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Rotation sets and monotone periodic orbits for annulus homeomorphisms

PHILIP BOYLAND*

Abstract. If f is a homeomorphism of the annulus and p/q is a rational in lowest terms that is contained in the rotation set of f then f has a (p, q)-topologically monotone periodic orbit. In addition, if f has a p/q-period orbit that is not topologically monotone then the Farey interval of p/q is contained in the rotation set of f.

Section 1

There are many theorems which give information about periodic orbits for maps of the annulus. The Poincaré–Birkhoff theorem implies that an area preserving homeomophism always has periodic orbits with rotation numbers equal to every rational number between the rotation numbers of the boundary circles. With the addition of the monotone twist hypothesis, one gets more information. In this case, the Aubry–Mather theorem yields the existence of periodic orbits whose radial order is preserved by the map. Such orbits are called monotone or Birkhoff ([K]).

Using topological techniques Hall showed that whenever a monotone twist map has a p/q-periodic orbit, it has a monotone p/q-periodic orbit ([H1]). In [H2] he pointed out the appropriate generalization of this notion to a homeomorphism of the annulus and asked whether the appropriate version of the Aubry–Mather theorem for periodic orbits was true in this more general context. Recently, Le Calvez answered this to the affirmative under the area preserving hypothesis ([LC]). This paper contains a proof of the general case. In addition, it is shown that if the homeomorphism has a p/q-periodic orbit that is not "topologically monotone" then a certain interval depending on simple arithmetic properties of p/q must be in the rotation set. This second result was proved for monotone twist maps without the area preserving hypothesis in [Bd1] using Lemma 4 of [B + H].

The main tool used in the proofs is the Thurston–Nielsen theory of surface automorphisms. The proofs are clarified and simplified by encoding the theorem in

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a statement about a partial order on the set of periodic orbits for homeomorphisms of the annulus. This partial order is the analog of the partial order in Sharkovski's Theorem. For background information on this partial order on braid types see [Bd2].

If p and q are relatively prime integers, define m/n to be the maximum of $\{r/s : r/s < p/q,\ s < q \text{ and } (r, s) = 1\}$ and k/l to be the minimum of $\{r/s : r/s > p/q,\ s < q \text{ and } (r, s) = 1\}$. The *Farey interval* of p/q is $I(p/q) = [m/n, k/l]$. The only property of Farey intervals we will use is that the endpoints of the Farey interval satisfy $mq - pn = -1$ and $kq - pl = 1$. The endpoints of the Farey interval are the last two convergents of the continued fraction of p/q if the last partial quotient is made equal to one (see [HW], chapter 3). The notation $\rho(f)$ means the rotation set of f.

MAIN THEOREM. *If f is an orientation and boundary preserving homeomorphism of the annulus and $p/q \in \rho(f)$ with p and q positive and relatively prime then f has a (p, q)-topologically monotone periodic orbit. If f has a (p, q)-orbit that is not topologically monotone then $I(p/q) \subseteq \rho(f)$.*

Recalling the interpretation of rotation number as a frequency of oscillation, one could make an analogy between this result and the fact that an oscillating physical system or field that supports a complicated vibration at a given frequency must also support the simplest, least excited state with that frequency (cf. [Hg]).

It is natural to ask whether there is a similar theorem about irrational numbers in the rotation set. One version of this might be: If α is an irrational number with $\alpha \in \rho(f)$ does f have a minimal set on which the dynamics are semiconjugate to rigid rotation on a circle? Under the monotone twist hypothesis the answer is yes. One way to prove this is by taking Hausdorff limits of monotone periodic orbits ([K]).

For a general homeomorphism the answer is no. In [Hn1], Handel constructed an area preserving, C^∞-diffeomorphism, h, with an irrational $\alpha \in \rho(h)$ such that every point with rotation number α is contained in a single minimal set, X. The set X is the Hausdorff limit of topologically monotone periodic orbits but h restricted to X is not semiconjugate to rigid rotation by α on a circle. The set X is topologically a pseudocircle. This leads to: What type of minimal sets with rotation number α must f possess when α is an irrational in the rotation set of f (cf [Bd4])?

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Section 2

We begin with some basic notation and definitions. A homeomorphism f of the annulus $A = S^1 \times [0, 1]$ will always be isotopic to the identity. A lift to the universal cover $\tilde{A} = \mathbb{R} \times [0, 1]$ is denoted $\tilde{f}$, and $T : \tilde{A} \to \tilde{A}$ given by $T(x, y) = (x + 1, y)$ is the deck transformation. The projection onto the first factor is $\pi_x : \tilde{A} \to \mathbb{R}$.

