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Doubling measures and quasiconformal maps

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1. Introduction

In this paper we study the relationship between quasiconformal maps and doubling measures. First recall the definition of a doubling measure. Let D be a domain in $\mathbb{R}^n$ and let μ be a Borel measure defined on D. Let $2Q$ denote that cube concentric with Q and of side length twice that of Q. We say that μ is *doubling* on D, $\mu \in \mathscr{D}(D)$, if there exists a constant $c > 0$ such that $\mu(2Q) \leq c\mu(Q)$ for all cubes Q with $2Q \subset D$.

In particular we examine the following problem. Suppose $f : D \to D'$ is a homeomorphism between domains in $\mathbb{R}^n$, $n \geq 1$. For each $\mu \in \mathscr{D}(D')$, consider the induced measure $v = \mu(f(\cdot))$ on D. Classify those f for which each such v is also a doubling measure.

Previous articles ([A], [J], [R], [S], [U]) have studied the analogous question with respect to the classes of BMO functions, Hardy–Littlewood maximal functions and A_∞-measures. For the higher dimensional cases, that is $n \geq 2$, the desired homeomorphisms f which preserve these classes prove to be quasiconformal maps. In the one dimensional case, Jones has shown that the precise class of homeomorphisms of the line which preserve BMO are those which satisfy $f' \in A_\infty$. This is not equivalent to the statement that f is quasisymmetric. Major differences thus exist in the results for the cases $n = 1$ and $n \geq 2$.

In each of the aforementioned articles, the authors needed to impose hypotheses on the homeomorphism f beyond the preservation of the given class in order to assure the quasiconformality of f. These extra assumptions have included differentiability assumptions and subdomain conditions in various forms.

Here we present proofs that quasiconformal maps preserve doubling measures in dimensions $n \geq 1$. We also show, using standard additional hypotheses similar to

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those mentioned above, that maps f which preserve doubling measures must be quasiconformal for $n \geq 2$ and quasisymmetric for $n = 1$. Note that in contrast to the situation with BMO functions ([J], [R]), similar theorems hold in both the cases $n = 1$ and $n \geq 2$. However, we will point out other observed differences for doubling measures dependent on this given dimensional break.

2. Notation and preliminary lemmas

Throughout this paper D and D' denote domains in $\mathbb{R}^n$, $n \geq 2$, and G and G' indicate subdomains of D and D' respectively. We use Q for any closed cube and by τQ we mean that cube concentric with Q which arises from expanding Q by a factor of $\tau \geq 1$. If the center of Q is specified to be x, we write $Q = Q(x)$; if, in addition, the side length of Q is $2r$, we write $Q = Q(x, r)$. Lebesgue measure is denoted by $|\cdot|$.

We recall the analytic definition of quasiconformality. A function $f: D \to \mathbb{R}^n$ is said to be *absolutely continuous on lines*, ACL, if f is continuous and if f is absolutely continuous on almost all line segments in R parallel to the coordinate axes. Here $R = \{x \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i\} \subset D$ is any closed n-interval in D.

Denote the Jacobian matrix of f at x by $F(x)$ and its determinant by $J(x, f)$. A homeomorphism $f: D \to D'$ is said to be *K-quasiconformal* if $f \in$ ACL, f is differentiable a.e. in D with respect to Lebesgue measure and

$$\sup_{h \in \mathbb{R}^n,\, |h| = 1} |F(x)h|^n \leq K|J(x, f)| \text{ a.e.} \tag{2.1}$$

A homeomorphism $f: D \to D'$ is said to satisfy the *condition* (N) if $|A| = 0$ implies $|fA| = 0$. It is well known that quasiconformal maps satisfy the condition (N).

The one dimensional analogues of quasiconformal maps are quasisymmetric maps. An increasing self-homeomorphism f of the real line is called *K-quasisymmetric* if

$$\frac{1}{K} \leq \frac{|f(x+t) - f(x)|}{|f(x) - f(x-t)|} \leq K \tag{2.2}$$

for all $x, t \in \mathbb{R}$, $t \neq 0$.

