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Normal forms for Hamiltonian systems with Poisson commuting integrals – elliptic case

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I. Introduction

In this paper we will consider the problem of normal forms of Hamiltonian systems near an elliptic stationary point. These systems are transformed as vector fields by local symplectic diffeomorphisms, and this transformation gives rise to an equivalence relation on the space of all such systems. In strict rigour, the normal form problem amounts to exhibiting a set of relatively simple systems (the *normal forms*) which has one and only one system in common with each equivalence class. A weaker version of this problem is to exhibit some set of relatively simple systems which intersects each equivalence class along a subset which is substantially smaller than the class itself.

Classifying Hamiltonian systems under local symplectic diffeomorphisms amounts to classifying functions under such diffeomorphisms. The problem is thus to find symplectic normal forms for functions. A nice non-symplectic normal form always exists. Indeed, a generic function is locally equivalent to its quadratic part. This is the content of *Morse's lemma*. However, such a normal form, though extremely simple, does not give much information about the Hamiltonian system itself.

It is otherwise with a symplectic normal form, for example *Birkhoff's normal* form which exists in a formal sense for most systems. If this normal form could be attained by a differentiable or analytic, and not only formal, symplectic transformation, the Hamiltonian system would be transformed to a very simple form which is easy to analyze. But in general this is not so. In the analytic case the transformation has in general convergence radius 0 – if the normal form itself is convergent or divergent is not known – and in the differentiable case one gets a restterm. Because of this disappointing result of Siegel, one must look for weaker normal forms. One such weaker form is given by the existence of the *centermanifolds of Lyapounov*.

In this article we will see what can be said about these problems when the Hamiltonian system has several integrals which commute for the Poisson bracket, and, in particular, when the system is integrable. Though such systems are very exceptional in any generic sense, there exist many well known examples. The 3-body problem and many rigid body problems have several Poisson commuting integrals. The 2-body problem, certain rigid body problems, and the Neumann problem are integrable, together with more recent examples, like the inverse square potential of Calogero and the lattice of Toda. We will restrict the discussion to the smooth, i.e. C^∞, case but there are corresponding results also in the analytic and the finite differentiable cases.

Preliminaries

Let (M, ω) be a smooth symplectic manifold of real dimension $2n$, and let $\mathbf{E}_p$ be the space of all germs of smooth real functions at some point p on M. We say that a germ f is *critical* if $df(p) = 0$.

The *Hamiltonian vector field* X_f of f is defined through the equation

$$\omega(X_f, Y) = df \rfloor Y$$

for any vector field Y, where $\rfloor$ is the interior product.

The *Poisson bracket* of two germs f and g is

$$\{f, g\} = df \rfloor X_g = \omega(X_f, X_g).$$

It defines a structure of a Lie algebra on $\mathbf{E}_p$, containing all the critical germs as a maximal ideal.

Two germs are said to be *Poisson commuting* (or to be *in involution*) if their Poisson bracket vanishes. This implies that their Hamiltonian vector fields commute. (For critical germs this is even equivalent.) The $\mathbf{R}$-linear span of a set of pairwise Poisson commuting germs is an abelian subalgebra, and their Hamiltonian vector fields generate a local abelian group action. When the dimension of the abelian subalgebra is half the dimension of M, we say that this subalgebra is *integrable*. (In this case all the Hamiltonian vector fields involved are integrable in the sense of Liouville.)

To each critical germ f we associate, in a unique way, a quadratic form

$$d_p^2 f \in S^2(T_pM, \omega_p)^*$$

where $\omega_p = \omega(p)$. (In local coordinates this quadratic form is just the Hessian of f at p.) The symplectic form provides $S^2(T_pM, \omega_p)^*$ with a Poisson bracket $\{\ ,\ \}_p$

making this space into a Lie algebra isomorphic to sp $(2n, \mathbf{R})$. This bracket can be defined by $\{d_p^2 f, d_p^2 g\}_p = d_p^2\{f, g\}$, so d_p^2 becomes a Lie algebra homomorphism for this structure.

A *Cartan subalgebra* (CSA) of $S^2(T_pM, \omega_p)^*$ is an n-dimensional subalgebra $\mathfrak{q}$ which is abelian and self-centralizing, i.e. the *centralizer*,

$$\text{Centr}(\mathfrak{q}) = \{\alpha \in S^2(T_pM, \omega_p)^* : \{\alpha, \mathfrak{q}\}_p = 0\}$$

is equal to $\mathfrak{q}$ itself. We say that it is *elliptic* if it is generated by $q_i = \frac{1}{2}(x_i^2 + y_i^2)$, $i = 1, \dots, n$, in some set of coordinates on T_pM such that $\omega_p = \sum dx_i \wedge dy_i$ (*symplectic coordinates*). Such a base is unique up to permutation of order, and $q_1, \dots, q_n$ will always denote this particular base for $\mathfrak{q}$, if not otherwise specified – it is *the* base for the CSA. (The elliptic CSA:s represent one conjugation class out of finitely many. In section VII we shall say someting about the other classes.)

Let $\alpha_1, \dots, \alpha_k$ be k quadratic forms. The set of all $X \in T_pM$ such that $\alpha_1 \rfloor X, \dots, \alpha_k \rfloor X$ has rank less than k is the *singular set*. It is the union of all

$$S_r(\alpha_1, \dots, \alpha_k) = \{X \in T_pM : \alpha_1 \rfloor X, \dots, \alpha_k \rfloor X \text{ have rank } \leq r\}, \qquad r < k.$$

This set only depends on the $\mathbf{R}$-linear span of $\alpha_1, \dots, \alpha_k$ and not on the particular basis.

If $\mathfrak{q}$ is an elliptic CSA of $S^2(T_pM, \omega_p)^*$, then the singular set $S_r(\mathfrak{q})$ is a union of $\binom{n}{r}$ symplectic subspaces of T_pM of dimension $2r$ – *the singular subspaces*. By abuse of notations we shall also let $S_r(\mathfrak{q})$ denote the set of all these subspaces.

DEFINITION. An abelian subalgebra $\mathbf{p}$ of $S^2(T_pM, \omega_p)^*$ is said to be *non-degenerate* if $\text{Centr}(\mathbf{p})$ is an elliptic CSA and $S_{k-1}(\mathbf{p}) = S_{k-1}(\text{Centr}(\mathbf{p}))$, $k = \dim \mathbf{p}$.

Since $\text{Centr}(\mathbf{p})$ is an elliptic CSA, the condition that $S_{k-1}(\mathbf{p}) = S_{k-1}(\text{Centr}(\mathbf{p}))$ says precisely that the restriction of $\mathbf{p}$ to any singular subspace E in $S_{k-1}(\text{Centr}(\mathbf{p}))$ is an elliptic CSA of $S^2(E, \omega_p/E)^*$. This is a strong maximality condition of $\mathbf{p}$, and clearly generic.

If dimp $\mathbf{p} = n$, then $\mathbf{p}$ is non-degenerate if and only if $\mathbf{p}$ is an elliptic CSA.

A Morse lemma for Poisson commuting functions

THEOREM A. *Let $h_1, \dots, h_k$ be k germs of smooth functions at p in (M, ω), all critical at p and pairwise Poisson commuting, and let $\mathbf{h}$ be their $\mathbf{R}$-linear span. Assume that $d_p^2\mathbf{h}$ is non-degenerate of dimension k.*

Then there exist a smooth diffeomorphism $\Phi:(T_pM, 0)\to(M, p)$, *and smooth functions* $\psi_1, \ldots, \psi_k$ *such that*

$$h_i \circ \Phi = \psi_i(q_1, \ldots, q_n), \qquad i \le k$$

where $q_1, \ldots, q_n$ *is the base for* Centr $(d_p^2\mathbf{h})$.

When $k=1$ this is just Morse's lemma. The case $k\ge 2$ seems to be new.

We must stress here that the theorem does *not* say that Φ is a symplectic mapping. In fact, Φ is not unique, but it seems very unlikely that it exists such a symplectic diffeomorphism in any generality. Indeed, if it did, then there would exist not k, but n commuting functions, and we would be in the integrable case. This is likely to be a very exceptional situation. (Though this has been shown rigorously only when $k=1$ [1, 2, 3, 4].)

In particular, the theorem gives no information about the action of $\mathbf{h}$, besides the evident fact that this action takes place on the common fibers of $h_1, \ldots, h_k$. But it gives a fairly nice description of these fibers themselves. In general, they are submanifolds of dimension $2n-k$ and fibrated into n-dimensional tori. Since the ψ_i's are not unique, the fibration is determined up to diffeomorphic equivalence by some special class of such functions. (We will supply some partial result on these equivalence classes in section III.)

Moreover, the theorem has as an immediate consequence the existence of *singular manifolds* on which the action of $\mathbf{h}$ is integrable.

COROLLARY. *There exist* $\binom{n}{k-1}$ *symplectic smooth submanifolds of dimension* $2k-2$ *at* p *which are invariant under the action of* $\mathbf{h}$, *i.e. they are invariant under each* X_{h_j}, $j\le k$.

Generalized centermanifolds

The singular manifolds of an abelian subalgebra $\mathbf{h}$ of dimension k are symplectic submanifolds of dimension $2k-2$. In many cases, however, there exist invariant submanifolds of dimension $2k$ on which $\mathbf{h}$ is integrable.

THEOREM B. *Let* $h_1, \ldots, h_k$ *be germs of smooth functions at* p *in* (M, ω) *which are all critical at* p *and pairwise Poisson commuting, and let* $\mathbf{h}$ *be their* $\mathbf{R}$*-linear span. Let* E *be a symplectic subspace of* (T_pM, ω_p) *of dimension* $2k$, *and* *assume that*

1) *there is an h in* $\mathbf{h}$ *such that E is invariant under j_1X_h – the linearized vector field at p – and such that a solution of j_1X_h is 2π-periodic if and only if it lies in E;*

2) *the restriction of* $d_p^2\mathbf{h}$ *to E is an elliptic* CSA *of $S^2(E, \omega_p/E)^*$.*

Then there exists a unique smooth submanifold N at p, $T_pN = E$, such that N is invariant under the action of $\mathbf{h}$.

