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Rational category of the space of sections of a nilpotent bundle

Y. FÉLIX*

Abstract. Denote by $\zeta : F \rightarrow E \xrightarrow{p} B$ a nilpotent fibration where F is a 1-connected space of finite category and B a finite c.w. complex with non trivial rational cohomology. In this note we compute the rational category of the space Γ_* of continuous pointed sections of ζ .

§1. Introduction

In 1956, R. Thom studied the homotopy type of the space F_f^X of continuous maps of X into F homotopic to a given map f . He computed explicitly the cohomology of F_f^X when F is a product of Eilenberg–Mac Lane spaces [12].

Later on, following ideas of Sullivan, A. Haefliger gave the rational minimal model of the space of sections of a nilpotent bundle [7]. This model has been extensively studied by K. Shibata and M. Vigué-Poirrier [14]. In particular, M. Vigué-Poirrier noted that, if the dimension of X is less than the connectivity of Y , then the rational homotopy Lie algebra of Y^X is isomorphic as a Lie algebra to $H^*(X; \mathbb{Q}) \otimes (\pi_*(\Omega Y) \otimes \mathbb{Q})$.

The aim of this paper is to show that the category of the space of continuous maps from X into Y , and more generally of pointed sections of a fibration, is often infinite.

To be more precise, we prove in fact the following two theorems.

THEOREM 1. *Let Γ_* be the space of continuous pointed sections of a nilpotent fibration $F \rightarrow E \rightarrow B$. Suppose that*

- (1) $\Gamma_* \neq \phi$
- (2) F is a nilpotent space of finite category
- (3) $H^+(B; \mathbb{Q}) \neq 0$ and $\dim H^*(B; \mathbb{Q}) < \infty$
- (4) $\dim (\pi_*(F) \otimes \mathbb{Q})$ is infinite.

Then, $\text{cat}(\Gamma_) = \infty$.*

* Chercheur qualifié FNRS.

In the case where the fibration is trivial, the result is more precise:

THEOREM 2. *Let X be a finite nilpotent c.w. complex and Y be a nilpotent space. We suppose that the rational cohomologies of X and Y are not trivial and one of the two following conditions is satisfied:*

- (1) $\dim \pi_*(Y) \otimes \mathbb{Q} = \infty$
- (2) $\dim \pi_*(Y) \otimes \mathbb{Q} < \infty$ and there are odd integers p and q such that $H^p(X; \mathbb{Q}) \neq 0$, $\pi_q(Y) \otimes \mathbb{Q} \neq 0$ and $q - p \geq 2$.

Then the functional space Y^X has infinite category.

This result clearly yields the following corollary previously proved by E. Fadell and S. Husseini.

COROLLARY [3]. *If Y is a 1-connected space of finite category, such that $\tilde{H}^*(Y; \mathbb{Q}) \neq 0$; then the free loop space Y^{S^1} has infinite category.*

The organization of the paper is as follows. We first recall the construction of the Sullivan–Haefliger model for the space of continuous (resp. pointed) sections Γ (resp. Γ_*). We then show how to deduce the two theorems from the model. We also deduce a way to compute explicitly the rational homotopy groups of Γ .

In fact, if X is a 1-connected space and X_0 its rationalization, we have the inequality $\text{cat } X_0 \leq \text{cat } X$ [13]. The integer $\text{cat}(X_0)$ is called the rational category of X and is denoted $\text{cat}_0 X$. Its relevance comes from the fact that $\text{cat}_0(X)$ can be obtained from the minimal model of the space [5].

§2. The Sullivan–Haefliger model

Let $\zeta : F \rightarrow E \xrightarrow{p} B$ be a fibration. We suppose that B is a finite nilpotent c.w. complex and F is a nilpotent space with finite Betti numbers. We suppose that $\Gamma_* \neq \phi$.

