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Large time behavior of the heat kernel: the parabolic λ-potential alternative

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Let M be a noncompact Riemannian manifold with Laplace–Beltrami operator Δ acting on functions on M, $\lambda =: \lambda(M)$ the bottom of spec $(-\Delta)$, and attendant minimal positive heat kernel $p(x, y, t)$ (where (x, y, t) is an element of $M \times M \times (0, +\infty)$). In this note we prove the following

THEOREM. *For all x, y in M we have the existence of the limit*

$$\lim_{t \uparrow +\infty} e^{\lambda t} p(x, y, t) =: \mathscr{F}(x, y), \tag{1}$$

for which we have the following alternative:

Either $\mathscr{F}$ vanishes identically on all of $M \times M$, in which case λ possesses no L^2 eigenfunctions; or $\mathscr{F}$ is strictly positive on all of $M \times M$ in which case λ possesses a positive normalized L^2 eigenfunction ϕ (normalized in the sense that its L^2 norm is equal to 1) for which

$$\lim_{t \uparrow +\infty} e^{\lambda t} p(x, y, t) = \phi(x)\phi(y) \tag{2}$$

locally uniformly on all of $M \times M$. Furthermore, if M is noncompact Riemannian complete with bounded geometry (to be explicated below), then

$$\lim_{x \to \infty} \phi(x) = 0. \tag{3}$$

The simplest example of the case $\mathscr{F} = 0$ is $\mathbb{R}^n$, $n \geq 1$, in which case we have

$$\lambda = 0, \quad p(x, y, t) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}.$$

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So

$$e^{\lambda t}p(x, y, t) = p(x, y, t) \to 0$$

as $t \uparrow +\infty$.

When M is noncompact with compact closure and smooth boundary, we denote its minimal positive (= Dirichlet) heat kernel by $q(x, y, t)$. Then one always has $\mathscr{F}$ strictly positive, and (2) follows from the Sturm–Liouville expansion of q:

$$q(x, y, t) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\phi_j(x)\phi_j(y), \tag{4}$$

where $\{\lambda_1, \lambda_2, \ldots, \uparrow +\infty\}$ denotes the spectrum of M (with eigenvalues repeated according to their multiplicity), and $\{\phi_1, \phi_2, \ldots\}$ is a complete orthonormal basis of $L^2(M)$ for which each ϕ_j is an eigenfunction of λ_j.

We also have the following easy consequences of our theroem.

COROLLARY 1. (P. Li [9]) *We always have*

$$\lim_{t\uparrow+\infty} \frac{\ln p(x, y, t)}{t} = -\lambda \tag{5}$$

locally uniformly on $M \times M$; *and when* M *has finite volume* V *we have*

$$\lim_{t\uparrow+\infty} p(x, y, t) = 1/V \tag{6}$$

locally uniormly on $M \times M$.

Indeed, we wish to show

$$\lim_{t\uparrow+\infty} \frac{\ln e^{\lambda t}p(x, y, t)}{t} = 0.$$

If the limit function $\mathscr{F}$ is positive, then the result is obvious. So we are only concerned with the situation where $\mathscr{F}$ is identically equal to 0.

For any domain D in M, let q_D denote the Dirichlet heat kernel of D and λ_D the lowest Dirichlet eigenvalue of D. Then, of course, we have

$$\frac{\ln e^{\lambda t}q_D(x, y, t)}{t} \le \frac{\ln e^{\lambda t}p(x, y, t)}{t}.$$

We let $t \uparrow +\infty$. Then

$$\lambda - \lambda_D \leq \liminf_{t \uparrow +\infty} \frac{\ln e^{\lambda t} p(x, y, t)}{t}.$$

Now let $D \uparrow M$. We conclude that

$$0 \leq \liminf_{t \uparrow +\infty} \frac{\ln e^{\lambda t} p(x, y, t)}{t}.$$

But since $\mathscr{F} = 0$, we have $\ln e^{\lambda t} p(x, y, t) < 0$ for large t, which implies

$$\limsup_{t \uparrow +\infty} \frac{\ln e^{\lambda t} p(x, y, t)}{t} \leq 0,$$

which implies (5).