We make the following slight extension in the definition of doubling measures. We say that a Borel measure μ defined on D is in $\mathscr{D}(D, \tau)$, $\tau \geq 1$, if there exists a constant $c > 0$ such that

$$\mu(2Q) \leq c\mu(Q) \tag{2.3}$$

for all cubes Q such that $2\tau Q \subset D$. If we wish to specify that a measure μ is doubling with an associated constant c, we write the constant in (2.3) as c_μ.

Similarly a Borel measure μ defined on D is in $A_\infty(D, \tau)$ [S], if there exist positive constants α and δ such that

$$\frac{\mu(E)}{\mu(Q)} \le \alpha\left(\frac{|E|}{|Q|}\right)^\delta \quad \text{and} \quad \frac{|E|}{|Q|} \le \alpha\left(\frac{\mu(E)}{\mu(Q)}\right)^\delta$$

for all cubes Q with $\tau Q \subset D$ and for every measurable set $E \subset Q$.

Note that the class $A_\infty(D, \tau)$ is contained in the class of doubling measures, $\mathscr{D}(D, \tau)$. This containment can be shown to be strict, i.e., there are examples of doubling measures which are not A_∞-measures ([FM], [W]). Thus the problem of examining which homeomorphisms preserve doubling measures is different from the analogous problem for A_∞-measures.

Reimann [R], has shown that the "pull back" of Lebesgue measure under a quasiconformal map is an A_∞-measure.

LEMMA 2.4 ([R], Corollary p. 262). *Let $f: D \to D'$ be K-quasiconformal. Then there exists a constant $\tau = \tau(K, n)$ such that $\mu \in A_\infty(D, \tau)$, where $\mu(E) \equiv |f(E)|$.*

Thus by our remark above the measure μ induced by a quasiconformal map f satisfies $\mu \in \mathscr{D}(D, \tau)$.

We now state three equivalent definitions of doubling measures. At any given point in the proofs in this paper we use whichever definition proves most convenient computationally. We say that two closed cubes are *neighboring* whenever $|Q| = |S|$ and $Q \cap S \neq \varnothing$.

LEMMA 2.5. *Let μ be a Borel measure defined on D. The following are equivalent*:

(2.6) *There exist constants $c > 0$ and $\tau \ge 1$, such that*

$$\mu(2Q) \le c\mu(Q)$$

for all cubes Q such that $2\tau Q \subset D$.

(2.7) *There exist constants $a > 0$, $b > 0$ and $\tau \ge 1$ such that*

$$\mu(mQ) \le am^b\mu(Q)$$

for all $m \ge 1$ and all cubes Q such that $2^k\tau Q \subset D$. Here $k = \min\{z \in \mathbb{Z} : z \ge \log_2 m\}$.

(2.8) *There exist constants $d > 0$ and $\sigma \geq 1$ such that*

$$\mu(S) \leq d\mu(Q)$$

for all pairs of neighboring cubes S and Q in D such that $\sigma Q \subset D$.

The equivalences above can be shown by standard geometric arguments. One can deduce (2.7) with $a = c$ and $b = \log_2 c$ by repeated applications of (2.6). For the proof of (2.7) $\Rightarrow$ (2.8), one clearly has $\mu(S) \leq \mu(mQ) \leq am^b\mu(Q)$ for $m = 1 + 2\sqrt{n}$. This will hold for all cubes Q satisfying $\sigma Q \subset D$, with $\sigma = 2^k\tau$, where $k = \min\{z \in \mathbb{Z} : z \geq \log_2(1 + 2\sqrt{n})\}$. Finally, to see (2.8) $\Rightarrow$ (2.6) we can decompose $2Q$ into 2^n cubes, each neighboring Q. Thus $\mu(2Q) \leq 2^n d\mu(Q)$ for all cubes Q such that $2Q\tau \subset D$, with $\tau = \sigma$. □

3. Main results

The theorems for the one dimensional case are a part of the known "folklore" for quasisymmetric maps. However, these results do not appear to be in print. Since they merely depend on the definition of quasisymmetry and a quick application of Lemma 2.5, we include them here for completeness.