Let $(A, d_A) \rightarrow (A \otimes \Lambda V, D) \rightarrow (\Lambda V, d)$ be a minimal K.S. model of ζ [9]. As $\Gamma_* \neq \phi$, we can also suppose that the differential D satisfies:

$$D(V) \subset A \otimes \Lambda^+ V.$$

B is a finite nilpotent c.w. complex. Therefore we can average A is a finite dimensional graded \mathbb{Q} -vector space. Denote by S a graded supplementary subspace of the graded vector space formed by the cocycles in A . This gives a direct sum decomposition of A : $A = S \oplus d(S) \oplus T$. We then choose a homogeneous basis

$(a_i)_{i \in I}$ of A by taking the union of homogeneous bases of S , $d(S)$ and T . The graded dual vector space of A will be denoted by A^\vee :

$$(A^\vee)_n = \text{Hom}(A^n, \mathbb{Q}).$$

A^\vee is naturally equipped with the dual basis a_i^* :

$$\langle a_i^*; a_j \rangle = \delta_{ij}.$$

We now look at the map of algebras defined by:

$$\varepsilon : A \otimes \Lambda V \rightarrow A \otimes \Lambda(A^\vee \otimes V) : \varepsilon(v) = \sum_{i \in I} a_i \otimes (a_i^* \otimes v); \quad \varepsilon(a) = a, \quad a \in A.$$

In [7], A. Haefliger shows how to put a uniquely defined differential $d_A \otimes \delta$ on $A \otimes \Lambda(A^\vee \otimes V)$ in such a way that ε becomes a morphism of commutative differential graded algebras. Let W be the quotient of $A^\vee \otimes V$ by the subspace of elements of degree < 0 , and by the subspace formed by the δ -cocycles in degree 0.

A short computation shows that $\delta(1 \otimes v) = 1 \otimes dv$, so that the injection $V \cong \mathbb{Q} \otimes V \hookrightarrow A^\vee \otimes V$ induces a K.S. extension:

$$\theta : (\Lambda V, d) \rightarrow (\Lambda W, \delta) \rightarrow (\Lambda(W/V), \delta).$$

THEOREM A (Haefliger, [7]). *θ is a model for the canonical fibration $\Gamma_* \rightarrow \Gamma \xrightarrow{p} F$ where p denotes the evaluation on the basis point of B .*

With this model, we can for instance give a rational analogue of the Cohen–Taylor theorem [2].

PROPOSITION. *Let X be a finite wedge of spheres of dimension less than m ($X = \vee_{i=1}^r S^{n_i}$) and Y be a $(m+2)$ -connected space, then we have a rational homotopy equivalence*

$$(Y^X)_* \cong \prod_{i=1}^r (Y^{S^{n_i}})_*.$$

Proof. The Haefliger model for $(Y^X)_*$ is $(\Lambda(H_*^+(X; \mathbb{Q}) \otimes (\pi_*(Y))^\vee), 0)$. □

§3. The rational homotopy Lie algebra of a space

If S is a nilpotent space with finite Betti numbers, the minimal model of S is a free commutative differential graded algebra $(\Lambda Z, d)$. The graded vector spaces Z^* and $\text{Hom}(\pi_*(S), \mathbb{Q})$ are then isomorphic [11].

The differential d always decomposes in the form $d = d_2 + d_3 + \dots$, where $d_i(Z) \subset \Lambda^i Z$. This gives on $s^{-1} \text{Hom}(Z, \mathbb{Q})$ a structure of Lie algebra by putting:

$$\langle d_2 z; f, g \rangle = (-1)^{\text{deg}(f)+1} \langle s z; [s^{-1} f, s^{-1} g] \rangle$$

$z \in Z; f, g \in \text{Hom}(Z, \mathbb{Q})$.