For one analytic function, theorem B is mainly due to Lyapounov, who proved the existence of a 1-parameter family of periodic solutions [5]. Siegel [6] proved the regularity at the origin, and the (first?) differentiable version can be found in [7].

Symplectic normal form for Poisson commuting functions

THEOREM C. *Let $h_1, \ldots, h_n$ be n germs of smooth functions at p in (M, ω) which are all critical at p and pairwise Poisson commuting, and let* $\mathbf{h}$ *be their* $\mathbf{R}$*-linear span. Assume that* $d_p^2\mathbf{h}$ *is non-degenerate of dimension n.*

Then there exist a smooth diffeomorphism $\Phi: (T_pM, 0) \to (M, p)$ which is symplectic, i.e. $\Phi^*\omega = \omega_p$, and smooth functions $\psi_1, \ldots, \psi_n$ such that

$$h_i \circ \Phi = \psi_i(q_1, \ldots, q_n), \qquad i \leq n$$

where $q_1, \ldots, q_n$ is the base for $d_p^2\mathbf{h}$.

The functions $\psi_1, \ldots, \psi_n$ are here uniquely determined in distinction to the case in theorem A, but Φ itself is not unique. It can be composed with any element of the linear action of $d_p^2\mathbf{h}$. In symplectic coordinates, its invariance group G consists of the rotations $(x, y) \mapsto (x', y')$ of the form

$$x_i' = x_i \cos a_i + y_i \sin a_i$$
$$y_i' = -x_i \sin a_i + y_i \cos a_i$$

for any function $a_i = a_i(q_1, \ldots, q_n)$.

Formally, theorem C is just a consequence of Birkhoff's normal form [8]. For analytic functions it has been proved by Rüssmann when $n = 2$ [9], and by Vey in the general case [10]. Vey's proof, however, does not carry over to the smooth case. (H. Ito has sharpened Vey's result, but his proof also only works in the analytical case [11].) We will deduce this result from theorem A, thus providing a unified approach to this problem in the smooth and the analytical cases.

One can use this result to construct *singular action and angle variables* near a lower dimensional torus. Theorem C represents, from this point of view, the case of a 0-dimensional torus.

Let $h_1, \ldots, h_n$ be Poisson commuting germs such that $dh_1(p), \ldots, dh_n(p)$ is of rank k. To the $\mathbf{R}$-linear span we can associate, in a natural way, an abelian subalgebra

$$d_p^2\mathbf{h} \subset S^2(K/L, \omega_p)^*$$

where $K = \bigcap \operatorname{Ker} dh_i(p)$ and L is the linear span of $X_{h_1}, \ldots, X_{h_n}$. (Suppose $h_{k+1}, \ldots, h_n$ are all critical at p. Introduce symplectic coordinates (x, y) near p, with $y_j = h_j$ for $j \le k$. Then $h_{k+1}, \ldots, h_n$ are independent of $x_1, \ldots, x_k$, so we just let $y_1 = \cdots = y_k = 0$ and take the Hessian at the origin of each of $h_{k+1}, \ldots, h_n$ as functions of $x_{k+1}, \ldots, x_n, y_{k+1}, \ldots, y_n$.)

THEOREM. *Let $h_1, \ldots, h_n$ be Poisson commuting smooth functions on M with $\mathbf{R}$-linear span $\mathbf{h}$, and let $c \in \mathbf{R}^n$ be such that $\Gamma = \bigcap_{1 \le i \le n} h_i^{-1}(c_i)$ is compact and connected. Assume that the* rank *of $dh_1, \ldots, dh_n$ is k on Γ, and that $d_p^2\mathbf{h}$ is non-degenerate of dimension $n-k$ at some point $p \in \Gamma$. Let $\mathbf{T}$ be the 1-torus $\mathbf{R}/(2\pi\mathbf{Z})$.*

Then there exist a neighbourhood U of Γ, a neighbourhood V of $\mathbf{T}^k \times 0$ in $T(\mathbf{T}^k \times \mathbf{R}^{n-k})$, a smooth diffeomorphism

$$\Phi: V \to U, \qquad \Phi(\mathbf{T}^k \times 0) = \Gamma$$

which is symplectic, i.e.

$$\Phi^*\omega = \sum_{1 \le i \le n} dx_i \wedge dy_i,$$

and smooth functions $\psi_1, \ldots, \psi_n$ such that

$$h_i \circ \Phi(x, y) = \psi_i(q_1, \ldots, q_n),$$

where we have put $q_i = y_i$, $i \le k$, and $q_i = \frac{1}{2}(x_i^2 + y_i^2)$, $i \ge k+1$.

The existence of non-singular action and angle variables was first proven by Arnold [12] under an extra assumption. Now, other proofs are available in the literature, for example [4]. One can prove the above theorem by, for example, adapting the proof in the non-singular case and using a parameter dependent version of theorem C. This is rather straight forward so we shall not carry it out here. (It has been done in [13]. J. P. Dufour and P. Molino have another proof of this theorem in the smooth case [14].)

Organization of the paper

In section II we give some more details on the elliptic CSA's.

In section III we study a division problem on the singular spaces $S_r(\mathbf{q})$ of some elliptic CSA $\mathbf{q}$. This division problem turns out to be the essential difficulty in the proof of theorem A.

In section IV we construct the singular manifolds. They will be constructed inductively, starting with those of lowest dimensions, and they are obtained as the solution of a set of equations which are singular. The division result from section III, however, will permit us to divide out this singularity and then to solve the equations by the implicit function theorem.

The construction of the diffeomorphism Φ in theorem A also involves a singularity problem, and it is only the existence of the singular manifolds which permits us to apply our division result and get rid of the singularity.

In section V we construct the generalized centermanifolds. This construction involves a singularity problem of the same kind as in section IV, and we shall treat it in the same way – we use the division result in order to divide out the singularity of the equations and then apply the implicit function theorem.

In section VI we formulate a version of Darboux's lemma for a given Lagrangian fibration using a deformation argument à la Moser [15]. This result fills the gap between theorem A and theorem C.

In section VII, finally, we discuss the corresponding results for other CSA's than the elliptic ones. These have been studied in [13], and, except for minor changes, theorem A and C remain true also for them. We also discuss briefly what is known for other types of Lie algebras.

NOTATIONS. The elementary result on linear symplectic algebra that we shall use can be found in [12] or [16].

Consider the real symplectic vector space (T_pM, ω_p). We define $J_p: T_p^*M \to T_pM$ by $\omega_p(J_p df, Y) = (df)Y$. Then $J_p^* = -J_p$, and the Hamiltonian vector field of a function f on T_pM, with respect to ω_p, is $X_f = J_p\, df$.

Given a symplectic base on T_pM and its dual base on T_p^*M, we have

$$J_p = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Such a choice of bases permits us to identify T_pM and T_p^*M and, hence, to consider J_p as an isomorphism on T_pM. Then $\omega_p(J_p, \ )$ is the standard euclidean metric in this base.

If E is a symplectic subspace of T_pM, then $E^\perp$ is the skew-orthogonal complement of E. Clearly $(E^\perp)^\perp = E$ and $T_pM = E + E^\perp$. We let ι_E and π_E be the natural injection and projection with respect to this decomposition.

We use the notation $f \in \mathbf{O}^{k+1}(z)$ to denote that f and all its derivatives (with respect to z) of order $\leq k$ vanish when $z = 0$.

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II. Algebraic preliminaries

On $S^2(T_pM, \omega_p)^*$ there is a natural Lie bracket defined in the following way. Let α and β be quadratic forms, and let $v, \omega \in T_pM$. Define $\alpha_v \in T_p^*M$ and $\hat{\alpha}_v \in T_pM$ by

$$\alpha_v(w) = \alpha(v, w) \quad \text{and} \quad \omega_p(\hat{\alpha}_v, w) = \alpha(v, w).$$

Then

$$\alpha(v, \hat{\beta}_w) = \omega_p(\hat{\alpha}_v, \hat{\beta}_w) = -\omega_p(\hat{\beta}_w, \hat{\alpha}_v) = -\beta(w, \hat{\alpha}_v).$$

If we now define

$$\{\alpha, \beta\}_p(v, w) = \alpha(v, \hat{\beta}_w) - \beta(v, \hat{\alpha}_w)$$

then clearly $\{\alpha, \beta\}_p$ is a quadratic form. And it is plain to verify that $\{\ ,\ \}_p$ is a Lie bracket, making $S^2(T_pM, \omega_p)^*$ into a Lie algebra isomorphic to sp $(2n, \mathbf{R})$.

A CSA $\mathbf{q}$ of $S^2(T_pM, \omega_p)^*$ is defined to be elliptic if it is generated by the quadratic functions $q_i = \frac{1}{2}(x_i^2 + y_i^2)$, $1 \leq i \leq n$, in some symplectic coordinate system $z = (x, y)$ on T_pM. Another way to describe the ellipticity in symplectic coordinates is the existence of a quadratic function $p(z) = z^*Az$ in $\mathbf{q}$ such that J_pA has distinct purely imaginary eigenvalues. This follows from the following lemma, which also establishes a certain stability property of an elliptic CSA.