When M has finite volume then $\lambda = 0$ with normalized L^2-eigenfunction $\phi(x) = 1/\sqrt{V}$ (for all x), which implies (6). □

COROLLARY 2. *For any M we have*

$$\lim_{t \uparrow +\infty} p(x, y, t) = 0 \tag{7}$$

if and only if M has infinite volume.

Indeed, if M has finite volume then (6) implies that $\lim p$, as $t \uparrow +\infty$, is nonzero. If, on the other hand, M has infinite volume, then (i) for $\lambda > 0$ simply use (1); and for (ii) $\lambda = 0$, then if $\mathscr{F}$ were positive, we would have the existence of an L^2 harmonic function on M, which is impossible by a theorem of Yau [14]. □

COROLLARY 3. *Suppose M noncompact is a covering of a compact Riemannian manifold. Then $\mathscr{F}$ is identically equal to zero. Consequently, if the covering is nonamenable – by* [1], $\lambda > 0$ *– then p tends to* 0 *faster than* $e^{-\lambda t}$.

Indeed, if $\lambda = 0$, one uses the above corollary, since M has infinite volume. If $\lambda > 0$ and $\mathscr{F} > 0$ then the L^2 eigenspace of λ, which is nontrivial, is 1-dimensional by Theorem 2.8 of [12]. But this is impossible, by the invariance of the eigenspace under the action of the deck transformation group. □

REMARK 1. We note that there are very few results valid, with any sharpness, for large time diffusion. The only cases we know, of sharp estimates from above and below, are those of [10] when the Ricci curvature of M is nonnegative. The only case of a precise limit (the result of sharp estimates from above and below), of which we are aware, is that of nonnegative Ricci curvature with maximal volume growth [9].

REMARK 2. Our theorem gives no discussion of the rate of convergence to $\mathcal{F}$ as $t \uparrow +\infty$. For some general results, see [4], [13]. In [4] it is shown that when M is λ-transient, then

$$e^{\lambda t}p(x, y, t) = o(t^{-1})$$

as $t \uparrow +\infty$; and in [13] is it shown that if M is complete noncompact with bounded geometry, then

$$p(x, y, t) = O(t^{-1/2+\epsilon})$$

for every $\epsilon > 0$, as $t \uparrow +\infty$. This last result was improved in [3] to

$$p(x, y, t) = O(t^{-1/2})$$

under the explicit bounded geometry hypothesis of Ricci curvature bounded from below, and positive injectivity radius. For a sampling of results under more explicit geometric hypotheses, we refer the reader to [9], [2], [3], and the references therein.

REMARK 3. When λ possesses an L^2 eigenfunction, then (2) has potential application to the "movement of hot spots" [5]. Simply put, for any $\psi \geq 0$ in $L_c^\infty(M)$,

$$(P_t\psi)(x) =: \int_M p(x, y, t)\psi(y)\, dV(y)$$

is the minimal positive solution of the heat equation on M satisfying

$$\lim_{t\downarrow 0} P_t\psi = \psi$$

at all points of continuity of ψ. For nonnegative ψ in $L_c^\infty(M)$, under not-so-restrictive hypotheses on the geometry of M (see [15], [7], [8]), the locus

$$H(t) = H(t; \psi) =: \{x : (P_t\psi)(x) = \max_y\, (P_t\psi)(y)\}$$

of "hot spots" is compact for all $t > 0$. If M possesses a compact set K such that $H(t)$ is contained in K for all $t > 0$, and if $\mathscr{F} > 0$, then (2) implies that the limiting locus of $H(t)$, as $t \uparrow +\infty$, is contained within the locus of maxima of ϕ – independent of the location of the support of the initial data ψ. (Of course, which maxima of ϕ are realized might very well depend on ψ). This contrasts to the examples of Euclidean and hyperbolic spaces. See [5] for background on the general question of "movement of hot spots".