It is a result of Andrews and Arkowitz [1] that this Lie algebra is isomorphic to the Lie algebra $L_S = \pi_*(\Omega S) \otimes \mathbb{Q}$ obtained on the rational homotopy groups by means of the Whitehead product. An extensive study of L_S has been made these last years with for instance the following result:

THEOREM B ([6], [4]). *If S is a space of finite category, then every nilpotent ideal I of L_S is finite dimensional.*

We now want to compute L_r for a given fibration. With the notations of §2, we decompose the differentials D and δ in the form

$$D = D_1 + D_2 + \dots \quad D_i(V) \subset A \otimes \Lambda^i(V)$$

$$\delta = \delta_1 + \delta_2 + \dots \quad \delta_i(A^\vee \otimes V) \subset \Lambda^i(A^\vee \otimes V)$$

δ_1 is completely defined by d_A and D_1 . In fact, put

$$D_1 v_r = \sum_s \alpha_{rs} v_s, \quad \alpha_{rs} \in A^+.$$

We then have:

$$(*) \sum_i (-1)^{\text{deg}(a_i)} a_i \otimes \delta_1(a_i^* \otimes v_r) = -\sum_i d_A(a_i) \otimes (a_i^* \otimes v_r)$$

$$+ \sum_i \sum_s \alpha_{rs} \cdot a_i \otimes (a_i^* \otimes v_s)$$

The homology of $(A^\vee \otimes V, \delta_1)$ and $(A^+ \otimes V, \delta_1)$ are respectively isomorphic to the vector spaces of indecomposable elements of the minimal models of Γ and Γ_* :

We thus have isomorphisms:

$$(1) H^*(A^\vee \otimes V, \delta_1) \cong (\pi_*(\Gamma) \otimes \mathbb{Q})^\vee$$

and

$$(2) H^*(A_+^\vee \otimes V, \delta_1) \cong (\pi_*(\Gamma_*) \otimes \mathbb{Q})^\vee.$$

Moreover, the short exact sequence of complexes

$$0 \rightarrow (V, 0) \rightarrow (A^\vee \otimes V, \delta_1) \rightarrow (A_+^\vee \otimes V, \delta_1) \rightarrow 0$$

induces in homology a long exact sequence isomorphic to the dual of the homotopy long exact sequence of the fibration $\Gamma_* \rightarrow \Gamma \rightarrow F$ [9]:

$$\begin{array}{ccccccc} \cdots \rightarrow H^q(A_+^\vee \otimes V, \delta_1) & \xrightarrow{\Delta} & V^{q+1} & \rightarrow & H^{q+1}(A^\vee \otimes V, \delta_1) & \rightarrow & H^{q+1}(A_+^\vee \otimes V, \delta_1) \rightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots \rightarrow (\pi_q(\Gamma_*))^\vee & \longrightarrow & (\pi_{q+1}(F))^\vee & \longrightarrow & (\pi_{q+1}(\Gamma))^\vee & \longrightarrow & (\pi_{q+1}(\Gamma_*))^\vee \rightarrow \cdots \end{array}$$

REMARK. “ D_1 is differential” can be expressed by the fact that the matrix \mathbf{a} consisting of the $\alpha_{r,s}$ satisfies $\mathbf{a}^2 + d\mathbf{a} = 0$.

§4. Proof of Theorem 1

We use the notations of §§2 and 3, $(AV, d) \rightarrow (AW, \delta) \rightarrow (A(W/V), \delta)$ is a model of the fibration $\Gamma_* \rightarrow \Gamma \xrightarrow{p} F$. We consider the linear map

$$D_1 : V \rightarrow A \otimes V.$$

There are two cases: Either there exists an infinite sequence of homogeneous linearly independent elements v_1, v_2, \dots belonging to V such that $D_1(v_i)$ doesn't belong to $D_1(A^+ \otimes V)$, or we can suppose that there exists an integer N such that for every v in V of degree larger than N , $D(v)$ belongs to $A \otimes A^{\geq 2}V$.

We take a K.S. basis $(v_i)_{i \geq 1}$ of V : $D(v_i) \in A \otimes (v_j)_{j < i}$. Write

$$D_1(v_n) = \sum_{r=1}^s \alpha_r \cdot v_r.$$

We obtain $d_A(\alpha_s) = 0$. If $[\alpha_s] = 0$, then $\alpha_s = d_A(b)$ and $D_1(v_n - b \cdot v_s) \in A \otimes (v_j)_{j < s}$.

