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Autor(en): **Ross, Marty**

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Complete nonorientable minimal surfaces in $\mathbf{R}^3$

MARTY ROSS

Abstract. We consider complete nonorientable minimal immersions $x(M) \subseteq \mathbf{R}^3$. Assuming the double cover N of M has finite total curvature, we generalize an argument of Lopez/Ros to give a sufficient condition for the instability of $x(M)$ in terms of the total curvature of M and the genus γ of $\bar{N}$. We apply this condition to prove that if the immersion is regular then $x(M)$ is unstable. We also consider the case where the immersion is finitely branched, and we classify the possibilities under the assumption that $\bar{N}$ is hyperelliptic.

1. Introduction

Let $x(M) \subseteq \mathbf{R}^3$ be a complete minimal immersion. $x(M)$ is *stable* if the 2nd variation of area is nonnegative for every compactly supported C^1 variation of $x(M)$ ([BC], §1). Fischer-Colbrie/Schoen [FS] and doCarmo/Peng [CP] independently proved that if M is orientable and $x(M)$ is stable then $x(M)$ is a plane. The corresponding question for nonorientable M remains open, and it is this question we investigate here.

One can study the case of nonorientable M by lifting to the double cover N of M. There is a natural $2:1$ projection $\tilde{\pi} : N \to M$ and an *antipodal map* $I : N \to N$, an orientation-reversing involution without fixed points satisfying

$$\tilde{\pi}(I(p)) = \tilde{\pi}(p), \qquad p \in N. \tag{1}$$

The immersion can be lifted to $x(N)$, and we are then interested in variations which are symmetric with respect to I (see Section 2). Restricting to oriented isothermal coordinates (with respect to the induced metric), N becomes a Riemann surface. In such coordinates the antipodal map is anticonformal. Also, if N has finite total Gauss curvature (again in the induced metric) then $N = \bar{N} - \{p_1, \ldots, p_k\}$ is conformally a finitely-punctured compact Riemann surface ([Os], §9). This characterization of N continues to hold if $x(M)$ has finitely many branch points, since the underlying result of Huber still holds ([Os], p. 89). However, if $x(M)$ is permitted to have infinitely many branch points, then the characterization can fail ([Os], p. 73). The above result of Fischer-Colbrie/Schoen/DoCarmo/Peng continues to hold if one allows finite branching ([M]), but the result is *false* for nonorientable surfaces. For example, Henneberg's surface, which possesses two branch points, is stable ([C]).

Since N is a Riemann surface, we can write $x(N)$ using the classical Weierstrass representation ([Os], §8):

$$x(p) = \mathrm{Re}\int^p \Phi, \qquad p \in N, \tag{2}$$

where

$$\Phi = (\phi_1, \phi_2, \phi_3) = (1 - g^2, i(1 + g^2), 2g)\alpha. \tag{3}$$

Here $g : N \to \bar{\mathbf{C}}$ is a meromorphic function and α is a holomorphic differential on N. Implicit in (2) is the fact that Φ has no real periods:

$$\mathrm{Re}\int_\gamma \Phi = 0 \qquad \text{for every closed curve } \gamma \subseteq N. \tag{4}$$

The branch points of $x(N)$, if they exist, occur at the zeros of Φ. Furthermore, if N has finite total curvature then g and w extend meromorphically to $\bar{N}$. $g : N \to \bar{\mathbf{C}}$ is the Gauss map of $x(N)$ in the following sense: if $C : N \to S^2$ is the (standard) Gauss map of $x(N)$, and if $\pi : S^2 \to \bar{\mathbf{C}}$ is stereographic projection then

$$g = \pi \circ G. \tag{5}$$

The fact that $x(N)$ is a double cover of a nonorientable immersion is equivalent (with (4)) to ([Ol], §1)