REMARK 4. In an addendum, we prove the C^∞ locally uniform convergence of $e^{\lambda t}p(x, y, t)$ to $\mathscr{F}(x, y)$ on $M \times M$, that is, that space derivatives on $M \times M$ of $e^{\lambda t}p(x, y, t)$ of all orders converge, as $t \uparrow +\infty$, locally uniformly to the corresponding derivatives of $\mathscr{F}(x, y)$.

Proof of the theorem.

Step 1. We start the proof by noting that the existence of a limit of $e^{\lambda t}p(x, y, t)$ as $t \uparrow +\infty$ is rather easy. The argument goes as follows:

First, for each $x \in M$ the function

$$e^{\lambda t}p(x, x, t)$$

is a decreasing function of t. Indeed, let D be a relatively compact domain in M with smooth boundary, and Dirichlet heat kernel q. Then one has $e^{\lambda t}q(x, x, t)$ is a decreasing function of t from the Sturm–Liouville expansion (4) of q. Now pick an exhaustion of M, $D_j \uparrow M$ as $j \uparrow +\infty$, by domains which are relatively compact in M and which possess smooth boundary. Let q_j denote the Dirichlet heat kernel of D_j. It is standard that

$$q_j \uparrow p.$$

The claim follows immediately.

So $e^{\lambda t}p(x, x, t)$ is strictly decreasing with respect to t. Next, note that the semigroup property of the heat kernel p implies

$$p(x, y, t + s) = \int_M p(x, z, t)p(z, y, s)\, dV(z). \tag{8}$$

Set

$$f_{x,t}(z) =: e^{\lambda t}p(x, z, t).$$

Then (8) implies

$$\begin{aligned}\|f_{x,t}-f_{x,T}\|_2^2 &= \|f_{x,t}\|_2^2 - 2(f_{x,t}, f_{x,T})_2 + \|f_{x,T}\|_2^2\\ &= e^{2\lambda t}p(x, x, 2t) - 2\, e^{\lambda(t+T)}p(x, x, t+T) + e^{2\lambda T}p(x, x, T)\\ &\to 0\end{aligned}$$

as $t, T \uparrow +\infty$ (where $(\ ,\ )_2$ and $\|\ \|_2$ refer to inner product and norm in $L^2(M)$, respectively). We therefore have the $L^2(M)$-limit F_x of $f_{x,t}$ as $t \uparrow +\infty$. Since $e^{\lambda t}p(x, x, t)$ decreases with respect to t, we also have the local uniform boundedness, with respect to x, of the convergence of $f_{x,t}$ in $L^2(M)$ as $t \uparrow +\infty$. (By *locally uniformly bounded convergence* of the functions $f_{x,t}$ to F_x we mean that $f_{x,t} \to F_x$ as $t \uparrow +\infty$, and to each x there exists a neighborhood U of x, $T > 0$, and a positive constant c_x such that $\|f_{z,t}\|_2 \le c_x$ for all $z \in U$ and $t > T$.)

But (8) implies

$$e^{\lambda t}p(x, y, t) = (f_{x,t/2}, f_{y,t/2})_2 \to (F_x, F_y)_2, \tag{9}$$

with the convergence locally uniformly bounded with respect to x and y. (Here the local uniform boundedness of the convergence is with respect to the sup norm.)

Therefore, the work lies in the careful study of the limit function

$$f(x; y) =: \lim_{t\uparrow+\infty} e^{\lambda t}p(x, y, t),$$

which we carry out below. Of course, $f(x; y)$ is symmetric with respect to x and y. (The function $f(x; y)$ is our $\mathscr{F}(x, y)$ in the statement of the theorem, but, since in what follows we wish to view y as a parameter, we temporarily stay with the notation $f(x; y)$.)

