10] and which fit rather well with this application. Knowing that $\tilde{e} \geq 2$ shows that there is a proper H-almost invariant subset B of G: our third step is to apply Dunwoody's work, [5] to use the subset B to cut up a certain graph associated to (G, H) and so to construct a G-tree which yields a splitting of G as an amalgam or *HNN*-extension over a subgroup commensurable with H. I would like to thank Martin Dunwoody for suggesting the method used in this step. Finally it becomes possible to deduce Theorem C by using a recent result of Bestvina and Feighn [1] which generalises the idea of accessibility (cf. Dunwoody's [4]). These steps are carried out in §§2–4.

What use is Theorem C? It should help to understand how Baumslag–Solitar relators can arise in groups of low cohomological dimension. Proposition 1 of [11] asserts that if x is an element of infinite order in a 3-manifold group and p, q are non-zero integers such that x^p and x^q are conjugate then $p = \pm q$. At the heart of this is the work [8] of Jaco and Shalen. In [8] the result is proved for Haken manifolds, and as they remark on p. 176, "this is essentially stronger than the result . . . that the Baumslag–Solitar group $G_{p,q} = \langle a, b : ab^p a^{-1} = b^q \rangle$ is not a 3-manifold group for $|p| \neq |q|$". On the other hand, if $y^{-1}x^p y = x^q$ then typically x and y generate a group G of cohomological dimension two (in a 3-manifold group) and certainly G belongs to $\mathfrak{X}$. Theorem C therefore permits an alternative approach to Theorem VI.2.1 of [8] by allowing a reduction to the case of a Baumslag–Solitar group. This proves nothing new about 3-manifold groups, but possibly Theorem C may be useful in helping to decide whether the Baumslag–Solitar relator can hold in an abstract 3-dimensional Poincaré duality group.

Conceivably Theorem C may be a starting point for an abstract theory of groups of cohomological dimension two. As a step towards this we mention one further corollary.

COROLLARY 4. *Let G be any group of cohomological dimension two, let H be an infinite cyclic subgroup and let $C = \mathrm{Comm}_G(H)$. Then for all g not in C, $C \cap C^g$ is either locally free or flat.*

Proof. If $H_0 = H \cap C^g$ is trivial then, by the argument used to prove Corollary 2, every finitely generated subgroup of $C \cap C^g$ is free. On the other hand, if H_0 is non-trivial then both H_0 and H^g are commensurised by $C \cap C^g$ and hence, by Corollary 3, either H_0 and H^g are commensurable or $C \cap C^g$ is flat. However, it cannot happen that H_0 and H^g are commensurable because g was chosen outside C.

1. Influence of torsion-free abelian subgroups on cohomology

In this section we establish two results needed to begin the proof of Theorem C, based on a simple observation: if A is a torsion-free abelian normal subgroup of a group G and F is a free G-module then $H^i(G, F) = 0$ for each i less than the rank of A. This follows from an application of the Lyndon–Hochschild–Serre spectral sequence since it is quite easy to see that $H^i(A, F) = 0$ for all these i. If A is not normal then nothing like this can be said: for example, if G is a free product $A * B$ then $H^1(G, F)$ is non-zero no matter what. In Lemma 1.1 we show that something can be recovered provided A is close to normal.

If H is a subgroup of G and X a subset, we write $\mathrm{core}_X(H)$ for the intersection

$$\bigcap_{g \in X} H^g.$$

and H^X for the subgroup generated by H^x, $x \in X$. For the sake of brevity we shall say that an abelian subgroup A of G is n-large if and only if $\mathrm{core}_X(A)$ has torsion-free rank at least n for each finite subset X.

LEMMA 1.1. *Let A be an n-large abelian subgroup of a countable group G, and F a free G-module. Then*

(i) $H^i(G, F) = 0$ *for* $i < n$, *and*

(ii) $H^n(G, F)$ *is non-zero if and only if G is (free abelian of rank n)-by-finite.*