$$\overline{I^*\Phi} = \Phi, \tag{6}$$

which is in turn equivalent to the two equations

$$g \circ I = -1/\bar{g}, \tag{7}$$

$$I^*\alpha = -g^2\alpha. \tag{8}$$

Condition (7) is of particular interest to us. Suppose that $N = \bar{N} - \{p_1, \ldots, p_k\}$ has finite total curvature. It is standard that the instability of $x(N)$ can be reduced to an eigenvalue estimate on $\bar{N}$, and thus to the existence of suitable test functions on $\bar{N}$. If we want to consider the instability of the *nonorientable* immersion $x(M)$, we have to then search for test functions on $\bar{N}$ with appropriate anti-symmetry with respect to I, corresponding to variations of $x(N)$ which don't separate the sheets – see Section 2. It turns out that certain functions into S^2 can be considered as a trio of such test functions. Composing with stereographic projection, the resulting functions have exactly the symmetry of the Gauss map given by (7). The idea in this context is due to Lopez and Ros [LR]; they use this formulation to prove that the catenoid and Enneper's surface are the only complete orientable minimal immersions into $\mathbf{R}^3$ of index 1. By a simple generalization of their argument we have the following sufficient condition for instability:

THEOREM 1. *Let $x(M) \subseteq \mathbf{R}^3$ be a complete, nonorientable, finitely branched minimal immersion of finite total curvature with double cover N and Gauss map $g : \bar{N} \to \bar{\mathbf{C}}$. Suppose there is a meromorphic function $\bar{h} : \bar{N} \to \bar{\mathbf{C}}$ with* $\deg h \leq \deg g$,

$$h \cdot I = -1/\bar{h}, \tag{7'}$$

and such that h is not obtained from g by composition with a Möbius transformation. Then $x(M)$ is unstable.

We prove Theorem 1, along with a generalization, Theorem 1′, in Section 2.

In order to apply this theorem, we need to know of the existence of meromorphic functions of relatively low degree satisfying (7)′. In connection with this, we would like to know something of the possible antipodal maps $I : \bar{N} \to \bar{N}$. In general, both of these questions appear to be difficult, but we do have the following (also proved in Section 2).

LEMMA 2. *Suppose $\bar{N}$ is a compact Riemann surface of genus γ with an anticonformal involution $I : \bar{N} \to \bar{N}$, and suppose there is a meromorphic function g on $\bar{N}$ satisfying (7)′. Then there is a meromorphic function h satisfying (7)′ with deg $h \leq \gamma + 1$.*

The point for us is that the Gauss map of a *regular* minimal immersion, as well as satisfying (7)′, always satisfies $\deg g \geq \gamma + 3$. (This follows readily from [HM], but for completeness we sketch a proof in Section 2). Thus, combined with the above results, we have

THEOREM 3. *Suppose that $x(M) \subseteq \mathbf{R}^3$ is a complete, regular, nonorientable minimally immersed surface of finite total curvature, Then $x(M)$ is unstable.* □

This theorem leaves open the possibility of the existence of stable finitely-branched immersions. In Section 3 we assume $\bar{N}$ is hyperelliptic and classify the possible antipodal maps on $\bar{N}$ (Theorem 5). This enables us to prove that except in two special cases, $x(M)$ must still be unstable (Corollary 6). One of the omitted cases allows Henneberg's Surface, which is a stable nonorientable surface with two branch points ([C]). In Section 4 we construct stable, finitely branched immersions permitted by the other case.

The above questions have been investigated by a number of authors. The stability of Henneberg's surface, and other nonorientable immersions of projective planes of total curvature -2π, has been observed, among others, by Choe [C] and Meeks. The instability of all other projective planes was independently proved by Choe [C] and Lima/daSilveira [LS]; Choe [C] also proved the instability of minimally immersed Klein bottles. Lima and daSilveira [LS] proved that if M has *infinite* total curvature and is finitely connected then the stability operator has infinite index.[1]