If for any

$$u : M \times (0, +\infty) \to \mathbb{R}$$

we define the operator

$$\mathscr{L}_\lambda u = \Delta u + \lambda u - \frac{\partial u}{\partial t},$$

then direct calculation verifies that

$$(\mathscr{L}_\lambda)_x(e^{\lambda t}p(x, y, t)) = (\mathscr{L}_\lambda)_y(e^{\lambda t}p(x, y, t)) = 0. \tag{10}$$

Set

$$u(x, t) = u(x, t; y) = p(x, y, t)$$

– so we are considering y as a parameter. Our first goal (in Steps 2 through 4 below) is to show that f is a solution of

$$\Delta f + \lambda f = 0. \tag{11}$$

Step 2. Fix $o \in M$, and $R > 0$ sufficiently large so that $y \in B(o; R/2)$, set

$$B = B(o; R), \quad B^* = B \backslash B(o; R/2),$$

and consider $u(x, t)$ on $B \times (0, +\infty)$. Let q denote the Dirichlet heat kernel of B, and pick $\varphi \in C_c^\infty(B)$, such that

$$\varphi \mid B(o; R/2) = 1.$$

Then for $x \in B(o; R/4)$ we have

$$u(x, t) = q(x, y, t) + \int_0^t ds \int_{B^*} u(z, s)\{2 \operatorname{div}_z (q(x, z, t-s)(\nabla\varphi)(z)) - (\Delta\varphi)(z) q(x, z, t-s)\}\, dV(z).$$

Indeed, Green's theorem implies, for $x \in B(o; R/4)$,

$$\begin{aligned} 0 &= \int_0^t ds \int_B \{(u\varphi)(z, s)(\Delta_z q)(x, z, t-s) - (\Delta_z(u\varphi))(z, s) q(x, z, t-s)\}\, dV(z) \\ &= -\int_0^t ds \int_B \frac{\partial}{\partial s} \{(u\varphi)(z, s) q(x, z, t-s)\}\, dV(z) \\ &\quad - \int_0^t ds \int_B \{2(\nabla_z u, \nabla\varphi) + u\, \Delta\varphi\}(z, s) q(x, z, t-s)\, dV(z) \\ &= -u(x, t) + q(x, y, t) \\ &\quad + \int_0^t ds \int_{B^*} u(z, s)\{2 \operatorname{div}_z (q(x, z, t-s)(\nabla\varphi)(z)) - (\Delta\varphi)(z) q(x, z, t-s)\}\, dV(z), \end{aligned}$$

which implies (12) as claimed.

We therefore have

$$\begin{aligned}
e^{\lambda t}\{u(x,t)-q(x,y,t)\} &= e^{\lambda t}\int_0^t ds\int_{B^*} u(z,s)\{2\,\mathrm{div}_z\,(q(x,z,t-s)(\nabla\varphi)(z)) \\
&\qquad - (\Delta\varphi)(z)q(x,z,t-s)\}\,dV(z) \\
&= \int_0^t ds\int_{B^*} e^{\lambda s}u(z,s)\,e^{\lambda(t-s)}\{2\,\mathrm{div}_z\,(q(x,z,t-s)(\nabla\varphi)(z)) \\
&\qquad - (\Delta\varphi)(z)q(x,z,t-s)\}\,dV(z) \\
&= \int_0^t ds\int_{B^*} e^{\lambda(t-s)}u(z,t-s)\,e^{\lambda s}\{2\,\mathrm{div}_z\,(q(x,z,s)(\nabla\varphi)(z)) \\
&\qquad - (\Delta\varphi)(z)q(x,z,s)\}\,dV(z) \\
&= \int_0^\infty I_{[0,t]}(s)\,ds\int_{B^*} e^{\lambda(t-s)}u(z,t-s)\,e^{\lambda s}Q(x,z,s)\,dV(z),
\end{aligned}$$

that is,

$$e^{\lambda t}u(x,t) = e^{\lambda t}q(x,y,t) + \int_0^\infty I_{[0,t]}(s)\,ds\int_{B^*} e^{\lambda(t-s)}u(z,t-s)\,e^{\lambda s}Q(x,z,s)\,dV(z), \tag{13}$$

where

$$Q(x,z,s) =: 2\,\mathrm{div}_z\,(q(x,z,s)(\nabla\varphi)(z)) - (\Delta\varphi)(z)q(x,z,s).$$

Step 3. Restrict x and y to $B(o; R/4)$, and consider $e^{\lambda t}u(x,t)$. Let λ_B denote the lowest Dirichlet eigenvalue of B. Since $\lambda < \lambda_B$ we have by (4), $e^{\lambda t}q(x,y,t) \to 0$ uniformly in x on $B(o; R/4)$ as $t \uparrow +\infty$.