We then replace v_n by $v'_n = v_n - b \cdot v_s$. If $D_1(v)$ does not belong to $D_1(A^+ \otimes V)$, we can suppose $[a_s] \neq 0$. In this case, formula (*) gives the equality

$$\delta_1(\alpha_s^* \otimes v_s) - (-1)^{\deg(a_s)} 1 \otimes v_n.$$

This means that the element $1 \otimes v_n$ belongs to the image of Δ in the dual homotopy long exact sequence. Recall that the elements in the image of Δ are called the Gottlieb elements of the fibration and let us come back to our dichotomy:

In the first case, the v_i are Gottlieb elements of Γ_* [5]. By [5], we know that the category of a space is greater or equal to the number of its linearly independent Gottlieb elements, so that $\text{cat}(\Gamma_*) = \infty$.

In the second case, put $n = \max \{p \text{ such that } A^p \neq 0\}$. For $q > n + N$, formula (*) yields the isomorphisms

$$H^q((A^\vee \otimes V), \delta_1) = (H(A, d_A)^\vee \otimes V)^q.$$

The injections $\delta(A_p^\vee \otimes V) \subset \Lambda(A_{\leq p}^\vee \otimes V)$, valid for $p > 0$, imply that the Lie algebra $L = (H(A, d_A)_+^\vee \otimes V)^{>n+N}$ is a nilpotent Lie algebra of infinite dimension, which is impossible by Theorem B. □

§5. Proof of Theorem 2

In this case, the fibration $(Y^X)_* \xrightarrow{i} (Y^X) \rightarrow Y$ admits a section and so $\pi_*(i) \otimes \mathbb{Q}$ is injective. It then results from [5] that $\text{cat}_0((Y^X)_*) \leq \text{cat}_0(Y^X)$. If (1) is satisfied, we deduce from Theorem 1 that $\text{cat}_0(Y^X)$ is infinite.

If (2) is satisfied, choose a non homologically trivial cycle α in A^p and a nonzero element v in V^q . We now have $D_1 = 0$. The formula (*) shows that $\alpha^* \otimes v$ is a δ_1 -cycle which is not a δ_1 -boundary. $\alpha^* \otimes v$ defines thus a nonzero indecomposable element in the minimal model of Y^X . The definition of δ , as given in §2 implies the following formulas (**) and (***):

$$(**) \sum_i (-1)^{\deg(a_i)} a_i \otimes \delta(a_i^* \otimes t) = -\sum_i d_A(a_i) \otimes (a_i^* \otimes t) + \varepsilon(D(t)).$$

$$(***) \varepsilon(v_1 \cdot v_2 \cdots v_r) = \sum_{i_1} \sum_{i_2} \cdots \sum_{i_r} a_{i_1} \cdot a_{i_2} \cdots a_{i_r} \otimes (a_{i_1}^* \otimes v_1)(a_{i_2}^* \otimes v_2) \cdots (a_{i_r}^* \otimes v_r).$$

As $\alpha^2 = 0$ formula (***) shows that $(\alpha^* \otimes v)^n$ can never appear in the decomposition of $\varepsilon(v_q \cdot v_2 \cdots v_r)$, and so by (**), in the differential of an element $\beta^* \otimes t$. It then results from ([8] Proposition 1) that $\text{cat}_0(Y^X) = \infty$. □

If the dimension of X is less than the connectivity of Y , the result we obtain is better.

THEOREM 3. *Let X be a nilpotent space such that there exists an integer $k \geq 1$ with $H^p(X; \mathbb{Q}) = 0$, $p > k$ and $H^k(X; \mathbb{Q}) \neq 0$ and let Y be a $(m - 1)$ -connected space, non contractible over \mathbb{Q} , with $m \geq k + 2$, then the functional space Y^X has finite rational category iff the three following conditions are satisfied:*

- (1) $\pi_*(Y) \otimes \mathbb{Q} = \pi_{\text{odd}}(Y) \otimes \mathbb{Q}$.
- (2) $H^*(X; \mathbb{Q}) = H^{\text{even}}(X; \mathbb{Q})$.
- (3) $\dim \pi_*(Y) \otimes \mathbb{Q} < \infty$.