LEMMA 1. *Let $A(\lambda)$ be a real symmetric matrix of class C^r, defined in some neighbourhood of the origin in $\mathbf{R}^m$, and such that $J_pA(0)$ have distinct purely imaginary eigenvalues. Let $f(z, \lambda) = z^*A(\lambda)z$.*

Then, for small λ, there is a C^r matrix $C(\lambda)$ which is symplectic, i.e. $C(\lambda)J_pC(\lambda)^* = J_p$, and such that

$$f(C(\lambda)z, \lambda) = \sum_{1\le i\le n} a_i(\lambda)q_i(z).$$

In particular, any quadratic function Poisson commuting with f is a linear combination of the q_i's.

Proof. If α is an eigenvalue of J_pA then $-\alpha$ also. In fact, $\det(\alpha I - J_pA) = \det(\alpha I + (J_pA)^*)$ since $J_p^* = J_p^{-1} = -J_p$, and a matrix and its transpose have the same eigenvalues.

Let $\pm(\alpha_1, \ldots, \alpha_n)$ be the eigenvalues of $J_pA(\lambda)$, and let w_j be an eigenvector that corresponds to α_j. These numbers and vectors are C^r in λ (i.e. the eigenvectors can be so chosen), the eigenvalues are pairwise different, and $\bar{w}_j$ is an eigenvector corresponding to $-\alpha_j$.

We first notice that $w_j^*J_pw_k = w_j^*J_p\bar{w}_k = 0$ for all $j \neq k$. In fact,

$$\alpha_j(w_j^*J_pw_k) = (J_pAw_j)^*J_pw_k = w_j^*Aw_k = -w_j^*J_p(J_pAw_k) = -\alpha_k(w_j^*J_pw_k),$$

which implies that $(\alpha_j + \alpha_k)w_j^*J_pw_k = 0$. Hence $w_j^*J_pw_k = 0$, and in the same way it follows that $w_j^*J_p\bar{w}_k = 0$.

Since $w_j^*J_pw_j = 0$ and $w_1, \ldots, w_n, \bar{w}_1, \ldots, \bar{w}_n$ is a base, it follows that $w_j^*J_p\bar{w}_j \neq 0$. Moreover,

$$w_j^*J_p\bar{w}_j = (w_j^*J_p\bar{w}_j)^* = -\bar{w}_j^*J_pw_j = -\overline{w_j^*J_p\bar{w}_j},$$

so $w_j^*J_p\bar{w}_j$ is purely imaginary and $\neq 0$. By eventually replacing w_j by $\bar{w}_j$, we can assume it is of the form $\sqrt{-1}\,b^2$ with b real, and, by dividing w_j by b, we can assume that $b = 1$.

If we now just let $\sqrt{2}\,u_j = w_j + \bar{w}_j$ and $\sqrt{2}\,v_j = \sqrt{-1}\,(w_j - \bar{w}_j)$, then $u_1, \ldots, u_n, v_1, \ldots, v_n$ is a symplectic base which depends in a C^r way on λ, and which takes f to the required form.

The *singular sets* of an elliptic CSA $\mathbf{q}$ in $S^2(T_pM, \omega_p)^*$ are easy to describe.

Indeed,

$$S_1(\mathbf{q}) = E_1 \cup \cdots \cup E_n$$

where E_i is the 2-dimensional symplectic subspace $\{X : q_i \rfloor X = 0\}^{\perp}$. (Notice that these subspaces are not of dimension 2 for an arbitrary base of $\mathbf{q}$, but only for the particular base $q_i = \frac{1}{2}(x_i^2 + y_i^2)$, $i = 1, \ldots, n$.)

$S_r(\mathbf{q})$, $r \leq n$, is the *union* of all products of r different spaces E_i. By *abuse of notation,* we let $S_r(\mathbf{q})$ also denote the *set* of all such products.

DEFINITION. Let B be a $k \times n$-matrix, $k \leq n$. We say that B is *non-degenerate* if all $k \times k$-minors are $\neq 0$.

When $k = n$ this just means that B is of maximal rank, but for $k < n$ it is a much stronger condition. When $k = 1$, for example, it means that all components of B are non-zero.

The relation of this concept to the non-degeneracy of abelian subalgebras of $S^2(T_pM, \omega_p)^*$ is the following. Suppose

$$p_j = \sum_{1 \leq i \leq n} b_{ji} q_i, \qquad j \leq k$$

where $q_1, \ldots, q_n$ is the base of an elliptic CSA. If $k \geq 2$, then $p_1, \ldots, p_k$ span a non-degenerate subalgebra if, and only if, the $k \times n$-matrix (b_{ji}) is non-degenerate. (This holds for the particular base $q_1, \ldots, q_n$ of $\mathbf{q}$ but not for an arbitrary base.) If $k = 1$, the non-degeneracy of B is necessary, but not sufficient, for the non-degeneracy of the subalgebra. In this case, a necessary and sufficient condition is that $|b_{11}|, \ldots, |b_{1n}|$ are $\neq 0$ *and* pairwise distinct.

LEMMA 2. *Let A be a non-degenerate $k \times n$-matrix. Then there exists a non-singular matrix C such that CA has a non-degenerate $(k-1) \times n$-submatrix.*

Proof. It suffices to show that there exists a non-singular $k \times k$-matrix C, such that all $(k-1) \times (k-1)$-minors of $B = CA$ are $\neq 0$.

Let $A = (a_{ji})$, and let A' be a $(k-1) \times (k-1)$-submatrix,

$$A' = (a_{ji})_{1 \leq j \leq k-1}^{1 \leq i < k-1}$$

say, which is singular. Let B' be the corresponding submatrix of B. Since there only are finitely many minors in A, it suffices to show that B' can be made non-singular for some choice of C arbitrarily close to the identity. It is now easy to verify that

$$C = I + \varepsilon(c_{ji}), \quad \text{with} \quad c_{ji} = \delta_l^j \delta_k^i,$$

some l, will arrange this for any $\varepsilon \neq 0$. This proves the lemma.

If $\mathbf{p}$ is a non-degenerate abelian subalgebra of $S^2(T_pM, \omega_p)^*$, $\dim \mathbf{p} = k$, then the lemma says that $\mathbf{p}$ has a non-degenerate subalgebra of dimension $k-1$.

III. The division

Let $\mathbf{R}^{2n} = \{z = (x, y)\}$ and let $q_i = \frac{1}{2}(x_i^2 + y_i^2)$ and $E_i = \{(x, y) \in R^{2n} : x_j = y_j = 0, j \neq i\}$ for $i = 1, \ldots, n$. Let $S_r = S_r(\mathbf{q})$, $r \leq n$.

Let J be the matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, and let $\langle\, ,\, \rangle$ be the standard euclidean metric on $\mathbf{R}^{2n}$. The metric permits us to identify dq_i with a vector field on $\mathbf{R}^{2n}$.

In this section we let C^r denote the r times differentiable functions, $r \in \mathbf{N} \cup \{\infty\}$, or the analytic functions.

LEMMA 3. *Let X be a germ of a C^r vector field on $(\mathbf{R}^{2n}, 0)$, $r \geq 1$, such that $\langle X, dq_i \rangle = 0$ for all i, Then there exist unique germs of C^{r-1} functions $c_1, \ldots, c_n$ such that $X = \sum c_i J\, dq_i$.*

Proof. It is sufficient to prove this for $n = 1$. Then

$$0 = \langle X(x, y), dq_1(x, y) \rangle = X_1(x, y)x + X_2(x, y)y,$$

which implies that $X_1(x, y) = \hat{X}_1(x, y)y$ and $X_2(x, y) = \hat{X}_2(x, y)x$, with $\hat{X}_1 + \hat{X}_2 = 0$ and $\hat{X}_1$ and $\hat{X}_2$ of class C^{r-1}. Now we just let $c_1 = \hat{X}_1$.

COROLLARY. *Let f be a germ of a C^{2r} function on $(\mathbf{R}^2, 0)$ which is rotation invariant. Then there exists a unique germ of a C^r function ψ on $(\mathbf{R}, 0)$ such that $f(x, y) = \psi(\frac{1}{2}(x^2 + y^2))$.*

Proof. ψ is defined by the equation $\psi(x) = f(\sqrt{2\,|x|}, 0)$, and of class C^{2r} for $x \neq 0$, and continuous everywhere. Since f is rotation invariant we have that $\langle J\, df, dq_1 \rangle = 0$, which, by lemma 3, implies that $df = c\, dq_1$ for some germ c of a C^{2r-2} function. Moreover, c is rotation invariant, and, since c is equal the derivative of ψ (outside the origin), the result follows by induction.

This argument is good except for an analytic function. If f is analytic, then it only shows that ψ is C^∞. But then the Taylor expansion of ψ clearly converges, so ψ must also be analytic.

LEMMA 4. *Let* $V_1, \ldots, V_m \in S_k$. *Let* f_i *be germs of* C^r *functions on* $(V_i, 0)$ *and let* Φ_i *be germs of* C^1*-diffeomorphisms on* $(V_i, 0)$.

If $f_i = f_j$ *on* $V_i \cap V_j$ *for all* i, j, *then there exists a (non-unique) germ of a* C^r *function* f *on* $(\mathbf{R}^{2n}, 0)$ *such that* $f \circ \iota_{V_i} = f_i$ *for all* i.

If Φ_i *leaves invariant the subspaces* $V_i \cap V_j$, *and if* $\Phi_i = \Phi_j$ *on* $V_i \cap V_j$ *for all* i, j, *then there exists a (non-unique) germ of a* C^1*-diffeomorphism* Φ *on* $(\mathbf{R}^{2n}, 0)$ *such that* $\Phi \circ \iota_{V_i} = \Phi_i$ *for all* i.

Proof. We construct f by induction on m. If $m = 1$, then the statement is true, so suppose it is true for $m - 1$. The problem then easily reduces to the case $f_2 = \cdots = f_m = 0$. So we let $E = V_1$ and we define $f(z) = f_1(z_E)$, $z_E = \pi_E z$. If now $z \in V_i$, $i \geq 2$, then $z_E \in V_i$ since $V_i = (V_i \cap E) + (V_i \cap E^\perp)$. Hence, $f(z) = f_1(z_E) = f_i(z_E) = 0$.