Given $\tilde{x} \in \tilde{A}$ we define its rotation number under $\tilde{f}$ as

$$\rho(\tilde{x},\tilde{f}) = \lim_{n\to\infty} \frac{\pi_x(\tilde{f}^n(\tilde{x})) - \pi_x(\tilde{x})}{n}$$

if the limit exists. Note that we include the dependence on the lift in our definition of rotation number. The set of rotation numbers of $\tilde{f}$ is

$$\rho(\tilde{f}) = \{\rho(\tilde{x},\tilde{f}) : \tilde{x} \in \tilde{A}\}.$$

A theorem of Handel ([Hn3]) states that $\rho(\tilde{f})$ is a closed set. A periodic orbit, $o(x)$, is called a (p, q)-periodic orbit if its period is q and there are lifts $\tilde{x}$ and $\tilde{f}$ with $T^{-p}\tilde{f}^q(\tilde{x}) = \tilde{x}$. Note that if x is a (p, q)-periodic orbit there always exists a p' with $0 \le p'/q < 1$ such that x is also a (p'/q)-periodic orbit.

Remark. One may define the rotation interval for any point $\tilde{x}$ as

$$\bar{\rho}(\tilde{x},\tilde{f}) = \left[\liminf_{n\to\infty} \frac{\pi_x(\tilde{f}^n(\tilde{x})) - \pi_x(\tilde{x})}{n}, \limsup_{n\to\infty} \frac{\pi_x(\tilde{f}^n(\tilde{x})) - \pi_x(\tilde{x})}{n}\right]$$

and then define $\bar{\rho}(\tilde{f}) = \bigcup \bar{\rho}(\tilde{x},\tilde{f})$. A theorem of Franks ([F1]) and Handel ([Hn2]) shows that $p/q \in \bar{\rho}(\tilde{f})$ with p and q relatively prime implies that f has a (p, q)-periodic orbit. Thus $\bar{\rho}(\tilde{f}) \cap \mathbb{Q} = \rho(\tilde{f}) \cap \mathbb{Q}$ and since $\rho(\tilde{f})$ is closed, a simple argument yields $\bar{\rho}(\tilde{f}) = \rho(\tilde{f})$.

We next define braid types and the partial order. Fix a copy of the annulus minus n interior points and call it A_n. The group of isotopy classes of orientation preserving homeomorphisms of A_n is denoted G_n. The isotopies fix the boundary setwise but not necessarily pointwise. If $f : A \to A$ is a homeomorphism with a period n-periodic orbit $o(x,f) \subseteq \mathrm{Int}\,(A)$, let $A_x = A - o(x,f)$ and $f_x = f_{|A_x}$. Pick a homeomorphism $h : A_x \to A_n$ and let $[hf_xh^{-1}]$ denote the isotopy class in G_n. Define the braid type of $o(x, f)$, denoted $bt(x, f)$, to be the conjugacy class of $[hf_x h^{-1}]$ in G_n. By passing to the conjugacy class we have made $bt(x, f)$ independent of the choice of homeomorphism h. For $o(x, f)$ contained in a boundary circle of A, we have to modify the definition of braid type slightly. If, for example, $o(x, f) \subseteq S^1 \times \{1\}$ we let $\bar{A} = S^1 \times [0, 1 + \epsilon]$ for some $\epsilon > 0$ and choose $\bar{f}: \bar{A} \to \bar{A}$, a homeomorphism extending f. Define $bt(x, f) = bt(x, \bar{f})$ and note that this is independent of the choice of the extension. The set of all possible braid types on the annulus is called BT. Given a homeomorphism $f: A \to A$ define its set of braid types as $bt(f) = \{bt(x, f)$: $o(x, f)$ is a periodic orbit$\}$.

We define a relation on BT as follows. If $\alpha, \beta \in BT$, $\alpha \geq \beta$ if and only if $\alpha \in bt(f)$ implies $\beta \in bt(f)$ for all homeomorphisms f. The first proposition says that this relation is, in fact, a partial order. This is stated without proof in [Bd2]. It relies on a result of Brunovsky [Br] which says that, given $N > 0$, an isotopy between Kupka–Smale diffeomorphisms f_0 and f_1 can always be approximated by a diffeotopy with the property that all orbits of period less than N undergo only saddle node and flip bifurcations. Further, all bifurcations involving orbits of period less than N occur at a finite number of distinct parameter values.