Proof. Choose a finite subset X of G. Let Z be a free abelian subgroup of $\mathrm{core}_X(A)$ of rank n, and note that Z is central in A^X. Then $H^i(Z, F) = 0$ for $i < n$, and $H^n(Z, F)$ is a free A^X/Z-module. Thus $H^i(A^X, F)$ vanishes for $i < n$ and when $i = n$ it can be identified with the subgroup of A^X-fixed points in the free A^X/Z-module $H^n(Z, F)$, so is non-zero only if Z has finite index in A^X. Now, regard the normal closure N of A as the (countable) direct limit of the A^X, where X varies through the finite subsets of G. Using the short exact sequence

$$\varprojlim{}^1 H^{i-1}(A^X, F) \to H^i(N, F) \to \varprojlim H^i(A^X, F)$$

one sees that $H^i(N, F)$ is zero for $i < n$. Moreover, for $H^n(N, F)$ to be non-zero one needs A^X to be centre-by-finite for all choices of X, so N must be locally (free abelian of rank n)-by-finite, and for such an N, $H^n(N, F)$ vanishes unless N is actually finitely generated, by Theorem 3.3 of [7]. The lemma now follows by applying the Lyndon–Hochschild–Serre spectral sequence to the group extension $N \to G \to G/N$.

As an amusing digression I want to point out how Lemma 1.1 can be used to establish the vanishing of cohomology for the group discussed by Brown and Geoghegan [3]. Their group, which I shall call G, has presentation

$$\langle x_0, x_1, x_2, \ldots ; x_i^{-1} x_n x_i = x_{n+1} \quad \text{for } i < n \rangle.$$

It is a remarkable discovery, being the first known example of a group of type $(FP)_\infty$ which has free abelian subgroups of infinite rank. Indeed the subgroup

$$A = \langle x_0 x_1^{-1}, x_2 x_3^{-1}, x_4 x_5^{-1}, \ldots \rangle$$

is clearly abelian and is in fact free on the displayed generators. Moreover if w is any word in the $x_i^{\pm 1}$ of zero exponent sum then $x_n x_{n+1}^{-1}$ commutes with w for all sufficiently large n, and consequently if X is any finite set of such words then $\mathrm{core}_X(A)$ has infinite rank. Therefore, using Lemma 1.1, it follows that $H^i(G_0, ZG)$ vanishes for all i, where G_0 is the subgroup comprising words of zero exponent sum. As G_0 is normal in G, we can recover one of the properties which Brown and Geoghegan proved by rather more topological methods; namely $H^i(G, ZG) = 0$ for all i.

LEMMA 1.2. *Let G be an $\mathfrak{X}$-group, H an infinite cyclic group satisfying* (*), *k a field and let V be a kG-module which is torsion-free and divisible qua kH-module. Then $H^i(G, V) = 0$ for all i.*

Proof. In effect H is a 1-large subgroup, and the Lemma is proved in a similar way to Lemma 1.1. Here one uses the fact that V is injective and without H-fixed points as a kH-module, so that $H^*(L, V) = 0$ for any non-trivial subgroup of H.

2. Strebel's criterion

Strebel's criterion [13] can be used to show that certain finitely generated groups of cohomological dimension two are of type $(FP)_2$. Let G be such a group. The criterion can be expressed as

$\varinjlim H^2(G, P_\lambda) = 0$ for every vanishing (meaning affording zero direct limit) direct limit system (P_λ) of projective G-modules. (**)

For the reader's convenience, and because our formulation is slightly different from Strebel's, we briefly outline why (**) is sufficient. To prove that a finitely generated group G is of type $(FP)_2$ it is enough to show that whenever (M_λ) is a vanishing direct limit system of G-modules then $\varinjlim H^2(G, M_\lambda) = 0$. Given a system (M_λ), one can form compatible short exact sequences

$$K_\lambda \to P_\lambda \to M_\lambda$$

with (P_λ) being a vanishing direct limit system of projective modules. Now, taking direct limits in the long exact sequence of cohomology we have the exact sequence

$$\varinjlim H^2(G, P_\lambda) \to \varinjlim H^2(G, M_\lambda) \to \varinjlim H^3(G, K_\lambda).$$

The right hand group here vanishes because $\mathrm{cd}\,(G) = 2$, and the left hand vanishes by (**). Therefore the central group vanishes and G is of type $(FP)_2$ as required.

For the remainder of this section, let G be an $\mathfrak{X}$-group and let H be an infinite cyclic subgroup satisfying (*). To apply Strebel's criterion to G we use the following ring theoretic result. The reader can find a good introduction to right Ore sets in group rings in Passman's book [12].