THEOREM 3.1. *Let $f: \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism of the real line onto itself. Then the following are equivalent.*

(3.2) *f is K-quasisymmetric.*

(3.3) *For every $\mu \in \mathscr{D}(\mathbb{R})$, $\nu = \mu(f(\cdot))$ belongs to $\mathscr{D}(\mathbb{R})$ with $c_\nu = c_\nu(c_\mu, K)$.*

Proof. Consider any two adjacent intervals Q and S with $|Q| = |S| = t$ and $|Q| \cap |S| = \{x\}$, the common endpoint of Q and S. We write $Q' = f(Q)$ and $S' = f(S)$.

Assume first that f is K-quasisymmetric and $\mu \in \mathscr{D}(\mathbb{R})$ with associated constant d from (2.8). By the definition of quasisymmetry in (2.2),

$$|Q'| \leq K|S'|.$$

We can cover Q' with $m = [K] + 1$ neighboring intervals $I_1, \ldots, I_m$ such that $|I_j| = |S'|$ for all $j = 1$ to m. Next we estimate $\mu(f(\cdot))$.

$$\mu(f(Q)) = \mu(Q') \leq \mu\left(\bigcup_{j=1}^{m} I_j\right) \leq \sum_{1}^{m} d^j\mu(S') = d'\mu(f(S))$$

where $d' = d'(d, K)$.

From this particular case of adjacent neighboring intervals we can deduce (2.6).

Now assume that condition (3.3) holds. In the one dimensional case the implication (3.3) $\Rightarrow$ (3.1) is almost trivial, since we need only apply (3.3) to the measure $\nu(Q) = |Q'|$ induced by Lebesgue measure. By (2.8) we have $\nu(S) \le d\nu(Q)$ as well as $\nu(Q) \le d\nu(S)$. In other words,

$$\frac{1}{d} \le \frac{|f(x+t) - f(x)|}{|f(x) - f(x-t)|} \le d. \qquad \square$$

We now proceed with the higher dimensional case; henceforth we assume $n \ge 2$. Here we state results for mappings between general domains $D, D' \subset \mathbb{R}^n$.

THEOREM 3.4. *If $f: D \to D'$ is a K-quasiconformal map, then $\varphi : \mu \to \nu = \mu(f(\,\cdot\,))$ is a monomorphism between $\mathscr{D}(D')$ and $\mathscr{D}(D, \tau)$, $\tau = \tau(K, n)$. Moreover, we can take $c_\nu = c_\mu^\beta$, $\beta = \beta(K, n)$.*

Note that in the case where $D = D' = \mathbb{R}^n$, we can take $\tau = 1$, and φ gives an automorphism of $\mathscr{D}(\mathbb{R}^n)$.

The proof depends on Lemma 2.4 along with the following lemma which derives from Lemma 4 in [G] and Lemma 4 in [R]. (See also [S], Lemmas 2.16 and 2.19.)

LEMMA 3.5. *Let $f: D \to D'$ be a K-quasiconformal map and let $\sigma \ge 1$ be given. Then there exists a constant $\tau = \tau(K, n, \sigma)$, $\tau \ge 1$, such that for every cube $Q(x) \subset D$ satisfying $\tau Q \subset D$, both of the following conditions are true.*

(3.6) *There exists a cube $P'(f(x)) \subset D'$ with $f^{-1}(P') = P \supset Q$ such that*

$$|P'| \le c_1 |Q'|, \qquad c_1 = c_1(K, n), \quad \text{and} \quad \sigma P' \subset D'.$$