The construction of Φ is completely analogous.

LEMMA 5. *Let* $f_1, \ldots, f_k$, $k \leq n$, *be germs of* C^r *functions on* $(\mathbf{R}^{2n}, 0)$ *such that all* $f_j = 0$ *on* S_{k-1}. *Let* $B = (b_{ji})$ *be a germ of a non-degenerate* $k \times n$*-matrix of class* C^s, $0 \leq s \leq r$, *on* $(\mathbf{R}^{2n}, 0)$.

Then there exist n *germs of* C^s *functions* $g_1, \ldots, g_n$ *on* $(\mathbf{R}^{2n}, 0)$, *such that*

$$f_j = \sum_{1 \leq l \leq n} b_{jl} g_l \quad \text{and} \quad g_i \in \mathbf{O}(z_i)$$

for all $1 \leq i \leq n$ *and* $1 \leq j \leq k$.

Proof. For any $E \in S_k$, $E = E_1 + \cdots + E_k$ say, we denote by D^E the set of all germs f on $(\mathbf{R}^{2n}, 0)$ such that $f \in \mathbf{O}(z_1) \cap \cdots \cap \mathbf{O}(z_k)$, i.e. f vanishes identically on all subspaces $E_j^\perp$, $j \leq k$.

It is easy to see that if f vanishes on S_{k-1}, then there is a (very non unique) decomposition

$$f = \sum_{E \in S_k} f^E, \qquad f^E \in D^E.$$

In fact, since $f = 0$ on S_{k-1} we have

$$f \circ \iota_E \circ \pi_E \in D^E \quad \text{for all } E \in S_k$$

and

$$f \circ \iota_{E \cap E'} = 0 \text{ for all } E, E' \in S_k, \qquad E \neq E'.$$

Hence

$$\Big(f - \sum_{E \in S_k} f \circ \iota_E \circ \pi_E\Big) = 0 \text{ on } S_k$$

and we can proceed by induction.

We can therefore reduce the problem to the case when each f_j belongs to some D^E, $E = E_1 + \cdots + E_k$ say, and then the proof is easy. Since B is non-degenerate, the equations

$$(f_1, \ldots, f_k) = (g_1, \ldots, g_n)B^*, \qquad g_{k+1} = \cdots = g_n = 0$$

determine $g_1, \ldots, g_n$ uniquely. By construction, $g_i \in \mathbf{O}(z_i)$ for all i.

PROPOSITION 1. *Let $X_1, \ldots, X_k$ be germs of C^r vector fields on $(\mathbf{R}^{2n}, 0)$, $r \geq 2k$, with linear part*

$$\sum_{1 \leq i \leq n} a_{ji}\, dq_i, \qquad j \leq k.$$

Assume that the $k \times n$-matrix $A = (a_{ji})$ is non-degenerate, and that $X_1, \ldots, X_k$ have rank less than m on S_m for all m.

i) *Then there exist germs of C^{r-2k+2} vector fields $Y_1, \ldots, Y_n$ on $(\mathbf{R}^{2n}, 0)$ and germs of C^{r-2k+2} functions b_{ji} on $(\mathbf{R}^{2n}, 0)$ such that $b_{ji}(0) = a_{ji}$ and*

$$Y_i = dq_i + \mathbf{O}^2(z) \quad \text{and} \quad Y_i \in \mathbf{O}(z_i), \qquad i \leq n$$

and such that

$$X_j = \sum_{1 \leq l \leq n} b_{jl} Y_l, \qquad j \leq k.$$

In particular, $X_1, \ldots, X_k$ have rank less than m only on S_m, $m \leq k-1$.

ii) *If $k = n$, and if Z is a germ of a C^{r-2n+1} vector field on $(\mathbf{R}^{2n}, 0)$ such that $Z, X_1, \ldots, X_n$ are linearly dependent, then there exist unique germs of C^{r-2n} functions $c_1, \ldots, c_n$ such that*

$$Z = \sum_{1 \leq i \leq n} c_i X_i.$$

iii) *If $k = n$, and if Z is a germ of a C^{r-2n+1} vector field on $(\mathbf{R}^{2n}, 0)$ such that $\langle Z, X_i \rangle = 0$ for all i, then there exist germs of C^{r-2n} functions $c_1, \ldots, c_n$ and a germ of a C^{r-2n+1} matrix C such that $C(0) = J$ and*

$$Z = \sum_{1 \le i \le n} c_i C X_i.$$

There is no uniqueness in this case.

Proof. We first prove that ii) and iii) follows from i). In these cases we can assume that $X_i = Y_i$ for all i. Then we can write $Y_i = Y_i' x_i + Y_i'' y_i$, and define a matrix M by

$$M^{-1} = (Y_1' \cdots Y_n' Y_1'' \cdots Y_n'').$$

Then $MY_i = dq_i$ for all i, and $M(0) = I$. Hence, $MZ, dq_1, \ldots, dq_n$ are linearly dependent everywhere, which implies that $\langle JMZ, dq_i \rangle = 0$ for all i. Now the existence follows from lemma 3, as well as the uniqueness.

In case iii) we get $\langle (M^{-1})^* Z, dq_i \rangle = 0$ for all i, and again the existence follows from lemma 3 with $C = M^* JM$.

We must now prove i). If $k = 1$, then i) follows from lemma 5 (applied to each component), so we can proceed by induction on k. By lemma 2 we can assume that the submatrix

$$A' = (a_{ji})_{1 \le j \le k-1}^{1 \le i \le n}$$

is non-degenerate. Since $X_1, \ldots, X_{k-1}$ have rank less than m on S_m we can apply induction. Therefore we can assume that

$$X_j = \sum_{1 \le i \le n} b_{ji} Y_i, \qquad j \le k - 1.$$

Let now E be a space in S_{k-1}, $E = E_1 + \cdots + E_{k-1}$ say. It then follows from ii) that

$$\pi_E X_k \circ \iota_E = \sum_{1 \le j \le k-1} c_j^E \pi_E X_j \circ \iota_E$$

for a unique set of germs $c_1^E, \ldots, c_{k-1}^E$. Since $X_1, \ldots, X_k$ are linearly dependent on E, it follows that

$$X_k \circ \iota_E = \sum_{1 \le j \le k-1} c_j^E X_j \circ \iota_E = \sum_{1 \le i \le k-1} d_i^E Y_i \circ \iota_E.$$

Hence, there exists a family of germs d_1^E, parametrized by all $E \in S_{k-1}$, $E_1 \subset E$. The uniqueness in lemma 3 implies now that $d_1^E = d_1^{E'}$ on $E \cap E'$ for any two spaces E and E'. So by lemma 4 there is a germ d_1 such that

$$d_1 = d_1^E \text{ on each space } E \in S_{k-1} \text{ such that } E_1 \subset E.$$

And the same is true for each i. Hence,

$$\hat{X}_j = X_j - \sum_{1 \le i \le n} b_{ji} Y_i = 0 \text{ on } S_{k-1} \qquad j \le k$$

where we have put $b_{ki} = d_i$.

The result now follows by applying lemma 5 to each component of the vector fields $\hat{X}_1, \dots, \hat{X}_k$. The last part in i) is obvious if we consider $MX_1, \dots, MX_k$, with M^{-1} defined as above.

The following lemma will permit us to apply the proposition to the case when the linear dependence occurs on certain submanifolds.

LEMMA 6. *Let* $V_1, \dots, V_k \in S_m$, *and let* $N_1, \dots, N_k$ *be germs of* C^r *submanifolds such that* $T_0 N_i = V_i$. *Then there exists a* C^r *diffeomorphism*

$$\Phi : (\mathbf{R}^{2n}, 0) \to (\mathbf{R}^{2n}, 0), \qquad D\Phi(0) = I$$

such that $\Phi(V_i) = N_i$ *for all* i.

Proof. We assume that $N_i \cap N_j = V_i \cap V_j$ for all i, j, and that $N_i = V_i$ for $i \ge 2$. Let $E = V_1$ and $F = E^{\perp}$. N_1 can be written as

$$z_F = \psi(z_E), \qquad \psi(0) = 0, \qquad D\psi(0) = 0$$

with ψ of class C^r. Since $N_1 \cap V_i = E \cap V_i$ for $i \ge 2$, we have that $\psi(z_E) = 0$ for $z_E \in E \cap V_i$. If we now define

$$\Phi(z) = (z_E, z_F + \psi(z_E)),$$

then $\Phi(V_1) = N_1$ and $\Phi / V_i = id$ for $i \ge 2$.

An obvious induction then gives the general result.

Parameter dependence

The preceding results have immediate analogies when the functions and vector fields depend on parameters λ. For example, if, in proposition 1, $X_1, \dots, X_k$

depend in a C^s way on some parameters λ defined near the origin in a euclidean space, then also the Y_i's and the b_{ij}'s depend on λ in a C^s way. Indeed, the Y_i's and the b_{ji}'s are not unique but the proof provides an explicit construction of such vector fields and functions, and the Y_i's and b_{ij}'s so constructed are C^s-dependent on λ. The same holds for $c_1, \ldots, c_n$ and the matrix C in ii) and iii), if Z is C^s in λ.

The results also remain true in the complex if we are considering holomorphic objects.

IV. A Morse lemma

Let $h_1, \ldots, h_k$ be germs of smooth functions at p on (M, ω) which are all critical at p and pairwise Poisson commuting, and let $\mathbf{h}$ be the $\mathbf{R}$-linear span of these germs.

Let $S_r(\mathbf{h})$ be the set of points where $dh_1, \ldots, dh_k$ have rank less than r.

Singular manifolds

LEMMA 7. *$S_r(\mathbf{h})$ is invariant under the Hamiltonian vector field X_f of any function f, Poisson commuting with* $\mathbf{h}$.