LEMMA 2.1. *Let k be an integral domain, and let Ω be the multiplicatively closed subset of kG generated by $\{x^g; 0 \neq x \in kH$ and $g \in G\}$. Then Ω is a right Ore set of non-zero-divisors in kG and $kG\Omega^{-1}$ is torsion-free and divisible as a kH-module. Moreover, if P is any projective kG-module then $H^i(G, P\Omega^{-1}) = 0$ for all i.*

Proof. I am indebted to the referee for correcting my original erroneous argument. We first prove that the set Π of non-zero elements of kH is a right Ore set. Given π in Π and x in kG we must find π' in Π and x' in kG such that $\pi x' = x\pi'$. Let X be the support of x and set $I = \mathrm{core}_X\,(H)$. We show that for each g in X there exist non-zero elements c_g in kH^g and d_g in kI such that $\pi^g c_g = d_g$. Since I has finite index in H^g, it follows that kH^g is integral over kI and hence there is a monic polynomial

$$f(Z) = Z^n + i_1 Z^{n-1} + \cdots + i_n$$

with coefficients i_j in kI, with $i_n \neq 0$ and satisfying $f(\pi^g) = 0$. We can therefore set

$$c_g = (\pi^g)^{n-1} + \cdots + i_{n-1}$$

and

$$d_g = -i_n.$$

Let $D_g = \Pi_{h \neq g, h \in X} d_h$.

If $x = \Sigma\, x_g g$, (x_g in k) then set

$$x' = \sum x_g g c_g D_g$$

and set

$$\pi' = \prod {}_g d_g.$$

Then $\pi x' = \pi(\Sigma\, x_g g c_g D_g) = \Sigma\, x_g g \pi^g c_g D_g = \Sigma\, x_g g d_g D_g = \Sigma\, x_g g \pi' = x\pi'$ as required.

The assertion that $kG\Omega^{-1}$ exists now follows. In fact $kG\Omega^{-1}$ can be identified with $kG\Pi^{-1}$.

It is clear that $kG\Omega^{-1}$ is torsion-free and divisible as a kH-module and the final remark follows from Lemma 1.2.

LEMMA 2.2. *G is of type $(FP)_2$ and has one end.*

Proof. Since H is a 1-large abelian subgroup of G, and G is not infinite cyclic, Lemma 1.1 shows that $H^1(G, -)$ vanishes on free modules. Thus G has one end. To prove that G is of type (FP) it is enough to check (**). Let (P_λ) be a vanishing direct limit system of projective modules, and consider the system of short exact sequences

$$P_\lambda \rightarrow P_\lambda \Omega^{-1} \rightarrow P_\lambda \Omega^{-1}/P_\lambda$$

formed by localising with the right Ore set of Lemma 2.1. Applying direct limits to the long exact sequence of cohomology yields the exact sequence

$$\varinjlim H^1(G, P_\lambda \Omega^{-1}/P_\lambda) \rightarrow \varinjlim H^2(G, P_\lambda) \rightarrow \varinjlim H^2(G, P_\lambda \Omega^{-1}).$$

Here, the right hand group vanishes by Lemma 2.1, and the left hand vanishes because G is finitely generated and $\varinjlim P_\lambda \Omega^{-1}/P_\lambda = 0$. Therefore (**) holds and the result follows.

As Bieri points out in proposition 9.17(c) of [2], the conclusion of Lemma 2.2 guarantees that G is a 2-dimensional duality group. For our purposes we need only consider this duality over the field $\mathbb{F}$ of two elements, in which case the dualising module is $D = H^2(G, \mathbb{F}G)$.

LEMMA 2.3. *The dualising module D is locally finite dimensional as an $\mathbb{F}H$-module, and $\tilde{e}(G, H) \geq 2$.*