For any $N = 0, 1, 2, \ldots$, we have

$$\Delta_x{}^N e^{\lambda t}u(x,t) = \Delta_x{}^N e^{\lambda t}q(x,y,t) + \int_0^\infty I_{[0,t]}(s)\,ds\int_{B^*} e^{\lambda(t-s)}u(z,t-s)\,e^{\lambda s}\,\Delta_x{}^N Q(x,z,s)\,dV(z), \tag{14}$$

with

$$\int_0^\infty e^{\lambda s}\,ds\int_{B^*} |\Delta_x{}^N Q(x,z,s)|\,dV(z)$$

locally bounded with respect to x, since

$$\Delta_x{}^N e^{\lambda s} Q(x, z, s) = O_N(e^{(\lambda - \lambda_B)s})$$

locally uniformly in x and z.

Let $t \uparrow +\infty$. Then (13) becomes

$$f(x; y) = \int_0^\infty ds \int_{B^*} f(z; y)\, e^{\lambda s} Q(x, z, s)\, dV(z) \tag{15}$$

since

$$e^{\lambda(t-s)} u(z, t-s) \to f(z; y)$$

locally uniformly boundedly, which shows f is smooth with respect to x for fixed y. Consequently,

$$\Delta_x{}^N f(x; y) = \int_0^\infty ds \int_{B^*} f(z; y)\, e^{\lambda s}\, \Delta_x{}^N Q(x, z, s)\, dV(z). \tag{16}$$

Now there exist positive constants c_1, c_2 such that for any function $w \in C^\infty$ we have, by the Sobolev lemma and interior L^2 estimates (see [11, pp. 197, 207]),

$$\begin{aligned} \|w\|_{C^k_{R/2}} &= \sup_{B(o;R/2);|\alpha| \le k} |\nabla^\alpha w| \\ &\le c_1 \|w\|_{W^{l,2}_{3R/4}} \\ &\le c_2\{\|\Delta^N w\|_{L^2_R} + \|w\|_{L^2_R}\}, \end{aligned} \tag{17}$$

for

$$k + \frac{n}{2} < l \le 2N.$$

(The subscripts $R/2$, $3R/4$, and R, indicate the radius of the disk centered at o on which the respective function spaces are defined.)

Apply the above estimate to $w(x) = e^{\lambda t} u(x, t)$; then

$$\|e^{\lambda t} u - f\|_{C^k_{R/2}} \le c_2\{\|\Delta_x{}^N (e^{\lambda t} u - f)\|_{L^2_R} + \|e^{\lambda t} u - f\|_{L^2_R}\}$$

for k and N as above. Now let $t \uparrow +\infty$; the representation (14) converges locally uniformly boundedly to the representation (16), which implies the local uniform convergence (with respect to x) of the x-derivatives of $e^{\lambda t}u$, of all orders, to the corresponding x-derivatives of f.

Step 4. We now wish to show that f is a solution of the eigenvalue equation (11). From (10) we know that the time derivatives of $e^{\lambda t}u$ converge locally uniformly (with respect to x) as $t \uparrow +\infty$. The question is: to what? Well, since $e^{\lambda t}u$ itself converges, and the time derivative converges, the time derivative must converge to 0. Thus f satisfies (11) with respect to x. Also, with respect to y.

So for each fixed y in M, we have the locally uniform convergence (with respect to x)

$$e^{\lambda t}p(x, y, t) \to f(x; y) \geq 0$$

as $t \uparrow +\infty$, with f a solution of (11) – symmetric with respect to x and y. This was our first task. We now continue our study f, using the methods of [12].