If N is a local symplectic submanifold of dimension $2r$, invariant under all X_{h_j}, then $N \subset S_r(\mathbf{h})$.

Proof. The first part follows from the fact that the pull back of X_{h_j} by the flow of X_f is X_{h_j} itself. The second part is true since the X_{h_j}'s span an isotropic subspace of the tangent space of N at any point.

PROPOSITION 2. *Let E be a symplectic subspace of (T_pM, ω_p) of dimension $2k-2$, and assume that* 1) *there exists a $h \in \mathbf{h}$ such that $d_p^2h = d_p^2h/E + d_p^2h/F$, $F = E^{\perp}$, with d_p^2h/F non-singular and $d_p^2h/E = 0$;*

2) *$d_p^2\mathbf{h}/E$ is an elliptic CSA of $S^2(E, \omega_p/E)^*$.*

Then there exists a unique smooth submanifold N, $T_pN = E$, which is invariant under all X_{h_j}.

Proof. We shall construct N as the unique submanifold N in $S_{k-1}(\mathbf{h})$ such that $T_pN = E$. The uniqueness of the construction will then imply the invariance.

Notice that condition 1) implies that each $d_p^2h_j$ splits over $E + F$ as a sum $d_p^2h_j/E + d_p^2h_j/F$. Moreover, there exists an elliptic CSA $\mathbf{q} \subset S^2(T_pM, \omega_p)^*$ such that $\mathbf{q}/E = d_p^2\mathbf{h}/E$. It is given on E by condition 2), and the extension is trivial. Also by 2), we can assume that $d_p^2h_j/E = q_j/E$, $j \leq k-1$.

Let $S_r = S_r(\mathbf{q})$ and let $E = E_1 + \cdots + E_{k-1}$, $E_i \in S_1$.

Since the problem is local we can identify (M, p) with $(T_pM, 0)$, but we don't identify ω and ω_p. $M = T_pM$ will have two symplectic structures, the linear ω_p and the non-linear ω. On (T_pM, ω_p) we introduce some symplectic base $z = (x, y)$ together with the corresponding euclidean metric $\langle\ ,\ \rangle$.

We shall construct N under the following assumption, which we shall justify by induction:

$(*)$ $S_{k-2} \cap E$ is invariant under each X_{hj}.

Observe that $(*)$ implies that $S_m \cap E$ is invariant under each X_{h_j} for all $m \leq k-2$.

Let h_k be the element whose existence is assumed in condition 1). Then $h_1, \ldots, h_k$ is a base for $\mathbf{h}$. Define

$$f^\varepsilon = \varepsilon_1 h_1 + \cdots + \varepsilon_{k-1} h_{k-1} + h_k, \qquad X^\varepsilon = X_{f^\varepsilon}.$$

Then

$$\pi_E X^\varepsilon = \sum_{1 \leq i \leq k-1} \varepsilon_i \pi_E J_p\, dq_i + \mathbf{O}^2(z) \tag{1}$$

(because each $d_p^2 h_j$ splits over $E + F$) and

$$\langle dh_j, X^\varepsilon \rangle = 0, \qquad j \leq k-1. \tag{2}$$

We now want to solve the $2n$ equations $X^\varepsilon = 0$. In order to do this we first consider

$$\pi_F X^\varepsilon = 0.$$

Since $d_p^2 h_k / F$ is non-singular and $X^\varepsilon(0) = 0$, these $2(n-k+1)$ equations in $2n + k - 1$ many unknowns can be solved by the implicit function theorem, and the solution is a family of submanifolds

$$N^\varepsilon : z_F = \phi^\varepsilon(z_E), \qquad \phi^\varepsilon(0) = 0$$

for ε sufficiently small.

Since each $d_p^2 h_j$ splits over $E + F$, it follows that $\pi_F X_{h_j}(z_E, 0) = \mathbf{O}^2(z_E)$. This implies that $D\phi^\varepsilon(0) = 0$ and, hence,

$$T_p N^\varepsilon = E \tag{3}$$

for all ε sufficiently small. By assumption ($*$) we have that $\pi_F X^\varepsilon = 0$ on $S_{k-2} \cap E$. So $\phi^\varepsilon = 0$ on $S_{k-2} \cap E$ and, hence,

$$N^\varepsilon \cap E = S_{k-2} \cap E \tag{4}$$

for all ε sufficiently small.

By ($*$) and lemma 7 it follows that $X_{h_1}, \ldots, X_{h_{k-1}}$ have rank less than m on $S_m \cap E$ for all $m \leq k-2$. Hence, this is true also for $dh_1, \ldots, dh_{k-1}$ and for $\pi_E\, dh_1, \ldots, \pi_E\, dh_{k-1}$. And, by (4), this is true also for $\pi_E\, dh_1 \circ \phi^\varepsilon, \ldots, \pi_E\, dh_{k-1} \circ \phi^\varepsilon$, where ϕ^ε here denotes the embedding $z_E \mapsto (z_E, \phi^\varepsilon(z_E))$.

By (3), the linearized vector fields are easy to compute:

$$\pi_E\, dh_j \circ \phi^\varepsilon = \pi_E\, dq_j + \mathbf{O}^2(z_E), \qquad j \leq k-1.$$

Since $\langle X, Y\rangle = \langle \pi_E X, \pi_E Y\rangle + \langle \pi_F X, \pi_F Y\rangle$ for all vectors X, Y, we get by (2)

$$\langle \pi_E\, dh_j \circ \phi^\varepsilon, \pi_E X^\varepsilon \circ \phi^\varepsilon\rangle = 0, \qquad j \leq k-1.$$

This permits us to apply proposition 1iii). Hence, on E there exist germs of functions $c_1^\varepsilon, \ldots, c_{k-1}^\varepsilon$, and a germ of a non-singular matrix C^ε such that

$$\pi_E X^\varepsilon \circ \phi^\varepsilon(z_E) = \sum_{1 \leq i \leq k-1} c_i^\varepsilon(z_E) C^\varepsilon(z_E) \pi_E\, dh_i \circ \phi^\varepsilon(z_E)$$

(with smooth dependence on ε).

Linearizing this relation at $z_E = 0$ is easy by (1) and (3). It gives

$$(c_1^\varepsilon(0), \ldots, c_{k-1}^\varepsilon(0)) = (\varepsilon_1, \ldots, \varepsilon_{k-1}).$$

Hence, by the implicit function theorem, there exist unique germs of functions $e_1, \ldots, e_{k-1}$ on $(E, 0)$ such that

$$c_i^{e(z_E)}(z_E) = 0, \qquad e_i(0) = 0$$

for all i. $e_1, \ldots, e_{k-1}$ are uniquely determined by $\pi_E X^\varepsilon \circ \phi^\varepsilon$ even though $c_1^\varepsilon, \ldots, c_{k-1}^\varepsilon$ are not.

Then the manifold

$$N : z_F = \phi^\varepsilon(z_E), \qquad \varepsilon = e(z_E),$$

is contained in $S_{k-1}(\mathbf{h})$ and intersects E along $S_{k-2} \cap E$. Clearly N is the unique manifold in $S_{k-1}(\mathbf{h})$ such that $T_pN = E$. Hence, N is invariant under all X_{h_i}.

It is obvious that $d_p^2\mathbf{h}$ satisfies conditions 1) and 2) for all singular spaces in $S_r \cap E$, $r \leq k-2$. So we can use the above procedure to verify $(*)$ by an easy induction, using lemma 6 to flatten out the submanifolds. The uniqueness follows from this construction.

COROLLARY. *Suppose that $d_p^2(\mathbf{h})$ is non-degenerate.*
Then there exists a smooth diffeomorphism $\Phi: (T_pM, 0) \to (M, p)$, $D\Phi(0) = I$, such that

$$\Phi^{-1}(S_{k-1}(\mathbf{h})) = S_{k-1}(\text{Centr}\,(d_p^2\mathbf{h})).$$

Proof. Let $S_r = S_r(\text{Centr}\,(d_p^2\mathbf{h}))$ and let $E \in S_{k-1}$. Then clearly condition 2) of proposition 2 is fulfilled, and, there exists an $h \in \mathbf{h}$ such that d_p^2h/E vanishes completely. If now $d_p^2h/E^\perp$ were singular, then there would exist some singular space $E_i \in S_1$, contained in $E^\perp$, such that $d_p^2h/E_i = 0$. But then $\mathbf{h}$ would be of rank less than $k-1$ on $E + E_i$, contradicting the assumption of non-degeneracy. Hence also 1) is fulfilled.

Proposition 2 together with lemma 6 give now Φ and the inclusion $\supset$. That equality holds is a consequence of proposition 1.

It follows from theorem C that we can choose Φ to be symplectic when $k = n$. Also, one singular manifold can always be straighten out with a symplectic diffeomorphism. In general, however, knowledge of the intersections of the singular manifolds is required in order to conclude the existence of a symplectic Φ.

Proof of theorem A

Let $S_r = S_r(\text{Centr}\,(d_p^2(\mathbf{h}))$.

Since the problem is local we can identify (M, p) and $(T_pM, 0)$, but we don't identify ω and ω_p. On (T_pM, ω_p) we introduce some symplectic base $z = (x, y)$ together with the corresponding euclidean metric $\langle\ ,\ \rangle$. By the corollary of proposition 2, we can assume that $S_{k-1}(\mathbf{h}) = S_{k-1}$. By lemma 2 we can assume that $d_p^2h_1, \ldots, d_p^2h_m$ generate a non-degenerate subalgebra for all $m \leq k$.