Proof. Now D naturally inherits a left $\mathbb{F}G$-module structure from $\mathbb{F}G$. Here we are interested only in the restriction of D to $\mathbb{F}H$-module: think of $\mathbb{F}G$ as a $(\mathbb{F}H, \mathbb{F}G)$-bimodule, its right $\mathbb{F}G$-structure being "used up" in forming D, and its left $\mathbb{F}H$-structure being inherited. Using the localisation of Lemma 2.1, $\mathbb{F}G$ can be imbedded in $\mathbb{F}G\Omega^{-1}$, and this imbedding respects the bimodule structure. Moreover, as a $(\mathbb{F}H, \mathbb{F}G)$-bimodule, $\mathbb{F}G\Omega^{-1}$ is isomorphic to $K \otimes_{\mathbb{F}H} \mathbb{F}G$, where K is the field of fraction of $\mathbb{F}H$. Lemma 2.1 therefore shows that

$$H^2(G, K \otimes_{\mathbb{F}H} \mathbb{F}G)$$

vanishes. Using Lemma 2.4 below we can conclude that

$$K \otimes_{\mathbb{F}H} H^2(G, \mathbb{F}G)$$

vanishes, and this is just a reformulation of the desired conclusion that $D = H^2(G, \mathbb{F}G)$ is locally finite dimensional qua $\mathbb{F}H$-module. Therefore the result follows in view of Lemma 2.4 below. That $\tilde{e}(G, H) \geq 2$ now follows from Lemma 4.1(iii) of [10].

LEMMA 2.4. *If G is any group of type $(FP)_\infty$, R a ring, M an (R, G)-bimodule, and V a right R-module, then there are natural isomorphisms*

$$H^i(G, V \otimes_R M) = V \otimes_R H^i(G, M).$$

Proof. This is easily seen by computing the cohomology of G using a projective resolution of the trivial module of finite type.

3. Dunwoody' method of cutting up graphs

We continue the study of the pair of groups G, H, as in §2. Now we aim to find a subset B of G such that for all $g \in G$ the symmetric difference $B + Bg$ is contained in a finite union of right cosets Hx of H. Such a B is called H-almost invariant, and it is called proper if neither B nor its complement is contained in a finite union of cosets.

LEMMA 3.1. *Let G and H be as in Lemma 2.2. Then G has a proper H-almost invariant subset B such that $B = BH$.*

Proof. As in [10], let $\mathcal{F}_H G$ denote the G-module of H-finite subsets of G. An easy variation on Lemma 2.2 of [9] yields a suitable B once one knows that the restriction map

$$H^1(G, \mathscr{F}_H G) \to H^1(H, \mathscr{F}_H G)$$

has non-zero kernel. Here, we have already shown that $H^1(G, \mathscr{F}_H G)$ is non-zero, for Lemma 1.2 of [10] shows that this is equivalent to the statement $\tilde{e}(G, H) \geq 2$ which was established in Lemma 2.3. In fact $H^1(H, \mathscr{F}_H G)$ is zero, because $\mathscr{F}_H G$ qua H-module is a direct sum of coinduced modules, and the Lemma follows.

LEMMA 3.2. *G splits over an infinite cyclic subgroup commensurable with H.*

Proof. Let X be a finite set of generators for G. I would like to thank Martin Dunwoody for suggesting the following construction of a graph Γ on which G acts: this graph does indeed play a crucial role in the proof by enabling the use of Dunwoody's results in [5]. The vertices are the left cosets $gH, g \in G$, and for each x in X there is an edge joining gH to GxH. The graph need not be locally finite, but it does admit a left action of G. Moreover, Γ is connected because X generates G, and $G\backslash\Gamma$ is finite because X is finite. Let B be the subset given by Lemma 3.1. As it is a union of left cosets of H, it determines a subset W of vertices of Γ in the obvious way, and in fact this subset is a *cut* in the sense that only finitely many edges of Γ have exactly one vertex in it. To see this, fix $x \in X$ and consider

$$\{g \in g;\ \text{exactly one of } gH \text{ and } gxH \text{ is contained in } B\}$$

$$= \{g \in G;\ \text{exactly one of } g \text{ and } gx \text{ belongs to } B\}$$

$$= \{g \in G;\ \text{belongs to the symmetric difference } B + Bx^{-1}\}.$$

This set is H-finite: it is contained in a finite union of right cosets of H, and since $\mathrm{Comm}_G\,(H) = G$ it is equally contained in a finite union of left cosets of H. Repeating this argument for each member of the finite set X, we conclude that W is indeed a cut. The upshot is that the graph Γ has more than one end. Theorem 1.1 of [5] now shows that there is a subset D of G such that $D = DH$ and for all $g \in G$ one of the four intersections $D \cap gD, D \cap gD^*, D^* \cap gD, D^* \cap gD^*$ is empty. (D^* denotes the complement of D.) As Dunwoody concludes in Corollary 4.2 of [5], this leads to a splitting of G as an amalgam or HNN-extension over a subgroup C which contains the stabiliser of an edge of Γ as a subgroup of finite index. Now the stabiliser of the typical edge, joining gH to gxH, is $gHg^{-1} \cap gxH(gx)^{-1}$ and so is commensurable with H. It follows that C is an infinite cyclic subgroup commensurable with H.