2. Instability of regular immersions

As stated in the introduction, the instability of regular nonorientable immersions follows immediately from Theorem 1 and Lemma 2. To prove Theorem 1, suppose $x(M)$ is a complete, nonorientable finitely-branched minimal immersion with Gauss curvature K, and let $x(N)$ be the induced immersion of the orientable double cover. By the 2nd variation formula ([S], [M]) $x(M)$ is stable iff

$$\int_N |\nabla^N f|^2 + 2Kf^2 \geq 0 \tag{9}$$

for every compactly supported C^1 function $f: N \to \mathbf{R}$ with the antipodal symmetry

$$f(I(p)) = -f(p), \qquad p \in N. \tag{10}$$

(f corresponds to a variation of $x(M)$ with initial velocitiy vector field $V = fG$. Thus (10) ensures $V \circ I(p) = V(p)$). If M, and thus N, has finite total curvature, then any C^1 function $f: \bar{N} \to \mathbf{R}$ satisfying (10) is a legitimate test function in (9) [F].

Suppose now that $h: \bar{N} \to \bar{\mathbf{C}}$ is a meromorphic function satisfying (7)$'$. Let $H = (H_1, H_2, H_3) = \pi^{-1} \circ h$ where $\pi: S^2 \to \bar{\mathbf{C}}$ is stereographic projection. Then H satisfies (10). Applying (9) to each component of H, the argument in [LR] proves that if $x(M)$ is stable then

$$0 \leq \int_{\bar{N}} |\nabla^{\bar{N}} H|^2 + 2K = 8\pi(\deg h - \deg g). \tag{11}$$

[1][LS] states only that $x(M)$ is unstable, but the proof appears to establish more.

This immediately gives a special case of Theorem 1: if $\deg h < \deg g$ then $x(M)$ is unstable. To obtain the theorem in full generality, we consider the case of equality in (11). It is shown in [LR] that equality in (11) implies

$$|\nabla^{\bar{N}} H| = |\nabla^{\bar{N}} G| \tag{12}$$

pointwise. To complete the proof of Theorem 1, we show that (12) implies G and H are related by an isometry of S^2. Let $U \subseteq \bar{N}$ be any open set small enough so that $G : U \to V_1$ and $H : U \to V_2$ are diffeomorphisms. Since G and H are conformal, $H \circ G^{-1} : V_1 \to V_2$ is a conformal diffeomorphism. By (12), $H \circ G^{-1} = T$ must in fact be an isometry of S^2, and thus a rigid motion of S^2. By connectedness, $H = T \circ G$ on all of $\bar{N}$. □

It is possible to obtain a slightly more general version of Theorem 1. (11) is a comparison of the energies of H and G, using the fact that

$$\int_{\bar{N}} 2K = -\int_{\bar{N}} |\nabla^{\bar{N}} G|^2 = -8\pi \deg G. \tag{13}$$

If we consider $H : \bar{N} \to S^2$ satisfying (10) but not necessarily conformal then the components of H are still legitimate test functions for (9), and we obtain

THEOREM 1′. *Let $x(M) \subseteq \mathbf{R}^3$ be a complete, nonorientable, finitely-branched minimal immersion of finite total curvature with double cover N and Gauss map $G : \bar{N} \to S^2$. Suppose there is a C^1 function $H : \bar{N} \to S^2$ satisfying* (10) *with less energy than G. Then $x(M)$ is unstable.* □

Next, to prove Lemma 2, we may as well assume $\deg g > \gamma + 1$. Then, by (7)′, we can write the divisor of g as

$$(g) = \frac{P_1 \ldots P_{\gamma+n}}{I(P_1) \ldots I(P_{\gamma+n})}.$$

Now consider the divisor

$$\mathscr{U} = \frac{P_{m+1} \ldots P_{\gamma+n}}{I(P_1) \ldots I(P_m)}, \qquad m = [n/2].$$