PROPOSITION 1. *The relation* $(BT, \leq)$ *is a partial order.*

Proof. It is obvious from the definition that $\alpha \leq \alpha$ and that $\alpha \leq \beta$ and $\beta \leq \gamma$ implies $\alpha \leq \gamma$. We must therefore show that $\alpha \leq \beta$ and $\beta \leq \alpha$ implies $\alpha = \beta$. For this it suffices to show that $\alpha \geq \beta$ and $\alpha \neq \beta$ implies the existence of some f with $\beta \in bt(f)$ but $\alpha \notin bt(f)$, i.e. $\beta \ngeq \alpha$.

Using standard constructions (e.g. section 6 of [F3]) we may find Axiom A (and thus Kupka–Smale) maps f_0 and f_1 with $\alpha \notin bt(f_0)$ and $\beta \notin bt(f_0)$ and $\beta \in bt(f_1)$. Let N be larger than the period of α or β. Use Brunovsky's theorem to obtain a nice diffeotopy, f_μ, between f_0 and f_1. Now let $\mu_0 = \inf\{\mu: \alpha \in bt(f_\mu)$ or $\beta \in bt(f_\mu)\}$. There must be a bifurcation occurring for f_{μ_0}. If this bifurcation is a saddle node, then the two orbits created as μ increases have the same braid type which must be β as $\alpha \geq \beta$. If the bifurcation is a flip, then the orbit that persists through the bifurcation was present for $\mu < \mu_0$ and thus cannot have type α or β. The doubled orbit must therefore have type β. In either case, picking $\epsilon > 0$ small enough to that no bifurcation of period less than N happens between μ_0 and $\mu_0 + 2\epsilon$, we have $\beta \in bt(f_{\mu_0+\epsilon})$ but $\alpha \notin bt(f_{\mu_0+\epsilon})$. □

Since a braid type is essentially an isotopy class, in order to understand this partial order one needs a good understanding of isotopy classes on surfaces. This is provided by the Thurston–Nielsen Theory (see [T] or [FLP] for more details). This theory provides a "prime decomposition theorem" for isotopy classes. If a class is irreducible, it is either finite order or pseudo-Anosov. In the first case there is a map

ϕ in the class that satisfies $\phi^n = \text{id}$ for some n. In the second case, there is a pseudo-Anosov map ϕ in the isotopy class. As the Thurston type of an isotopy class is unchanged by conjugacy, it makes sense to speak of reducible, irreducible, finite order and pseudo-Anosov braid types.

Following Handel ([Hn3]) we say that a map is "pseudo-Anosov relative to a finite invariant set K if it satisfies all the properties of a pseudo-Anosov homeomorphism except the associated stable and unstable foliations may have 1-pronged singularities at points in K". Note that these maps are the same as the "generalized pseudo-Anosov maps" of Geber and Katok ([GK]).

There are several features of pseudo-Anosov maps that will be important here. The first two follow from the existence of a Markov partition ([FLP], exposé 10). First, pseudo-Anosov maps have finitely many periodic orbits of each period and second, each point is nonwandering under iteration. Another useful property is that $\phi^n(\Gamma) \not\simeq \Gamma$ for all $n \neq 0$ when Γ is any homotopically nontrivial simple closed curve that is not boundary parallel ([M] and [HT]). This implies that if $\phi : A \to A$ is pseudo-Anosov rel a periodic orbit, $o(x,f)$, and γ is a simple arc connecting two points on $o(x,f)$, then $\phi^n(\gamma) \not\simeq \gamma$ for all $n \neq 0$. One can see this by choosing a smooth model for ϕ (using [GK]), blowing up the points of $o(x,f)$ to boundary circles (see [Hn4]) and then observing that the behaviour of γ under iteration is easily understood by examining the behaviour of a simple closed curve Γ obtained from γ as shown in figure 1.

