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Autor: Palmer, Bennett
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Stability of minimal hypersurfaces

BENNETT PALMER

This paper is concerned with the stability of minimal hypersurfaces of $n+1$ dimensional Euclidean space $\mathbb{E}^{n+1}$. For $n=2$ it was shown independently by Do Carmo-Peng and Fischer Colbrie-Schoen that a smooth, orientable, complete, stable minimal surface must be a plane. In higher dimensions, $n \geq 7$, the celebrated result of Bombieri, De Giorgi and Giusti on the existence of minimal graphs indicates that the situation is more complicated since these hypersurfaces are minimizing and hence stable.

With the existence of minimal, stable graphs in mind it is natural to seek a topological restriction which implies instability. We show:

THEOREM I. *Let $M^n \subset \mathbb{E}^{n+1}$ be a complete, orientable minimal hypersurface. Suppose there exists a codimension one cycle C in M which does not separate M. Then M is unstable*

Proof of Theorem I. Suppose C exists. By the results of [D], Cor 1, M supports a non-trivial L^2-harmonic vector field ξ. Let ∇ denote the naturally defined connection on M and let ∇_i denote ∇_{e_i} where e_i is a locally defined orthonormal frame for TM. Compute

$$\nabla\|\xi\|^2 = 2\Big(\sum_i (\langle \nabla_i \nabla_i \xi, \xi\rangle + \langle \nabla_i \xi, \nabla_i \xi\rangle)\Big)$$

$$= 2(\langle \mathrm{tr.}\, \nabla\nabla\xi, \xi\rangle + \|\nabla\xi\|^2)$$

Since ξ is harmonic, one has

$$\langle \mathrm{tr.}\, \nabla\nabla\xi, \xi\rangle = \mathrm{Ricci}(\xi, \xi)$$

so that

$$\Delta\|\xi\|^2 - 2\,\mathrm{Ricci}(\xi, \xi) = 2\|\nabla\xi\|^2. \tag{1}$$

For a minimal hypersurface,

$$-\text{Ricci} = \langle dv, dv \rangle$$

where $v : M \to S^n$ is the Gauss map. Therefore

$$\|dv\|^2\|\xi\|^2 \geq \langle dv(\xi), dv(\xi)\rangle = -\text{Ricci}(\xi, \xi). \tag{2}$$

By diagonalizing dv at an arbitrary $p \in M$ and using the fact that M is minimal, one easily sees that equality holds in (2), at P, if and only if $\|dv(P)\|^2 = 0$ or $\xi(P) = 0$.

We claim ξ cannot vanish identically on any open set in M. One way to see this is to associate to ξ the one form $\omega \equiv \langle \xi, \cdot \rangle$. Using a theorem of Andreotti and Vesentini stated in [D], one has $d\omega = 0 = d * \omega$. The lift of ω to the simple connected universal cover $\tilde{M}$ and M is hence the differential of a globally defined harmonic function U. If $\xi \equiv 0$ on an open set then on the lift of this set $U \equiv \text{const}$, on $\tilde{M}$ by unique analytic continuation for the Laplacian. This implies $\xi \equiv 0$ on M. Using these facts we write (1) as

$$\Delta \|\xi\|^2 + 2\|dv\|^2\|\xi\|^2 = 2\|\nabla\xi\|^2 + 2W^2 \tag{3}$$

where $W^2 \geq 0$ and $w^2 = 0$ a.e. implies $\|dv\|^2 \equiv 0$.
Away from the zeroes of ξ, one has

$$\Delta \|\xi\|^2 = 2(\|\xi\|\Delta\|\xi\| + \|\nabla\|\xi\|\|^2). \tag{4}$$

In fact this equality holds globally in M in the sense of distributions, which is how it will be used. Recalling Kato's inequality

$$\|\nabla\xi\| \geq \|\nabla\|\xi\|\|,$$

we obtain from (3) and (4),

$$\|\xi\|\Delta\|\xi\| + \|\nabla\|\xi\|\|^2 + \|dv\|^2\|\xi\|^2 \geq \|\nabla\xi\|^2 + W^2 \geq \|\nabla\|\xi\|\|^2 + W^2$$

and hence

$$\|\xi\|\Delta\|\xi\| + \|dv\|^2\|\xi\|^2 \geq W^2. \tag{5}$$

Next recall that M is stable provided

$$0 \leq \int (\|\nabla \psi\|^2 - \|dv\|^2 \psi^2) \tag{6}$$

for all $\psi \in W_c^{1,2}$, the space of $W^{1,2}$ functions with compact support in M. We take $\psi = \omega \|\xi\|$ in (6). Here ω denotes a "standard" cutoff function of geodesic distance r with the following properties:

(i) $\omega \in C^\infty$,

(ii) $0 \leq \omega \leq 1$

(iii) $\omega \equiv 1$ on Br/2, $\omega \equiv 0$ off Br

(iv) $\|\nabla \omega\|^2 \leq c/r^2$, where $c =$ const. independent of r.

Br denotes the geodesic ball of radius r centered at a point $P \in M$.
We have

$$\begin{aligned}
0 &\leq \int (\|\nabla(\omega \|\xi\|)\|^2 - \|dv\|^2 \|\xi\|^2 \omega^2) \\
&= \int (-\|\xi\| \omega \Delta(\|\xi\| \omega) - \|dv\|^2 \|\xi\|^2 \omega^2) \\
&= \int (-\|\xi\| \omega (\|\xi\| \Delta \omega + \omega \Delta \|\xi\| + 2\langle \nabla \omega, \nabla \|\xi\| \rangle - \|dv\|^2 \|\xi\|^2 \omega^2) \\
&= -\int \omega^2 \|\xi\| (\Delta \|\xi\| + \|dv\|^2 \|\xi\|) - 2 \int \|\xi\| v \langle \nabla \|\xi\|, \nabla \omega \rangle - \int \|\xi\|^2 \omega \, \Delta \omega \\
&\leq \int \|\xi\|^2 \|\nabla \omega\|^2 - \int \omega^2 W^2 \\
&\leq c/r^2 \int_{\mathrm{Br}} \|\xi\|^2 - \int_{\mathrm{Br}/2} W^2
\end{aligned}$$

Letting $r \to \infty$ and using the assumption $\xi \in L^2$ one finds $W^2 \equiv 0$ on M. This implies that $\|dv\|^2 \equiv 0$ which implies M is a hyperplane. A contradiction is reached by the existence of the cycle C. Q.E.D.

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Technische Universität Berlin

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