Step 5. Fix $y \in M$. If f is a nonnegative solution of (11) on M, then for any relatively compact Ω in M with smooth boundary, $x \in \Omega, f(x)$ is given by the λ-Poisson integral formula:

$$f(x) = \int_0^{+\infty} e^{\lambda t}\, dt \int_{\partial\Omega} -\frac{\partial q}{\partial \nu}(x, w, t) f(w)\, dA(w),$$

where q is (now) the Dirichlet heat kernel of Ω. Then Green's theorem implies, for any $T > 0$,

$$\begin{aligned} f(x) &\geq \int_T^{+\infty} e^{\lambda t}\, dt \int_{\partial\Omega} -\frac{\partial q}{\partial \nu}(x, w, t) f(w)\, dA(w), \\ &= e^{\lambda T} \iint_\Omega q(x, z, T) f(z)\, dV(z). \end{aligned}$$

By taking an exhaustion of M by relatively compact domains Ω with smooth boundary, we obtain

$$f(x) \geq e^{\lambda T} \int_M p(x, z, T) f(z)\, dV(z). \tag{18}$$

Since p is always positive, we conclude that either $f > 0$ on all of M, or $f = 0$ on all of M.

Since f is symmetric with respect to x and y, we have either f positive on all $M \times M$ or f identically equal to 0 on all of $M \times M$.

Step 6. We now assume f is positive on all of $M \times M$. Then for any compact K in M we have by (18)

$$\begin{aligned} f(x; y) &\geq \int_M e^{\lambda t} p(x, z, t) f(z; y)\, dV(z) \\ &\geq \int_K e^{\lambda t} p(x, z, t) f(z; y)\, dV(z) \end{aligned}$$

for all $t \geq 0$, which implies

$$f(x; y) \geq \int_K f(x; z) f(z; y)\, dV(z).$$

Now set $x = y$ and use the fact that $f(x; y)$ is symmetric in x and y. Then

$$f(y; y) \geq \int_K f^2(z; y)\, dV(z) \tag{19}$$

for all compact K in M. So for each $y \in M, f(\ ; y) \in L^2(M)$.

This implies, by Theorem 2.8 of [12], that λ has precisely a 1-dimensional eigenspace, each of whose nontrivial elements never vanish on M. So there exists a positive normalized L^2 eigenfunction ϕ of λ so that

$$f(x; y) = c(y)\phi(x).$$

But the symmetry of f with respect to x and y implies

$$f(x; y) = c(y)\phi(x) = c(x)\phi(y),$$

which implies

$$c(x) = \alpha\phi(x)$$

for some constant $\alpha > 0$. Thus

$$f(x; y) = \alpha\phi(x)\phi(y). \tag{20}$$

Step 7. We note that (20) implies, whether f is identically equal to 0 or always positive, that $f: M \times M \to \mathbb{R}$ is continuous.

Let $x = y$. Then $e^{\lambda t}p(x, x, t)$ decreases to the continuous function $\alpha\phi^2(x)$ as $t \uparrow +\infty$; so Dini's theorem implies the convergence is locally uniform. Said differently, (see Step 1 for the notation) $f_{x,t} \to F_x$ as $t \uparrow +\infty$, locally uniformly with respect to x. But (9) then implies that $e^{\lambda t}p(x, y, t)$ converges as $t \uparrow +\infty$, locally uniformly with respect to $(x, y) \in M \times M$.

Step 8. Since ϕ is a positive L^2 eigenfunction of λ, we have, by Theorem 2.10 of [12], for all $t > 0$,

$$\phi(x) = \int_M e^{\lambda t}p(x, y, t)\phi(y)\, dV(y), \tag{21}$$

which implies

$$\phi(x) \geq \int_K e^{\lambda t}p(x, y, t)\phi(y)\, dV(y)$$

for all compact K in M and $t > 0$, which implies

$$\phi(x) \geq \int_K \alpha\phi(x)\phi^2(y)\, dV(y)$$

for all compact K in M. Since ϕ has L^2 norm equal to 1, we conclude $\alpha \leq 1$.