A braid type, β, is said to be of type (p, q) if one (and hence all) periodic orbits, $o(x,f)$, with $bt(x,f) = \beta$ are of type (p, q). If β is a pseudo-Anosov (p, q)-braid type we say that the map ϕ with lift $\tilde{\phi}$ *represents* β if $\phi : A \to A$ has a periodic orbit, $o(x)$, with $bt(x, \phi) = \beta$ and $T^{-p}\tilde{\phi}^q(\tilde{x}) = \tilde{x}$ for any lift $\tilde{x}$ of x and ϕ is pseudo-Anosov rel $o(x)$. Handel has shown ([Hn3]) that $\rho(\tilde{\phi})$ is a closed interval (closely related results were proved by Fried [Fr1], Lemma 3 and [Fr2], Theorem 4). We may

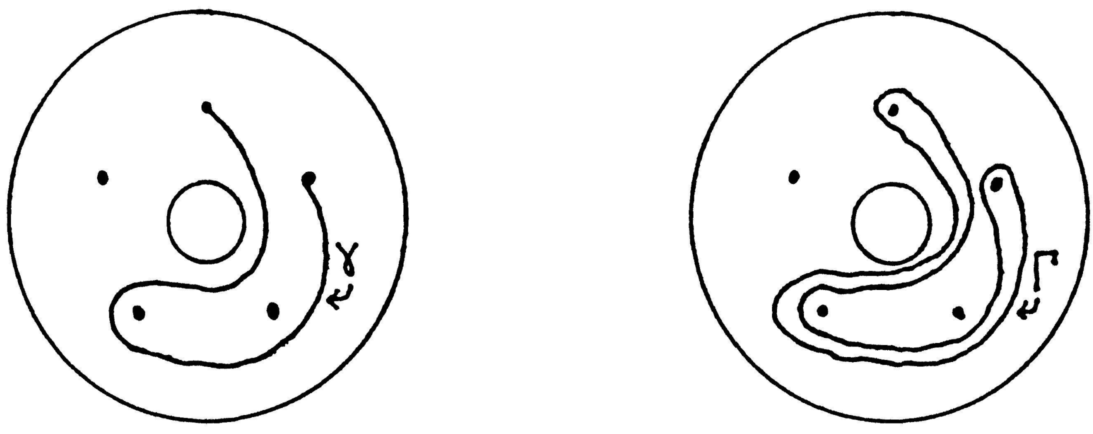

Figure 1

thus define the rotation interval of β as $pi(\beta) = \rho(\tilde{\phi})$. Since any two isotopic pseudo-Anosov maps are conjugate ([FLP], exposé 12), this definition is independent of the choice of ϕ, however, it does depend on the choice of $\tilde{\phi}$. More precisely, it depends on the fact that we have treated β as a (p, q)-braid type and ignored the fact that β is also a $(p + kq, q)$-braid type for any $k \in \mathbb{Z}$. To avoid this ambiguity we shall always assume that the p and q in the notation "(p, q)-braid type" and "(p, q)-perodic orbit" satisfy $0 < p/q < 1$.

Braid types will be used to define the notion of a topologically monotone periodic orbit. If $\tilde{T}_{p/q}: \tilde{A} \to \tilde{A}$ is defined by $z \mapsto z + (p/q, 0)$, let $T_{p/q}: A \to A$ be the projection. Define $\alpha_{p/q} \in BT$ to be the braid type of a periodic orbit of $T_{p/q}$. A periodic orbit is topologically monotone if $bt(x, f) = \alpha_{p/q}$. One can check that if f is monotone twist, a periodic orbit is *topological monotone* if and only if it is Birkhoff in the usual sense ([H2]). For the braid type $\alpha_{p/q}$, we define $pi(\alpha_{p/q}) = p/q$.

The next proposition collects together some useful properties of the objects we have defined. Parts (1) and (2) are essentially folklore. They are contained in [Bd3] and were also known to Smillie. However, they do not seem to have a published proof.

PROPOSITION 2. *Let $\beta \in BT$ be a (p, q)-braid type with p and q positive and relatively prime then*

(1) *β is irreducible.*

(2) *$\beta = \alpha_{p/q}$ if and only if β is finite order.*

(3) *$\beta \in bt(f)$ implies $pi(\beta) \subseteq \rho(f)$.*

(4) *If β is pseudo-Anosov and ϕ represents β then $bt(\phi) = \{\sigma \in BT: \sigma \leq \beta\}$.*

Proof. (1) Assume that β is reducible. This implies the existence of a homeomorphism $\psi: A \to A$ with a periodic orbit $o(x, \psi)$ with $bt(x, \psi) = \beta$ and a family of simple closed curves $\Gamma = \{\Gamma_1, \ldots, \Gamma_k\}$ that are pairwise disjoint and nonhomotopic and satisfy $\Gamma \subseteq A - o(x, f)$ and $\psi(\Gamma) = \Gamma$. In addition, if D_i is the disk bounded by Γ_i, D_i contains at least 2 but no more than $q - 1$ elements of $o(x, f)$. Renumbering if necessary, let Γ_1 be such that $x \in B_1$ and $\Gamma_i \cap B_1 = \emptyset$ if $i \neq 1$.