(3.7) *There exists a cube $S'(f(x)) \subset D'$ with $f^{-1}(S') = S \subset Q$ such that*

$$|Q'| \le c_2 |S'|, \qquad c_2 = c_2(K, n).$$

Proof of Theorem 3.4. Let $\tau = \max(\tau_1, \tau_2)$ where τ_1 is the constant for τ given by Lemma 2.4 and τ_2 is similarly that for Lemma 3.5. Consider any cube Q satisfying $2\tau Q \subset D$. Apply Lemma 3.5 with $\sigma = 2^k$, $k = \min\{z \in \mathbb{Z} : z \ge \log_2 m^{1/n}\sqrt{n}\}$ to find cubes $P' \supset (2Q)' = f(2Q)$ and $S' \subset Q' = f(Q)$ such that

$$|P'| \le c_1 |(2Q)'| \tag{3.8}$$

and

$$|Q'| \le c_2 |S'|. \tag{3.9}$$

From Lemma 2.4 we also deduce

$$|(2Q)'| \le c_3|Q'|, \qquad c_3 = c_3(K, n). \tag{3.10}$$

Combining (3.8), (3.9) and (3.10) yields

$$|P'| \le m|S'|, \qquad m = m(K, n). \tag{3.11}$$

Choose any doubling measure $\mu \in \mathscr{D}(D')$. We can make use of definition (2.7) since $P' \subset m^{1/n}\sqrt{n}S'$ and $2^k S' \subset \sigma P' \subset D'$. By (2.6) we see that $\nu = \mu(f(\,\cdot\,)) \in \mathscr{D}(D, \tau)$, since for the cube Q under consideration

$$\frac{\mu(f(2Q))}{\mu(f(Q))} \le \frac{\mu(P')}{\mu(S')} \le c_\mu(m^{1/n}\sqrt{n})^{\log_2 c_\mu} = (c_\mu)^\beta, \qquad \beta = \beta(K, n). \qquad \square$$

Note that since the restriction, $f|_G$, remains K-quasiconformal for any subdomain G of D, we can also state Theorem 3.4 in terms of subdomains G and $G' = f(G)$. In Theorem 3.12 we assume such subdomain conditions hold.

We show (Example 3.23) that is does not suffice to deal exclusively with the doubling measure induced by Lebesgue measure. Instead we construct a specific doubling measure on D' to arrive at our result.

THEOREM 3.12. *Let $f: D \to D'$ be a homeomorphism which is* ACL *and differentiable a.e. and which along with f^{-1} satisfies the condition* (N). *Suppose there exists a constant $\tau = \tau(f)$ such that the induced mapping $\varphi: \mu \to \nu = \mu(f(\,\cdot\,))$ is a monomorphism from $\mathscr{D}(G')$ to $\mathscr{D}(G, \tau)$, whenever G is a subdomain of D and $G' = f(G)$. Assume, in addition, that $c_\nu = c_\nu(c_\mu, f)$. Then f is a quasiconformal map.*

Proof. The condition (N) above guarantees that the set $J_0 = \{x \in D : J(x, f) = 0\}$ has Lebesgue measure zero. Note that this is the only place in the proof where we need to invoke this condition.

Consider now any point $x_0 \in D$ such that f is differentiable at x_0 and $J(x_0, f) \neq 0$. The Jacobian matrix $F(x_0)$ can be written in the form $F(x_0) = \rho A \sigma$, where $\rho, \sigma \in O(n)$ and

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \qquad \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n.$$

By (2.1) it suffices to show that

$$\lambda_n \leq a\lambda_1, \tag{3.13}$$

where a is a constant independent of x_0.

Observe that the rotation maps σ^{-1} and ρ^{-1} provide isomorphisms preserving doubling constants from $\mathscr{D}(D, \tau)$ to $\mathscr{D}(\sigma D, \tau)$ and from $\mathscr{D}(D')$ to $\mathscr{D}(\rho^{-1}D')$ respectively. Thus we can assume without loss of generality that $F(x_0)$ is a diagonal matrix. It is equally apparent we can reduce to the case $\lambda_1 = 1$.

The principal idea in the proof rests on the construction of a specific doubling measure which will guarantee (3.13). Since all of the estimates in the construction are translation invariant, we can assume without loss of generality that $x_0 = f(x_0) = 0$.