Let's consider the following statement for $0 \leq m \leq k-1$:

(P_m) There exist a smooth diffeomorphism $\Phi = \Phi^m$ and functions $\psi_j = \psi_j^m$

such that $\Phi(S_{k-1}) = S_{k-1}$ and

$$h_j \circ \Phi = \psi_j(q_1, \ldots, q_n) \text{ on } S_m, \qquad j \leq k.$$

(P_{k-1}) *implies the theorem.*

We can assume without restriction that Φ^{k-1} is the identity. We want to construct a non-autonomous vector field Z_t such that

$$\langle d(\phi_j + t(h_j - \phi_j)), Z_t \rangle = -(h_j - \phi_j), \qquad j \leq k$$

for all $0 \leq t \leq 1$, where we have put $\phi_j = \psi_j(q_1, \ldots, q_n)$.

Let $\alpha_j = d(\phi_j + t(h_j - \phi_j))$. $dh_1, \ldots, dh_k$ and $d\phi_1, \ldots, d\phi_k$ both have rank less than m on S_m, and, by assumption (P_{k-1}), this is also true for $\alpha_1, \ldots, \alpha_k$. By the assumption of non-degeneracy, the conditions of proposition 1 are fulfilled. Hence, we can write

$$\alpha_j = \sum_{1 \leq i \leq n} b_{ji} Y_i, \qquad j \leq k$$

with

$$Y_i \in \mathbf{O}(z_i) \quad \text{and} \quad Y_i = dq_i + \mathbf{O}^2(z), \qquad i \leq n$$

and, by lemma 5,

$$(h_j - \phi_j) = \sum_{1 \leq i \leq n} b_{ji} g_i, \qquad j \leq k$$

with

$$g_i \in \mathbf{O}(z_i), \qquad i \leq n.$$

α_j, b_{ji}, Y_i and g_i all depend smoothly on t.

It is now sufficient to solve

$$\langle Y_i, Z_t \rangle = g_i, \qquad i \leq n$$

for all $0 \leq t \leq 1$, which we can do in the following way. Write

$$Y_i = Y_i^1 x_i + Y_i^2 y_i \quad \text{and} \quad g_i = g_i^1 x_i + g_i^2 y_i,$$

and let M be the matrix $(Y_1^1, \ldots, Y_n^1, Y_1^2, \ldots, Y_n^2)$ and g the column vector $(g_1^1, \ldots, g_n^1, g_1^2, \ldots, g_n^2)$. Then M is invertible for $0 \le t \le 1$, and $Z_t = M^{-1}g$ solves the equations.

Let now Φ be the "time-1-map" of Z_t. Then

$$h_j \circ \Phi = \psi(q_1, \ldots, q_n) \tag{5}$$

for all $j \le k$, and this proves that theorem A follows from (P_{k-1}). Moreover, Φ preserves S_{k-1} and each singular space in S_{k-1} (since it preserves the rank of $dh_1, \ldots, dh_k$ and $D\Phi(0) = I$). Hence

$$q_i \circ \Phi \circ \iota_E = q_i \circ \iota_E \text{ for all } E \in S_{k-1}, \qquad i \le n. \tag{6}$$

Proof of (P_m).

Since (P_0) is obvious, we can assume (P_{m-1}) and apply induction. We can also assume without restriction that Φ^{m-1} is the identity.

So we let E be a singular space in S_m, $E = E_1 + \cdots + E_m$ say, $E_i \in S_1$. Then there exist a local diffeomorphism Φ^E, $\Phi^E(S_{k-1}) = S_{k-1}$, and functions $\psi_1^E, \ldots, \psi_k^E$ such that

$$h_j \circ \Phi^E = \psi_j^E(q_1, \ldots, q_n) \text{ on } E, \qquad j \le k \tag{7}$$

and

$$q_i \circ \Phi^E \circ \iota_{E'} = q_i \circ \iota_{E'} \text{ for any } E' \in S_m, \qquad E' \ne E, \qquad i \le n. \tag{8}$$

Φ^E and $\psi_1^E(q_1, \ldots, q_n), \ldots, \psi_m^E(q_1, \ldots, q_n)$ are constructed on E by applying theorem A to $h_1 \circ \iota_E, \ldots, h_m \circ \iota_E$, and then extended in a trivial way to $E + E^\perp$. Φ^E preserves $S_{m-1} \cap E$ and, hence, S_{k-1}. For $m < j \le k$, ψ_j^E follows from the corollary of lemma 3, since all $h_j \circ \Phi^E \circ \iota_E$ are constant on the fibers of $h_1 \circ \Phi^E \circ \iota_E, \ldots, h_m \circ \Phi^E \circ \iota_E$. Finally, (8) holds by assumption (P_{m-1}), just as (6) above holds by assumption (P_{k-1}).

We can do this construction for each singular space in S_m, and we can let Φ be the composition of all the Φ^E. (8) now implies that (7) holds for all $E \in S_m$, with Φ^E replaced by Φ. Moreover, by (P_{m-1}), we know that there exist ψ_j's such that

$$h_j = \psi_j(q_1, \ldots, q_n) \text{ on } S_{m-1}.$$

(8) also implies that $h_j \circ \Phi = h_j$ on S_{m-1}, and therefore

$$\psi_j^E = \psi_j^{E'} \text{ on } E \cap E' \text{ for all } E, E' \in S_m, \qquad j \le k.$$

Now we can apply lemma 4, to get functions ψ_j' such that

$$\psi_j' \circ \iota_E = \psi_j^E \text{ for all } E \in S_m, \qquad j \leq k.$$

This proves (P_m), and completes the proof of theorem A.

Parameter dependence

Suppose that $h_1, \ldots, h_k$ depend smoothly on parameters λ defined near 0 in some euclidean space, and that they verify the assumptions of theorem A for each λ. Suppose also that $d_p^2 h_1, \ldots, d_p^2 h_k$ does not depend on λ. It is then clear, from the explicit construction given in the proof, that also Φ and the ψ_i's depend on λ smoothly.

Remarks

The fact that the germs are Poisson commuting is, of course, essential. If, for example, $h_1 = x_1^2 + y_1^2 + x_2^2 y_2$ and $h_2 = x_2^2 + y_2^2 + (x_1^2 + y_1^2)^2$, then $S_1(\mathbf{h})$ is the union of one 2-dimensional subspace and three 1-dimensional subspaces near the origin. In particular, it follows from theorem A that there exists no symplectic structure on $\mathbf{R}^4$ for which h_1 and h_2 commute in the sense of Poisson.

The non-degeneracy condition cannot be relaxed without caution. For example, if $h_1 = (x_1^2 + y_1^2) + (x_2^2 + y_2^2)$ and $h_2 = (x_1 y_2 - x_2 y_1)^2$, then $S_1(\mathbf{h})$ is the union of two 2-dimensional spaces and the set $h_2^{-1}(0)$. These germs commute but the non-degeneracy condition is not fulfilled.

The theorem gives a fairly nice description of the common fibers of $h_1, \ldots, h_k$. On the singular manifolds, the fibers are tori (in general of half the dimension of the manifold), but outside these manifolds the fibers are sub-manifolds of dimension $2n - k$, each of which is fibrated into n-dimensional tori. And this fibration is determined (in a non unique way) by the ψ_i's.

In case $k = 1, n$ there is only on such fibration (up to diffeomorphism), but not for $1 < k < n$. In particular, for $k = 2$ we have the following normal forms for ψ_1, ψ_2:

$$d_p^2 h_1 + \sum_{7 \leq i \leq n} P_i(q_i), \qquad d_p^2 h_2,$$

where P_i is a polynomial of degree $\left[\dfrac{i-1}{2}\right] - 1$. So in this case the fibrations are finitely determined, and for $n \leq 6$ they are completely determined by the quadratic part $d_p^2(\mathbf{h})$.

The proof also works in the analytic and the C^r cases. In the C^r case, Φ will undergo a loss of differentiability which depends on k. A crude computation shows that one looses not more than 7 derivatives at each induction step. Hence Φ is at least of class C^{r-7k}.

V. Centermanifolds

We shall construct N as the set of all 2π-periodic solutions of a family of vector fields X^ε, all of which commute with all the vector fields X_{h_j}, $j \leq k$. Since the flow of X_{h_j} takes one 2π-periodic solution of X^ε onto another, N must be invariant under all X_{h_j}.

We first notice that condition 1) implies that E is invariant under $j_1 X_{h_j}$ – the linearized vector field at p – for each j. There exists an elliptic CSA $\mathbf{q}$ of $S^2(T_pM, \omega_p)^*$ such that $d_p^2\mathbf{h}/E = \mathbf{q}/E$. By condition 2), we can assume that $d_p^2 h_j/E = q_j/E$, $j \leq k$.

Let $S_r = S_r(\mathbf{q})$ and let $E = E_1 + \cdots + E_k$, $E_i \in S_1$.

Since the problem is local we can identify (M, p) with $(T_pM, 0)$, but we don't identify ω and ω_p. On (T_pM, ω_p) we introduce some symplectic base $z = (x, y)$ together with the corresponding euclidean metric $\langle\ ,\ \rangle$.

We shall first assume that

$(*)$ $\quad S_{k-1} \cap E$ is invariant under each X_{h_j}

– an assumption which we will justify by induciton. Notice that $(*)$ implies that $S_m \cap E$ is invariant under X_{h_j}, for each $m \leq k-1$.