Proof. By Theorem 2, Condition 3 is necessary. In this case, we have $\dim \pi_{\text{even}}(Y) \otimes \mathbb{Q} \leq \dim \pi_{\text{odd}}(Y) \otimes \mathbb{Q}$ ([8], Proposition 1), so that $\pi_{\text{odd}}(Y) \otimes \mathbb{Q} \neq 0$. By Theorem 2, the second condition is thus also necessary.

Suppose thus that $H^*(X; \mathbb{Q}) = H^{\text{even}}(X; \mathbb{Q})$. We can suppose that $A^{>k} = 0$. If $\pi_{\text{even}}(Y) \otimes \mathbb{Q} \neq 0$, let's choose a cycle α of A^k defining a nonzero element of $H^k(X; \mathbb{Q})$ and let's choose a nonzero element v in V^{even} . Then, formula (***) shows that no power of $a^* \otimes v$ can appear in the decomposition of $\varepsilon(v_1 \cdot v_2 \cdots v_r)$ for any choice of v_1, v_2, \dots, v_r . Now by (**) no power of $a^* \otimes v$ appear in the expression of a boundary, so that, by ([8]) the category of Y^X has to be infinite.

On the other hand, if the three conditions are satisfied,

$$\pi_*(Y^X) \otimes \mathbb{Q} \cong \pi_{\text{odd}}(Y^S) \otimes \mathbb{Q}$$

is finite dimensional and concentrated in odd degrees. The minimal model of Y^X is thus finite dimensional. This implies that Y^X has the rational homotopy type of a finite c.w. complex. \square

REFERENCES

- [1] P. ANDREWS and M. ARKOWITZ, *Sullivan's minimal models and higher order Whitehead products*. Can. J. of Math. 30 (1978), 961–982.
- [2] F. R. COHEN and L. R. TAYLOR, *Homology of function spaces*. Math. Z. 198 (1988), 299–316.
- [3] E. FADELL and S. HUSSEINI, *A note on the category of the Free loop space*. (Preprint) 1988.
- [4] Y. FÉLIX, *La dichotomie Elliptique-Hyperbolique en homotopie rationnelle*. Asterisque n° 179, 1989, Société Mathématique de France.
- [5] Y. FÉLIX and S. HALPERIN, *Rational L.S. category and its applications*. Trans. Amer. Math. Soc. 273 (1982), 1–37.
- [6] Y. FÉLIX, S. HALPERIN, C. JACOBSON, C. LÖFWALL and J. C. THOMAS, *The radical of the homotopy Lie algebra*. Amer. J. of Math. (1988), 301–322.
- [7] A. HAEFLIGER, *Rational homotopy of the space of sections of a nilpotent bundle*. Trans. Amer. Math. Soc. 273 (1982), 609–620.
- [8] S. HALPERIN, *Finiteness in the minimal models of Sullivan*. Trans. Amer. Math. Soc. 230 (1977), 173–199.

- [9] S. HALPERIN, *Lectures on minimal models*. Mémoire Soc. Math. France n° 9/10 (1983).
- [10] K. SHIBATA, *On Haefliger's model for the Gelfand–Fuchs cohomology*. Japan J. Math. 7 (1981), 379–415.
- [11] D. SULLIVAN, *Infinitesimal computations in topology*. Publ. IHES 47 (1977), 269–331.
- [12] R. THOM, *L'homologie des espaces fonctionnels*. Colloque Top. Alg. Louvain (1956), 29–39.
- [13] G. TOOMER, *Two applications of homology decompositions*. Can. J. Math. vol. XXVII (1975), 323–329.
- [14] M. VIGUÉ-POIRRIER, *Sur l'homotopie rationnelle des espaces fonctionnels*. Manuscripta Math. 56 (1986), 177–191.

Institut Mathématique
2, chemin du Cyclotron
1348 Louvain-la-Neuve
Belgique

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