Since $\deg \mathscr{U} = \gamma + n - 2m \geq \gamma$, the Riemann-Roch theorem ([FK, §3.4]) implies there is a meromorphic function f with $(f) \geq 1/\mathscr{U}$. Relabelling the P_j if necessary, we can write

$$(f) = \frac{I(P_1)\ldots I(P_m)Q_1\ldots Q_s}{P_{m+1}\ldots P_r}, \qquad s = r - 2m.$$

Now define

$$h = g\frac{f}{\bar{f}\circ I}.$$

One easily checks that h satisfies (7)′ and that $\deg h \leq s + (\gamma + n - r) \leq \gamma + 1$. □

The final ingredient in the proof of theorem 3 is the claim that $\deg g \geq \gamma + 3$ for the Gauss map of a regular nonorientable immersion. This comes from the Gauss-Bonnet formula for a complete minimal immersion in $\mathbf{R}^3$ ([JM, Th 4]):

$$\int_M K = 2\pi(X(M) - n(M)),$$

where $n(M)$ is the total number of ends of M at ∞, counting multiplicity. Now, if M is nonorientable with double cover N of genus γ, then since M has at least one puncture, $X(M) \leq -\gamma$. As well, $n(M) \geq 3$ ([HM, Th 6], [K, Cor 1]). Together with (13), this gives

$$-2\pi \deg g \leq 2\pi(-\gamma - 3)$$
$$\Rightarrow \deg g \geq \gamma + 3,$$

as desired. □

3. Antipodal maps on hyperelliptic surfaces

A compact Riemann surface $\bar{N}$ is said to be *hyperelliptic* if there is a degree 2 meromorphic function $z : \bar{N} \to \bar{\mathbf{C}}$. Our intention is to classify the possible antipodal maps on $\bar{N}$ (Theorem 5), which together with Theorem 1 will give instability results for hyperelliptic minimal immersions (Corollary 6).

To begin, we recall some elementary properties of hyperelliptic Riemann surfaces ([FK], III·7). If $\bar{N}$ is a hyperelliptic surface of genus γ, and if a degree 2 function $z : \bar{N} \to \bar{\mathbf{C}}$ is chosen to not have branch points at its poles, then $\bar{N}$ can be represented by the polynomial

$$w^2 = (z - p_1)(z - p_2)\cdots(z - p_{2\gamma+2}). \tag{14}$$

Here $w : \bar{N} \to \bar{\mathbf{C}}$ is a meromorphic function on $\bar{N}$ of degree $2\gamma + 2$, and every point $P \in \bar{N}$ is determined uniquely by $z(P)$ and $w(P)$. In particular, the (distinct) points $p_1, \ldots, p_{2\gamma+2}$ can be identified with points on $\bar{N}$, the branch points of z. Any meromorphic function $f : \bar{N} \to \bar{\mathbf{C}}$ can be written as a rational function of z and w, and if $\deg f \leq \gamma$ then f can be written as a rational function of z alone.

Any compact Riemann surface of genus $\gamma \leq 2$ is hyperelliptic (of course if $\gamma = 0$ then $\bar{N} \approx \bar{\mathbf{C}}$ and $\bar{N}$ admits degree 1 functions). If $\gamma \geq 2$ then any degree 2 function $z_1 : \bar{N} \to \bar{\mathbf{C}}$ can be written as $z_1 = T \circ z$ where T is a Möbius transformation. There is then a meromorphic function w_1 and a polynomial Q such that $w_1^2 = Q(z_1)$. Q is of degree $2\gamma + 2$ unless T sends some p_k to ∞, in which case Q is of degree $2\gamma + 1$.