Consider now the restriction $\mathbf{h} \circ \iota_{E_1^\perp \cap E}$, and notice that $E_1^\perp \cap E \in S_{k-1}$. Then we get k Poisson commuting functions on the symplectic submanifold $E_1^\perp \cap E$ of dimension $2(k-1)$. Hence, there exists a function f_1 such that $f_1(h_1, \ldots, h_k)$ vanishes on $E_1^\perp \cap E$, and $d_p^2(f_1(h_1, \ldots, h_k)) = d_p^2 h_1$. (This is easy to see by introducing, by theorem A, coordinates on $E_1^\perp \cap E$ such that each $h_j \circ \iota_{E_1^\perp \cap E}$ is of the simple form $\psi_j(q_2, \ldots, q_k)$, $j \leq k$.) Doing this construction for each $E_j^\perp$, we can assume that h_j vanishes on $E_j^\perp \cap E$ for each j. Now, since $E_j^\perp \cap E \in S_{k-1}$ is invariant under X_{h_j} by $(*)$, it follows that

$$dh_j \circ \iota_{E_j^\perp \cap E} = 0, \qquad j \leq k.$$

Let now h be the function whose existence is assumed in condition 1), and consider the vector field

$$X^{\varepsilon} = X_h + \varepsilon_1 X_{h_1} + \cdots + \varepsilon_k X_{h_k}$$

and its time-2π-map φ^{ε}. Then

$$\pi_E(\varphi^{\varepsilon} - id) \circ \iota_E(z_E) = \pi_E \sum_{1 \le i \le k} (e^{2\pi\varepsilon_i J_p} - I)\, dq_i + \mathbf{O}^2(z_E) \tag{9}$$

since E is invariant under $j_1 X_{h_i}$.

Moreover, if we Taylor expand h_j at z and use that $h_j \circ \varphi^{\varepsilon} = h_j$, we get

$$\langle v_j^{\varepsilon}(z), \varphi^{\varepsilon}(z) - z \rangle_p = 0, \tag{10}$$

where

$$v_j^{\varepsilon}(z) = \int_0^1 dh_j(z + s(\varphi^{\varepsilon}(z) - z))\, ds, \qquad j \le k.$$

We now want to solve the $2n$ equations $\varphi^{\varepsilon}(z) = z$, and in order to do this we will argue as in the proof of proposition 2. So we first consider

$$\pi_F(\varphi^{\varepsilon} - id) = 0.$$

Since $d_p^2 h/F$ has no 2π-periodic solutions and $\varphi^{\varepsilon}(0) = 0$, these $2(n-k)$ equations, in $2n + k$ unknowns, can be solved by the implicit function theorem, and the solution is a family of submanifolds

$$N^{\varepsilon}: z_F = \phi^{\varepsilon}(z_E), \qquad \phi^{\varepsilon}(0) = 0$$

for ε sufficiently small.

Since E is invariant under $j_1 X_{h_i}$, it follows that $\pi_F(\varphi^{\varepsilon} - id)(z_E, 0) \in \mathbf{O}^2(z_E)$. This implies that $D\varphi^{\varepsilon}(0) = 0$, and, hence,

$$T_p N^{\varepsilon} = E \tag{11}$$

for ε sufficiently small. By assumption $(*)$, it follows that $\pi_F(\varphi^{\varepsilon} - id) = 0$ on $S_{k-1} \cap E$, so

$$N^{\varepsilon} \cap E = S_{k-1} \cap E \tag{12}$$

for ε sufficiently small.

By (10) we get

$$\langle \pi_E v_j^\varepsilon \circ \phi^\varepsilon, \pi_E(\varphi^\varepsilon - id) \circ \phi^\varepsilon \rangle = 0, \qquad j \le k$$

where ϕ^ε here denotes the embedding $z_E \mapsto (z_E, \phi^\varepsilon(z_E))$.

We know that dh_j vanishes on $E_j^\perp \cap E$. By $(*)$ it follows that v_j^ε and, hence, $\pi_E v_j^\varepsilon$ vanish on $E_j^\perp \cap E$. Finally, by (12) it follows that $\pi_E v_j^\varepsilon \circ \phi^\varepsilon$ vanishes on $E_j^\perp \cap E$. Moreover, by (11), it is easy to compute the linearized vector field:

$$\pi_E v_j^\varepsilon \circ \phi^\varepsilon = \tfrac{1}{2}\pi_E(e^{2\pi\varepsilon_j J_p} + I)\, dq_j + \mathbf{O}^2(z_E), \qquad j \ge k.$$

The right hand side can be written, for all j, as $D_\varepsilon \pi_E\, dq_j$, where D_ε is a $2k \times 2k$-matrix with $D_0 = I$. This permits us to apply proposition 1iii). Hence, there exist germs of functions $c_1^\varepsilon, \ldots, c_k^\varepsilon$ and a germ of a non-singular matrix C^ε on E such that

$$D_\varepsilon^* \pi_E(\varphi^\varepsilon - id) \circ \phi^\varepsilon(z_E) = \sum_{1 \le i \le k} c_i^\varepsilon(z_E) C^\varepsilon(z_E) D_\varepsilon^{-1} \pi_E v_i^\varepsilon \circ \phi^\varepsilon(z_E)$$

(with smooth dependence on ε).

Linearizing this equality at $z_E = 0$ is easy by (9) and (11). It gives

$$(c_1^\varepsilon(0), \ldots, c_k^\varepsilon(0)) = 2\pi(\varepsilon_1, \ldots, \varepsilon_k) + \mathbf{O}^2(\varepsilon).$$

Hence, by the implicit function theorem there exist unique germs of functions $e_1, \ldots, e_k$ on $(E, 0)$ such that

$$c_j^{e(z_E)}(z_E) = 0, \qquad e(0) = 0$$

for all j. Moreover, $e_1, \ldots, e_k$ are uniquely determined by $\pi_E(\varphi^\varepsilon - id) \circ \phi^\varepsilon$ even though $c_1^\varepsilon, \ldots, c_k^\varepsilon$ are not.

Hence, if

$$N : z_F = \phi^\varepsilon(z_E), \qquad \varepsilon = e(z_E)$$

then $\varphi^{e(z_E)}(z) = z$ for all $z = (z_E, z_F) \in N$. If N' were another manifold with this property, then it would follow from the uniqueness of the construction that $N = N'$. So N is invariant under all vector fields X_{h_j}.

This proves the theorem modulo $(*)$. If we now only observe that $d_p^2\mathbf{h}$ satisfies conditions 1) and 2) for all singular spaces in $S_r \cap E$, $r \le k-1$, then it is clear that we can fulfill (*) by an obvious induction. This proves the existence of N, and the uniqueness follows from this construction.

VI. The symplectic normal form

Darboux's lemma with a Lagrangian fibration

Consider $\mathbf{R}^{2n} = \{(x, y)\}$ with the symplectic structure $\omega_p = \sum dx_i \wedge dy_i$. Let $q_i = \frac{1}{2}(x_i^2 + y_i^2)$ for all $i \le n$.

LEMMA 8. *Let $g_1, \ldots, g_n$ be germs of smooth functions at $(\mathbf{R}^{2n}, 0)$ such that*

$$dg_i \rfloor J_p\, dq_j = dg_j \rfloor J_p\, dq_i, \qquad i, j \le n.$$

Then there is a germ of a smooth function f at $(\mathbf{R}^{2n}, 0)$, and there are unique germs of smooth functions $\psi_1, \ldots, \psi_n$ at $(\mathbf{R}^n, 0)$ such that

$$df \rfloor J_p\, dq_i = g_i - \psi_i(q_1, \ldots, q_n), \qquad i \le n.$$

Proof. We give an explicit formula for the solution (due to J. Moser). Let φ_i^t be the flow map of $J_p\, dq_i$ (the Hamiltonian vector field of q_i with respect to ω_p), and define

$$M_i g(x, y) = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi_i^t(x, y))\, dt,$$

$$L_i g(x, y) = \frac{1}{2\pi} \int_0^{2\pi} t(g(\varphi_i^t(x, y)) - M_i g(x, y))\, dt$$

for any function g.

Then one verifies easily that $M_i g_i = (M_1 \cdots M_n) g_i$, and then $M_i g_i = \psi_i(q_1, \ldots, q_n)$ by to the corollary of lemma 3. Now we just take $f = \sum_{1 \le i \le n} M_1 \cdots M_{i-1} L_i g_i$.

PROPOSITION 3. *Let ω be a smooth symplectic form on $(\mathbf{R}^{2n}, 0)$ with $\omega(0) = \omega_p$, such that*

$$\{q_i, q_j\} = 0, \qquad i, j \le n$$

i.e. the fibration $\bigcap \{q_i = \text{const.}\}$ is Lagrangian for ω.

Then there exists a diffeomorphism $\Phi:(\mathbf{R}^{2n},0)\to(\mathbf{R}^{2n},0)$ *such that* $\Phi^*\omega=\omega(0)$, *and* Φ *respects the fibration, i.e.* Φ *maps fibers into fibers.*

Proof. Let α be a primitive to ω. We can assume without restriction that the linear part $j_1\alpha$ of α is $\frac{1}{2}\sum x_i\,dy_i - y_i\,dx_i$.

The vector fields $J_p\,dq_i$ and $J_p\,dq_j$ commute and are tangential to the fibration. Therefore, the assumption on the fibration, which can be formulated

$$\omega(J_p\,dq_i, J_p\,dq_j)=0, \qquad i,j\le n,$$

implies that

$$d(\alpha\rfloor J_p\,dq_i)\rfloor J_p\,dq_j = d(\alpha\rfloor J_p\,dq_j)\rfloor J_p\,dq_i.$$

Notice also that these relations still hold with α replaced by $j_1\alpha$ since the fibration is Lagrangian also for $\omega(0)$.

Now we need a function f such that

$$df\rfloor J_p\,dq_i=(\alpha-j_1\alpha)\rfloor J_p\,dq_i$$

for all i. It follows from lemma 8 that such a function exists if and only if the mean value of $(\alpha-j_1\alpha)\rfloor J_p\,dq_i$ vanishes for all i, as we now assume.