Let g denote the linear map $g(x) = Ax$. Consider the adjacent neighboring cubes $Q_1(x_r, r) \equiv Q_1(r)$ and $Q_2(x_{-r}, r) \equiv Q_2(r)$ with $x_r = (0, 0, \ldots, r)$ and $x_{-r} = (0, 0, \ldots, -r)$. Note that $x_0 = 0$ is in $Q_1 \cap Q_2$.

We build a doubling measure $\mu(E) = |h(E)|$ on the union of the rectangular boxes $R_1(r) \cup R_2(r)$, where $R_1(r) = g(Q_1(r))$ and $R_2(r) = g(Q_2(r))$, and h is a K_0-quasiconformal map to be constructed. Here in an effort to make the explanation clearer and more concise, we give the precise details and computations only for the case $n = 2$. However, all of these ideas in the construction can be generalized to higher dimensions. Note that all of the estimates are scale invariant with respect to r and if $\lambda_2 \notin \mathbb{N}$ our construction provides an upper bound for $[\lambda_2]$ which, in turn, gives (3.13). Thus to ease computations we momentarily assume $r = 1$ and $\lambda_2 \in \mathbb{N}$.

Let $K_0 > 1$ and define the K_0-quasiconformal map h on $R_1(1) \cup R_2(1) \equiv R_1 \cup R_2$ as follows. First, let h be the identity on R_2. Divide R_1 into λ_2 adjacent neighboring cubes $S_1, \ldots, S_{\lambda_2}$ of side length 2 with centers $y_i = (0, 2i-1)$, $i = 1$ to λ_2. Further decompose each cube S_i into four regions $T_{i,1}$, $T_{i,2}$, $T_{i,3}$ and $T_{i,4}$, where $T_{i,1}$ is the closed triangle with vertices $\{y_i, v_i = (-1, 2i-2), u_i = (1, 2i-2)\}$, $T_{i,2}$ is the closed triangle with vertices $\{y_i, v_i, w_i = (-1, 2i-1)\}$, $T_{i,3}$ is the closed triangle with vertices $\{y_i, u_i, z_i = (1, 2i-1)\}$ and $T_{i,4} = \{x = (x_1, x_2) : |x_1| \leq 1, 2i-1 \leq x_2 \leq 2i\}$ is the top half of S_i.

Define h in a piecewise manner on each cube S_i as a radial stretching map with respect to y_i followed by a suitable translation. On S_1, let $h(y_1) = y_1$, and let the radial stretching factor be 1 in $T_{1,1}$ (i.e., $h(x) = x$ in $T_{1,1}$), and let the radial stretching factor be K_0 in $T_{1,4}$. As the rays sweep from y_1v_1 to y_1w_1 and from y_1u_1 to y_1z_1, let the radial stretching factor increase continuously from 1 to K_0. In particular, if we denote the angle between a ray y_1x in $T_{1,2}$ and y_1v_1 (or similarly the angle between a ray in $T_{1,3}$ and y_1u_1) as θ, then we denote the radial stretching factor on that ray by $H(\theta)$, $1 \leq H(\theta) \leq K_0$.

We continue to define h inductively. Assume that h has been defined on $S_1, \ldots, S_{i-1}$. We define h on S_i as follows. On $T_{i,1}$ let h be that radial stretching map with constant stretching factor K_0^{i-1} followed by a suitable translation such that $h(\partial T_{i,1}) = h(\partial T_{(i-1),4})$. In other words we glue together consecutive maps on cubes $\{S_i\}$ in such a way that the stretching factors agree on $\partial S_i \cap \partial S_{i-1}$. Note that this uniquely determines $h(y_i)$. Now in $T_{i,4}$ we let the radial stretching factor be K_0^i and in $T_{i,2}$ and $T_{i,3}$ we let the stretching factor be $K_0^{i-1}H(\theta)$, where θ is defined in a way analogous to that above. This completes the construction of the desired map h.