The key to classifying the antipodal maps on $\bar{N}$ is the following:

LEMMA 4. *Suppose $T : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ is a Möbius transformation satisfying*

$$\overline{T(\overline{T(z)})} = z. \tag{15}$$

Then there is a Möbius transformation S such that either

$$S \circ \bar{T} \circ S^{-1}(z) = \bar{z} \tag{16}$$

or

$$S \circ \bar{T} \circ S^{-1}(z) = -1/\bar{z}. \tag{17}$$

Proof. ([AG], 1·9·4). □

THEOREM 5. *Suppose $\bar{N}$ is a hyperelliptic Riemann surface of genus $\gamma \geq 2$, and suppose $I : \bar{N} \to \bar{N}$ is an antipodal map. Then $\bar{N}$ and I can be written either in the form*

$$w^2 = (z - p_1)(z - \bar{p}_1) \cdots (z - p_{\gamma+1})(z - \bar{p}_{\gamma+1}), \qquad \operatorname{Im} p_j \neq 0, \tag{18}$$

$$\begin{cases} z \circ I = \bar{z}, \\ w \circ I = -\bar{w}, \end{cases} \tag{19}$$

or

$$w^2 = (z - p_1)(z + 1/\bar{p}_1) \cdots (z - p_{\gamma+1})(z + 1/\bar{p}_{\gamma+1}), \tag{20}$$

$$\begin{cases} z \circ I = -1/\bar{z}, \\ w \circ I = -\bar{w}/\bar{z}^{\gamma+1}. \end{cases} \tag{21}$$

If γ is even then the 2nd case does not occur.

REMARK. This result also holds for $\gamma = 0$ and 1, but has less meaning in these cases. (As well, the proof given below does not apply in these cases.) If $\gamma = 0$ then it follows from Lemma 4 that $\bar{N}$ admits a degree 1 function z satisfying (7)$'$. If $\gamma = 1$ then the easiest way to obtain a suitable classification is to identify $\bar{N} \approx \mathbf{C}/L$ where L is a lattice in the plane. The existence of the antipodal map I means that we can assume $L = \langle 2, 2ci \rangle$ is rectangular and I is given by

$$I(z) = \overline{z + 1}. \tag{22}$$

([AG]), Th. 1·9·8). Notice that the case where L is rhomboidal is ruled out by the condition that I have no fixed points). Now let

$$\mathscr{F}(z) = \frac{1}{b}(\mathscr{P}(z - ci/2) - e_3), \tag{23}$$

where $\mathscr{P}$ is the Weierstrass $\mathscr{P}$-function, e_1, e_2, e_3 are the values of $\mathscr{P}$ at the zeros of $\mathscr{P}'$, and

$$b = \sqrt{(e_1 - e_3)(e_3 - e_2)} \in \mathbf{R}^+, \tag{24}$$

([A], pp. 277–279). A calculation shows that $\deg \mathscr{F} = 2$ and $\mathscr{F}$ satisfies (7)$'$. Furthermore, by the functional equation for $\mathscr{P}$,

$$(\mathscr{F}'(z))^2 = 4b\mathscr{F}(z)(\mathscr{F}(z) - k)\left(\mathscr{F}(z) + \frac{1}{k}\right), \tag{25}$$

$$k = \sqrt{\frac{e_1 - e_3}{e_3 - e_2}} \in \mathbf{R}^+. \tag{26}$$

Proof of Theorem 5. Let z_1 be a degree 2 function on $\bar{N}$. Then $\overline{z_1 \circ I}$ is also of degree 2 and so, because $\gamma \geq 2$, there is a Möbius transformation such that $\overline{z_1 \circ I} = T \circ z_1$. Since I is an involution, T satisfies (15). Let $z = S \circ z_1$ where S is the Möbius transformation given by Lemma 2. According to whether (16) or (17) is satisfied, we have either $z \circ I = \bar{z}$ or $z \circ I = -1/\bar{z}$.

If $z \circ I = \bar{z}$ then the branch points of z come in conjugate pairs and we can find w such that (18) is satisfied. (The branch points of z cannot occur on the extended real axis, since these would then be fixed points of I.) (18) then implies $(w \circ I)^2 = \bar{w}^2$, and thus $w \circ I = \pm \bar{w}$. The plus sign cannot occur, since then I would fix the real z axis.