To show that $\alpha \geq 1$, note that by the Cauchy–Schwarz inequality and the semigroup property we have

$$\phi(x) = \int_M e^{\lambda t}p(x, y, t)\phi(y)\, dV(y) \tag{22}$$

$$\leq \left\{\int_M \{e^{\lambda t}p(x, y, t)\}^2\, dV(y)\right\}^{1/2}\left\{\int_M \phi^2(y)\, dV(y)\right\}^{1/2} \tag{23}$$

$$= \{e^{2\lambda t}p(x, x, 2t)\}^{1/2}.$$

Let $t \uparrow +\infty$; we obtain

$$\phi(x) \leq \sqrt{\alpha}\phi(x),$$

which implies $\alpha \geq 1$. Therefore $\alpha = 1$, and we have (4) on all of M.

Step 9. We now wish to prove (3) under the hypothesis of "bounded geometry". Our "bounded geometry" will be a uniform lower bound on the Ricci curvature of M, and the uniform positivity of the injectivity radius on all of M. For any compact K in M we have

$$\begin{aligned}\phi(x) &= \int_M e^{\lambda t} p(x, y, t)\phi(y)\, dV(y) \\ &= \int_K e^{\lambda t} p(x, y, t)\phi(y)\, dV(y) \\ &\quad + \int_{M\setminus K} e^{\lambda t} p(x, y, t)\phi(y)\, dV(y) \\ &\leq \int_K e^{\lambda t} p(x, y, t)\phi(y)\, dV(y) \\ &\quad + e^{\lambda t}\sqrt{p(x, x, 2t)}\, \|\phi\|_{2, M\setminus K}.\end{aligned}$$

Then the fact that the Ricci curvature is bounded uniformly below implies

$$\int_K e^{\lambda t} p(x, y, t)\phi(y)\, dV(y) \to 0$$

as $x \to \infty$ ([15], [7], [8]); from the bounded geometry hypotheses on the Ricci curvature and the injectivity radius of M we have

$$p(x, x, t) \leq \text{const.}_t$$

uniformly on all of M (by an easy argument from [6] and [10]). This then implies (3).

Step 10. The last thing we must prove is that if λ possesses an L^2 eigenfunction ϕ, then $e^{\lambda t} p(x, y, t)$ is bounded away from 0 as $t \uparrow +\infty$.

This simply follows from (23), and fact that $\mathscr{F}$ either vanishes identically or is everywhere positive.

An alternative argument goes as follows: Let P_t denote the heat semigroup, and E_μ the spectral family, associated to the Laplacian on L^2. Then the spectral theorem implies, for any positive u in L^2,

$$\begin{aligned}(e^{\lambda t}P_t u, u) &= \int_{\lambda}^{+\infty} e^{(\lambda-\mu)t}(dE_\mu u, u)\\ &= (u, \phi)^2 + \int_{\lambda}^{+\infty} e^{(\lambda-\mu)t}(d\bar{E}_\mu u, u)\\ &\geq (u, \phi)^2,\end{aligned}$$

where

$$\bar{E}_\mu = E_\mu - E_\lambda.$$

This implies $e^{\lambda t}p(x, y, t)$ is bounded away from 0 as $t \uparrow +\infty$. □

Addendum. We give here an alternative proof of the continuity of $\mathscr{F}(x, y)$ on $M \times M$. It will also have the added feature that the argument also proves the C^∞ smoothness of $\mathscr{F}$ on $M \times M$, and the locally uniform convergence on $M \times M$ of derivatives of $e^{\lambda t}p(x, y, t)$ to the corresponding derivatives of $\mathscr{F}$ on $M \times M$.

Consider the natural Riemannian product structure on $M \times M$. The associated Laplace–Beltrami operator acting on functions on $M \times M$ is given by

$$\Delta_{M \times M} = \Delta_1 + \Delta_2,$$

where Δ_j is the Laplacian on the j-th variable, $j = 1, 2$. Then $p(x, y, t)$ satisfies the "heat equation" on $M \times M$:

$$\Delta_{M \times M}p = 2\frac{\partial p}{\partial t}.$$

We have (see Steps 2 and 3)