4. Conclusion

In order to complete the proof we need a finiteness theorem proved by Bestvina and Feign [1]. The following is a special case of their Main Theorem. Although they state the result only for finitely presented groups, they point out in Remark (8) following the statement that it holds equally for groups of type $(FP)_2$.

THEOREM 4.1. *Let G be a group of type $(FP)_2$. Then there is a number δ such that if T is a reduced G-tree with abelian edge stabilisers then T has at most δ orbits of edges.*

Proof of Theorem C. Rather than trying the reader's patience with the details of the complete proof I shall just illustrate what happens in a simple example. Let G be in $\mathfrak{X}$ and let H be an infinite cyclic subgroup satisfying (*). If G is not infinite cyclic then it splits over a subgroup commensurable with H. Let us suppose that $G = K *_H L$ is an amalgam, and let T be the corresponding G-tree. Suppose furthermore that δ is 1. Then the Theorem above shows that there is no refinement of this splitting. Since both G and H are of type (FP), standard results imply that K and L are of type (FP). In particular both K and L belong to $\mathfrak{X}$. If K is not infinite cyclic then Lemma 3.2 shows that it splits over an infinite cyclic subgroup J commensurable with H. For the sake of argument, suppose that $K = M *_J N$ is an amalgam, and let U be the corresponding K-tree. Now J meets H and so in U there must be a vertex fixed by H. Hence there is an element $x \in K$ such that $x^{-1}Hx$ is contained in M or N. Now replace M, J, N by their conjugates by x^{-1}. Then, without loss of generality, we may assume that H is contained in M. But this means that G can be expressed as the double amalgam $L *_H M *_J N$. This yields a G-tree with two orbits of edges, and contradicts the assumption $\delta = 1$. Therefore both K and L are indeed infinite cyclic.

REFERENCES

[1] M. BESTVINA and M. FEIGHN, Bounding the complexity of simplicial group actions on trees (preprint).
[2] R. BIERI, *Homological dimension of discrete groups*, Queen Mary College Maths. Notes, 2nd ed. 1981.
[3] K. S. BROWN and R. GEOGHEGAN, *An infinite-dimensional FP_∞ group*, Invent. Math. *77*, 367–381 (1984).
[4] M. J. DUNWOODY, *Accessibility and groups of cohomological dimension one*, Proc. London Math. Soc. *38*, 193–215 (1979).
[5] M. J. DUNWOODY, *Cutting up graphs*, Combinatorica *2*, 15–23 (1982).
[6] D. GILDENHUYS, *Classification of soluble groups of cohomological dimension two*, Math. Z. *166*, 21–25 (1979).
[7] D. GILDENHUYS and R. STREBEL, *On the cohomological dimension of soluble groups*, Canad. Bull. Math. *24*, 385–392 (1981).

[8] W. H. JACO and P. B. SHALEN, *Seifert fibered spaces in 3-manifolds*, Memoirs Amer. Math. Soc. vol. 21, no. 220 (1979).
[9] P. H. KROPHOLLER and M. A. ROLLER, *Splittings of Poincaré duality groups*, Math. Z. *197*, 421–438 (1988).
[10] P. H. KROPHOLLER and M. A. ROLLER, *Relative ends and duality groups*, J. Pure Appl. Algebra *61*, 197–210 (1989).
[11] P. H. Kropholler, *A note on centrality in 3-manifold groups*, Math. Proc. Camb. Philos. Soc. *107*, 261–266 (1990).
[12] D. S. Passman, *The algebraic structure of group rings*, Wiley-Interscience, 1977.
[13] R. Strebel, *A homological finiteness criterion*, Math. Z. *151*, 263–275 (1976).

School of Mathematical Sciences
Queen Mary and Westfield College
Mile End Road
GB-London E1 4NS

Received September 28, 1989