Now suppose $z \circ I = -1/\bar{z}$. Replacing z by $(z+e^{i\theta})/(z-e^{i\theta})$ if necessary, we can assume z does not have a branch point at ∞. Pairing the branch points of z as above, we can find w such that (20) is satisfied. (20) then gives

$$(w \circ I)^2 = (-1)^{\gamma+1} \prod_{j=1}^{\gamma+1} \frac{p_j}{\bar{p}_j} \frac{\bar{w}^2}{\bar{z}^{2\gamma+2}}. \tag{27}$$

Replacing z by a suitable $e^{i\alpha}z$, we obtain (21).

It remains to show that if γ is even then (21) is in fact inconsistent. Since I is supposed to be an involution, we have

$$w = (w \circ I) \circ I = -(-z)^{\gamma+1}\overline{w \circ I} = (-1)^{\gamma+1} w = -w,$$

giving the desired contradiction. □

COROLLARY 6. *Suppose $x(M) \subseteq \mathbf{R}^3$ is a complete, finitely branched, non-orientable minimal immersion of finite total curvature, and suppose that the double cover $\bar{N}$ of $\bar{M}$ is hyperelliptic of genus γ. Then $x(M)$ is unstable in the following cases*:

(i) *if $\gamma = 0$ and the total curvature of M is less than -2π*;

(ii) *if $\gamma = 1$*;

(iii) *if $\gamma \geq 2$ and γ is even*;

(iv) *if $\gamma \geq 3$, γ is odd, and the total curvature of M is less than -4π.*

Proof of Corollary 6. Cases (i) and (ii) follow from Theorem 1, (13), and the remark following the statement of Theorem 3: for case (ii) we define

$$\mathscr{G}(z) = \frac{\sqrt{k}\mathscr{F}'(z)}{2\sqrt{b}(\mathscr{F}(z)-k)}, \tag{28}$$

and prove that $\deg \mathscr{G} = 2$, $\mathscr{G}$ satisfies (7)′, and $\mathscr{G}$ is not related to $\mathscr{F}$ by a Möbius transformation.

To prove case (iii), let

$$f_1 = \frac{w}{(z-p_1)(z-p_2)\cdots(z-p_{\gamma+1})},$$

$$f_2 = \frac{w}{(z-\bar{p}_1)(z-p_2)\cdots(z-p_{\gamma+1})}.$$

It is easy to show $\deg f_1 = \deg f_2 = \gamma + 1$, f_1 and f_2 satisfy (7)′, and that f_1 and f_2 are not related by a Möbius transformation. The proof of (iii) will now follow if we can show $\bar{N}$ does not admit functions of degree $\leq \gamma$ satisfying (7)′. Supposing f is such a function, we know $f = R(z)$ is a rational function of z. If $z(p) = 0$ then by (19),

$$f(p) \cdot \bar{f}(I(p)) = R(0) \cdot \overline{R(0)} \in \mathbf{R}^+,$$

contradicting (7)′.

Finally, the proof of case (iv) divides into two subcases. The 1st subcase, when (18) and (19) apply, is identical to case (iii). For the 2nd subcase, z itself satisfies (7)′, and the result follows from Theorem 1 and (13). □

4. Stable branched immersions

Let $\bar{N}$ be the compact Riemann surface given by

$$w^2 = (z^4 + p^4)(z^4 + 1/p^4)(z^2 - q_1^2 i)(z^2 - i/q_1^2) \cdots (z^2 - q_n^2 i)(z^2 - i/q_n^2), \tag{29}$$

where $n \geq 0$, $p, q_1, \ldots, q_n \in \mathbf{R}$, and

$$1 < p < q_1 < \cdots < q_n. \tag{30}$$

$\bar{N}$ is a hyperelliptic surface of genus $\gamma = 3 + 2n$, and we can define an antipodal map $I : \bar{N} \to \bar{N}$ by (21).