$$\begin{aligned}e^{\lambda t}p(x, y, t) &= e^{\lambda t}q(x, y, t)\\ &\quad + \int_0^\infty I_{[0,t]}(s)\, ds \int_{B^*} e^{\lambda(t-s)}p(z, y, t-s)\, e^{\lambda s}Q(x, z, s)\, dV(z),\end{aligned} \tag{24}$$

where

$$Q(x, z, s) =: 2\, \mathrm{div}_z\, (q(x, z, s)(\nabla\varphi)(z)) - (\Delta\varphi)(z)q(x, z, s),$$

and, using the symmetry of $p(x, y, t)$ with respect to x and y,

$$\Delta_{M\times M}{}^{N} e^{\lambda t}p(x, y, t) = \Delta_{M\times M}{}^{N} e^{\lambda t}q(x, y, t) + \int_0^\infty I_{[0,t]}(s)\, ds \int_{B^*} \{e^{\lambda(t-s)}p(z, y, t-s)\, e^{\lambda s}\, \Delta_1{}^N Q(x, z, s) + e^{\lambda(t-s)}p(z, x, t-s)\, e^{\lambda s}\, \Delta_1{}^N Q(y, z, s)\}\, dV(z), \tag{25}$$

for any $N = 0, 1, 2, \ldots$, with

$$\int_0^\infty e^{\lambda s}\, ds \int_{B^*} |\Delta_1{}^N Q(\zeta, z, s)|\, dV(z)$$

locally bounded with respect to ζ, since

$$\Delta_1{}^N e^{\lambda s}Q(\zeta, z, s) = O_N(e^{(\lambda - \lambda_B)s})$$

locally uniformly in ζ and z.

Let $t \uparrow +\infty$. Then (25) becomes, for $N = 0$,

$$2\mathscr{F}(x, y) = \int_0^\infty ds \int_{B^*} \{\mathscr{F}(z, y)\, e^{\lambda s}Q(x, z, s) + \mathscr{F}(z, x)\, e^{\lambda s}Q(y, z, s)\}\, dV(z).$$

Each integrand is smooth in (x, y), since $\mathscr{F}$ is smooth as a function of each variable separately (as already shown), which implies $\mathscr{F}$ is smooth on $M \times M$. Consequently,

$$\Delta_{M\times M}{}^{N}\mathscr{F}(x, y) = \int_0^\infty ds \int_{B^*} \{\mathscr{F}(z, y)\, e^{\lambda s}\, \Delta_1{}^N Q(x, z, s) + \mathscr{F}(z, x)\, e^{\lambda s}\, \Delta_1{}^N Q(y, z, s)\}\, dV(z). \tag{26}$$

There exists a positive constant such that for any function $\mathscr{W} \in C^\infty(M \times M)$ we have

$$\|\mathscr{W}\|_{C^k_{R/2}} \leq \text{const.}\, \{\|\Delta_{M\times M}{}^{N}\mathscr{W}\|_{L^2_R} + \|\mathscr{W}\|_{L^2_R}\},$$

for

$$k + n < 2N.$$

(The subscripts $R/2$, $3R/4$, and R, now refer to radii of disks, centered at (o, o), in $M \times M$.)

Apply the above estimate to $\mathscr{W}(x) = e^{\lambda t}p(x, y, t)$; then

$$\|e^{\lambda t}p - \mathscr{F}\|_{C^k_{R/2}} \leq c_2\{\|\Delta^N_{M\times M}(e^{\lambda t}p - \mathscr{F})\|_{L^2_R} + \|e^{\lambda t}p - \mathscr{F}\|_{L^2_R}\}$$

for k and N as above. Now let $t \uparrow +\infty$; the representation (25) converges locally uniformly boundedly to the representation (26), which implies the local uniform convergence of $e^{\lambda t}p$ and its spatial derivatives, of all orders, to $\mathscr{F}$ and its corresponding derivatives. □

Added in proof: We refer the reader to the recent preprint of Y. Pinchover, *Large time behavior of the heat kernel and the behavior of the Green function near criticality for nonsymmetric elliptic operators*, where the author generalizes our result to nonsymmetric operators on domains in Euclidean space.

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