Now the proof is straight forward. Let's consider the equation

$$(\omega(0)+s(\omega-\omega(0)))\rfloor Z_s = -(\alpha-j_1\alpha-df).$$

It defines a non-autonomous vector field Z_s for $0\le s\le 1$, whose "time-1-map" pulls ω back to $\omega(0)$. Moreover, since

$$(\omega(0)+s(\omega-\omega(0)))(Z_s, J_p\,dq_i) = -(\alpha-j_1\alpha-df)\rfloor J_p\,dq_i=0$$

for all i, and since the fibration is Lagrangian both for ω and $\omega(0)$ (and therefore also for their interpolation), it follows that Z_s is tangential to the fibers. Hence, the "time-1-map" leaves the fibers invariant.

In order to complete the proof we must show that the assumption on the meanvalue can be fulfilled. So let φ^t, $t\in\mathbf{T}^n$, be the group action generated by integrating the commuting vector fields $J_p\,dq_1,\ldots,J_p\,dq_n$, and let M denote the operation of taking the mean value under this group action. Notice that α is so chosen that $M\alpha$ and α has the same linear part, and notice also that M commutes with the exterior differential d.

Suppose now that $M\omega = \omega(0)$. Then we have

$$dM\,(\alpha - j_1\alpha) = d(M\alpha - j_1\alpha) = 0,$$

so there is a function f such that $M(\alpha - j_1\alpha) = df$. Since $df = M\,df = d\,Mf$, we can assume that $f = Mf$, hence $df \rfloor J_p\,dq_i = 0$ for all i. Now

$$M((\alpha - j_1\alpha) \rfloor J_p\,dq_i) = M(\alpha - j_1\alpha) \rfloor J_p\,dq_i = df \rfloor J_p\,dq_i = 0, \qquad i \leq n$$

so the assumption is fulfilled.

It now suffices to find a diffeomorphism Φ respecting the fibration and such that $M\Phi^*\omega = \omega(0)$, but this is easy. We just let Z_s be the unique solution of

$$(\omega(0) - s(M\omega - \omega(0))) \rfloor Z_s = -M(\alpha - j_1\alpha).$$

The "time-1-map" Φ of Z_s pulls $M\omega$ back to $\omega(0)$, and, since it commutes with φ^t (because $(\varphi^t)^*Z_s = Z_s$), we have that $M\Phi^*\omega = \Phi^*M\omega$. This completes the proof.

Parameter dependence

It is clear from the construction that if ω depends smoothly on some parameters λ, defined near the origin in an euclidean space, and if $\omega(0)$ is independent of λ, then also Φ will depend smoothly on the parameters near $\lambda = 0$.

Proof of theorem C

Theorem C now follows immediately from theorem A and proposition 3. In fact, if $h_1, \ldots, h_n$ are pairwise Poisson commuting for the symplectic form ω, then the fibration $\bigcap \{h_i = const.\}$ is Lagrangian for ω. If, moreover, $h_i = \psi_i(q_1, \ldots, q_n)$, then this fibration is precisely $\bigcap \{q_i = \text{const.}\}$. This proves the theorem.

Parameter dependence

We shortly discus a parameter dependent version of theorem C. So let $h_1, \ldots, h_n$ depend on the parameters λ, defined near the origin in an euclidean space. Assume that, for all λ, they are critical at p and pairwise Poisson commuting, and that, for $\lambda = 0$, $d_p^2\mathbf{h}$ is non-degenerate. Then theorem C remains true for all λ near the origin, and Φ and all ψ_i's depend smoothly on these parameters.

Indeed, if each $d_p^2 h_i$ is independent of λ, then this follows immediately from the remarks on parameter dependence in theorem A and in proposition 3. And the general case can always be reduced to this particular case, as follows from lemma 1.

VII. Various generalizations

Other CSA

As we mentioned in the introduction, all CSA's are conjugate in the complex. In the real, however, the elliptic ones constitute only one conjugation class out of finitely many. A general CSA in $S^2(\mathbf{R}^{2n}, \sum dx_i \wedge dy_i)^*$ has a base consisting of *elliptic* functions $q_i = \frac{1}{2}(x_i^2 + y_i^2)$, of *hyperbolic* functions $q_i = x_i y_i$, and pairs of functions of the type $q_i = x_i y_i + x_{i+1} y_{i+1}$, $q_{i+1} = x_i y_{i+1} - x_{i+1} y_i$. (See for example [17].)

It is easy to generalize our results to include also hyperbolic functions. The reason for this is that the singularities $S_k(\mathbf{q})$ remain the same as for elliptic functions. In the analytic case, everything goes through in the same way, and theorem A and C are still true. In C^∞ everything goes through too, with two little exception. Lemma 8 remains true, without uniqueness, but the proof is different. The corollary of lemma 1, however, is not true for flat functions since the fibers $xy = const.$ are not connected. In theorem A and C the conclusion therefore becomes slightly weaker, namely that $h_i \circ \Phi$ is invariant under the linear action of Lie algebra generated by $q_1, \ldots, q_n$.

Of course, the purely hyperbolic case in C^∞ is not very interesting since one knows, by a theorem of Sternberg, that a single hyperbolic Hamiltonian (in general) is integrable in C^∞ [18, 19].

In the case when the CSA contains pairs of functions q_i, q_{i+1}, as described above, the situation is more involved since the structure of the singular sets is different. For example, dq_i, dq_{i+1} has rank 0 when $x_i = y_i = x_{i+1} = y_{i+1} = 0$, but rank 2 everywhere else, and the rank-1-spaces only exist in the complex. Theorem A and C, however, still remains true. We shall just in few words indicate how this can be proven.

In the analytic case we can complexify, and reduce the problem to the elliptic holomorphic version plus a reality condition which can be verified. This procedure also makes sense in a formal way if we are considering formal power series. In the C^∞ case we don't have this possibility, and the division problem then becomes much more delicate. However, it can be solved if we know that the functions are flat at the singularities (modulo functions of normal form $\psi(q_1, \dots, q_n)$).

In general, the functions are not flat (modulo normal forms) at the singularities, but this can be achieved by considering Taylor expansions of the functions, not in all variables $z = z_E + z_F$, but only certain directions, in z_F say. If now the functions are flat at the singularities in E, then we can combine the formal and the flat procedure and solve the problem in this "semi-formal" way, and obtain that the functions are flat (modulo normal forms) at E. By a finite induction, one can then achieve that all functions are flat (modulo normal forms) at the singularities. Such a procedure is technically quite involved, but it has been carried out in some detail in [13].

There should also be a generalization of theorem B in the case when the CSA contain pairs q_i, q_{i+1}. In fact, $\alpha q_i + \beta q_{i+1}$ has complex eigenvalues, and for appropriate values of α and β there exists a 4-dimensional "centermanifold" as is described in [20].

An example

The systems which are neither elliptic nor hyperbolic has not been much considered in the literature. Birkhoff, for example, seems to ignore their existence in [8]. We shall therefore describe the example of the Lagrangian spinning top where they appear. (See [12, 16].)

This top has principal moments of inertia $I_1 = I_2 \neq I_3$, is rotational invariant around the third principal axis of inertia, and lives in a gravitational field which is rotational invariant around the vertical. It can be described by a Hamiltonian system on $T^*SO(3)$, and the Hamiltonian H and the two infinitesimal generators Q_3^S and Q_3^B, of the rotational invariance, are all in involution.

The vertical positions is a circle $\mathbf{T}$ in configuration space. A neighbourhood of such a position can be parameterized by symplectic coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ such that $Q_3^B(x, y) = y_3$ and

$$H(x, y) = \frac{1}{2I_1}(y_1^2 + y_2^2)$$
$$+ \left(\frac{1}{2I_1} y_3^2 - \frac{m}{2}\right)x_1^2 - \frac{m}{2}x_2^2 + \frac{1}{I_1} y_3 x_1 y_2 + \frac{1}{2I_3} y_3^2 + \mathbf{O}^3(x_1, x_2, y_1, y_2)$$
$$Q_3^S(x, y) = x_1 y_2 - x_2 y_1 + \tfrac{1}{2} y_3(x_1^2 - x_2^2) + y_3 + \mathbf{O}^3(x_1, x_2, y_1, y_2).$$

(m is the mass of the top.) That H and Q_3^S commute with Q_3^B is reflected in the fact that they are independent of x_3.

Fixing the value of y_3, H and Q_3^S become functions in (x_1, x_2, y_1, y_2)-space, and the linearized Hamiltonian vector fields at the origin have eigenvalues

$$\lambda^2 = \frac{1}{4I_1}(-a \pm \sqrt{a^2 - m^2}), \qquad a = \frac{1}{2I_1} y_3^2 - m$$

for H, and

$$\lambda^4 + \tfrac{1}{2}\lambda^2 + \tfrac{1}{16} = 0$$

for Q_3^S.

Hence, if $y_3^2 > 4mI_1$, then we can apply theorem C and introduce symplectic coordinates, in a neighbourhood of $x_1 = x_2 = y_1 = y_2 = 0$, such that the algebra H, Q_3^S is generated by

$$x_1^2 + y_1^2, \qquad x_2^2 + y_2^2.$$

But if $y_3^2 < 4mI_1$, then the quadratic algebra is generated by

$$x_1 y_1 + x_2 y_2, \qquad x_1 y_2 - x_2 y_1$$

so we are in the non-elliptic-hyperbolic case.

Other Lie algebras

Another class of Lie algebras where the question of normal forms can reasonably be asked is the semisimple ones. There is a result of Hermann–Guillemin–Sternberg–Kushnirenko that says that a semisimple Lie algebra of analytic vector fields can be simultaneously linearized near a stationary point [21, 22, 23]. If the vector fields are Hamiltonian, then it is not hard to show that this can be done by a symplectic diffeomorphism, so for analytic systems the problem is solved.

It is otherwise with C^∞ systems. In [22] there is a counter example for arbitrary vector fields, but if linearization is possible for Hamiltonian vector fields is an unsolved problem.

Of course, if the Lie algebra is compact and semisimple, then the problem reduces to the linearization a compact group action, near a fixed point. A problem which can always be solved [19].

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