We shall construct complete finitely-branched nonorientable immersions $x(M) \subseteq \mathbf{R}^3$ with double cover $x(N)$, antipodal map I, and Gauss map

$$g = z. \tag{31}$$

(Note (7) is automatically satisfied.) If q_n is close enough to p then any such immersion will be stable, the proof of which we now sketch: for more details, see [R].

By [F], the stability of $x(M)$ is equivalent to the nonnegativity of certain eigenvalues of $\Delta + 2$ on the branched cover of S^2 obtained by the map $\pi^{-1} \circ z : \bar{N} \to S^2$. To be more precise, we initially assume $n = 0$ (in which case $\bar{N}$ is the underlying Riemann surface of the Schwarz surfaces considered in [R]). Let g_h,

g_1, g_2, g_3 be the following automorphisms of $\bar{N}$:

$$\begin{cases} g_0(z, w) = (z, -w) \\ g_1(z, w) = (i\bar{z}, \bar{w}) \\ g_2(z, w) = (-i\bar{z}, \bar{w}) \\ g_3(z, w) = (1/\bar{z}, \bar{w}/\bar{z}^4). \end{cases} \tag{32}$$

Notice that

$$I = g_0 \circ g_1 \circ g_2 \circ g_3. \tag{33}$$

These automorphisms are commuting isometries on the branched cover of S^2. Thus, when considering Rayleigh quotients to estimate eigenvalues of $\Delta + 2$, it is enough to consider test functions f which are odd or even with respect to g_0, g_1, g_2, g_3. Further, by (10) and (33), we can assume f is odd with respect to g_0 and even with respect to g_3: if not then the other symmetries ensure that the zero set of f includes a great circle, making the eigenvalue estimate straight-forward.

We have thus reduced the case $n = 0$ to considering functions f which are odd with respect to g_1 (say) and g_0, and even with respect to g_2 and g_3. In [R] we show that the lowest eigenfunction f of $\Delta + 2$ with these symmetries has (strictly) positive eigenvalue (the key to the proof is that the zero set of f intersects every circle of latitude in diametrically opposite points). This completes the proof of stability for $n = 0$.

If $n > 0$ then we assume q_n is close to p. All the branch points of z are then contained in the lift of eight small disks on S^2. Removing these disks gives us a Riemann surface independent of $n \geq 0$. Then, by the $n = 0$ case, the lowest symmetric eigenvalue of $\Delta + 2$ on this surface is positive and bounded away from zero, independent of the size of the disks. By a standard logarithm cut-off argument [F], removing the disks can only raise the eigenvalue slightly. Thus the original eigenvalue must also have been positive, and we have stability of $x(M)$ for all n.

It remains to show there is a symmetric immersion of $\bar{N}$ with Gauss map $g = z$. For any integer $k \geq 0$ let

$$\alpha_k = \frac{(i)^l w^k \, dz}{z^l}, \tag{34}$$

where

$$l = 2 + \frac{k(\gamma + 1)}{2}. \tag{35}$$

Then

$$\overline{I^*\alpha_k} = \frac{(-i)^l(-z)^l w^k(-1/z^2)\,dz}{z^{k(\gamma+1)}} = -g^2\alpha_k.$$

Thus α_k satisfies (8) and if

$$\alpha = \sum_{k=0}^{K} A_k\alpha_k, \qquad A_k \in \mathbf{R}, \tag{36}$$

then α also satisfies (8). Thus we just have to show that (4) can be satisfied by suitable choice of the constants A_k. By making K huge, this is in fact very easy. The immersion $x(N)$ with Weierstrass g and α will have $2\gamma + 4$ poles, and we must ensure that Φ has no real periods if we wind around one of these poles. As well, the homology group of $\bar{N}$ has a basis of 2γ elements we need to consider. Since there are three differentials to integrate, this gives a total of $12\gamma + 12$ period conditions to be satisfied. These conditions are linear in the A_k, so $K = 12\gamma + 13$ is large enough for the A_k to be selected.

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Department of Mathematics
Rice University
Houston, TX 77251
USA

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