Let τ be the constant given in the hypothesis and let $\sigma = \sigma(\tau)$ be the constant for which (2.8) is satisfied. Now since f is differentiable at $x_0 = 0$ in D, given any $\varepsilon > 0$, there exists an $r > 0$ such that all of the following are satisfied:

$$\sigma Q_1(r) \cup \sigma Q_2(r) \subset D, \tag{3.14}$$

$$f(\sigma Q_1(r) \cup \sigma Q_2(r)) \subset R_1(\sigma r + \varepsilon r) \cup R_2(\sigma r + \varepsilon r) \subset D', \tag{3.15}$$

$$P_1(r, -\varepsilon) \subset f(Q_1(r)), \tag{3.16}$$

and

$$f(Q_2(r)) \subset P_2(r, +\varepsilon). \tag{3.17}$$

Here by $P_1(r, -\varepsilon)$ (similarly $P_2(r, +\varepsilon)$), we mean that rectangle concentric with $R_1(r)$ having edge lengths $(2-2\varepsilon)r$ and $(2\lambda_2 - 2\varepsilon)r$.

Consider now the K_0-quasiconformal map h defined on $R_1(\sigma r + \varepsilon r) \cup R_2(\sigma r + \varepsilon r) = E'$ and the doubling measure $\mu(\,\cdot\,) = |h(\,\cdot\,)|$ in $\mathscr{D}(E')$ it induces. We take our subdomains G and G' to be $\sigma Q_1(r) \cup \sigma Q_2(r)$ and $f(\sigma Q_1(r) \cup \sigma Q_2(r))$ respectively. Note that $\mu|_{G'}$ gives a doubling measure on G' which we also denote by μ, with $c_\mu = c_\mu(K_0, n)$.

Applying our hypothesis along with (2.8) we have

$$\mu(f(Q_1(r))) \leq d\mu(f(Q_2(r))), \qquad d = d(K_0, f). \tag{3.18}$$

Now assume $\lambda_2 \geq 2$ and $\varepsilon < 1/2$ and estimate both sides of (3.18). Condition (3.17) yields

$$\mu(f(Q_2(r))) \leq \mu(P_2(r, +\varepsilon)) \leq 2((r+\varepsilon)^2 K_0^2 + r^2(1+\varepsilon)(2\lambda_2 + \varepsilon)), \tag{3.19}$$

while (3.16) gives

$$\mu(f(Q_1(r))) \geq \mu(P_1(r, -\varepsilon)) \geq 2(1-\varepsilon)^2 K_0^{2(\lambda_2 - 1)} r^2. \tag{3.20}$$

In (3.20) the quantity on the far right arises from that part of the tail end cube of $R_1(r+\varepsilon)$ which is in $P_1(r, -\varepsilon)$.

Combining these with (3.18) and letting $\varepsilon \to 0$ we have

$$K_0^{2(\lambda_2-1)} \leq d(K_0^2 + 2\lambda_2), \qquad d = d(K_0, f). \tag{3.21}$$

Finally, since $K_0 > 1$, we conclude from (3.21) that λ_2 is bounded, namely

$$\lambda_2 \leq a = a(K_0, f). \tag{3.22}$$

This completes the proof. □

Note that a more general case of Theorem 3.1 can be proven with only minor modifications in the original argument. The theorem cannot be stated analogously to Theorem 3.12 in terms of intervals I and I' of $\mathbb{R}$ and $\mathscr{D}(I')$ and $\mathscr{D}(I, \tau)$ though, since we lack the theorems of Gehring [G] and Reimann [R] in the one dimensional case.

As we mentioned earlier the proof of the higher dimensional case involves more than examining the doubling measure induced by Lebesgue measure in D'. It is necessary to consider some non-trivial doubling measure on D' as the following example shows.

EXAMPLE 3.23. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be of the form

$$f(x_1, x_2, \ldots, x_n) = (f_1(x_1), f_2(x_2), \ldots, f_n(x_n)),$$

where each f_i is K_i-quasisymmetric. Then the induced measure $\mu(E) = |f(E)|$ is a doubling measure, whereas f need not be quasiconformal.

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