Let ∂_1 denote the inner boundary of A. Now if ∂_1 was contained in B_1 then since $\psi(\partial_1) = \partial_1$ we would have $\psi(B_1) = B_1$ and thus $o(x, \psi) \subseteq B_1$, a contradiction. We may therefore lift B_1 to a compact $\tilde{B} \subset \tilde{A}$. If we let k be the least positive integer with $\psi^k(\Gamma_1) = \Gamma_1$, then $\tilde{\psi}^k(\tilde{B}) = \tilde{B} + (m, 0)$ for some integer m. It is easy to see that this implies that $\rho(x, \tilde{\psi}) = m/k$. But since x is (p, q)-periodic for $\tilde{\psi}$, $p/q = m/k$ and thus since p and q are relatively prime, $q = k$. But using the definition of k, this implies that $o(x, \psi) \cap B_1 = x$, a contradiction.

(2) Once again, let $\psi: A \to A$ be a homeomorphism with a periodic orbit $o(x, \psi)$ with $bt(x, \psi) = \beta$. If $\beta = \alpha_{p/q}$ then $\psi^q \simeq \mathrm{id}$ rel $o(x, \psi)$, and thus β is finite

order. On the other hand, if β is finite order, we may find a ψ that represents β with $\psi^q = \mathrm{id}$. If we let D denote the disk obtained by collapsing the inner boundary of A to a point, the induced map $\psi' : D \to D$ is a homeomorphism that satisfies $\psi'^q = \mathrm{id}$. Using a theorem of Brouwer ([Bw]), Karekjarto ([Kj]) and Eilenberg ([E]) this implies that ψ' is topologically conjugate to $R_{p/q}$ and thus $\beta = \alpha_{p/q}$.

(3) This is a direct consequence of Proposition 1.2 of [Hn3].

(4) This is a direct consequence of the application on page 531 of [TH]. □

Section 3. The main result on $(BT, \leq)$ is

THEOREM 1. *If β is a (p, q)-braid type then $\beta \geq \alpha_{p/q}$. Further,* if *$\beta \neq \alpha_{p/q}$ then $I(p/q) \subset pi(\beta)$.*

The main theorem stated in the introduction is an easy consequence of Theorem 1, the definition of $\leq$ and Proposition 2. For the proof of Theorem 1 we shall need the following lemma. Its proof came from a discussion with John Franks.

LEMMA 1. *If $f: \tilde{A} \to \tilde{A}$ is an orientation and boundary preserving homeomorphism with $fT = Tf$ and $0 \notin$ convex hull $(\rho(f))$ then $\tilde{A}/f$ is an annulus.*

Proof. A theorem of Franks ([F1]) states that if f has a nonwandering point it has a fixed point. Thus $0 \notin \rho(f)$ implies that given $x \in \tilde{A}$ there exists an $\varepsilon > 0$ so that for all $n \neq 0$, $f^n(B_\varepsilon(x)) \cap B_\varepsilon(x) = \emptyset$. This gives charts for $\tilde{A}/f$.

The main work lies in showing that $\tilde{A}/f$ is Hausdorff. For this we must show that given x and $y \notin o(x, f)$, there exists $\varepsilon > 0$ so that $f^n(B_\varepsilon(x)) \cap f^m(B_\varepsilon(y)) = \emptyset$ for all m and n. Equivalently, $B_\varepsilon(y) \cap f^n(B_\varepsilon(x)) = \emptyset$ for all n. We assume that $\rho(f) \subset (0, \infty)$ (the other case being similar) and claim there exists an M such that for all z and $|n| > M$,

$$|\pi_x(f^n(z)) - \pi_x(z)| > 2.$$

This claim easily implies that for any $\epsilon < 1/2$ and $|j| > M$, $B_\varepsilon(y) \cap f^j B_\varepsilon(x)) = \emptyset$. While for $|j| \leq M$, continuity and the fact that $x \notin o(y)$ imply that there exists an $\varepsilon < 1/2$ with $f^j(B_\varepsilon(x)) \cap B_\varepsilon(y) = \emptyset$. Thus granted the claim we are done.

To prove the claim, note that $\rho(f) \subset (0, \infty)$ implies that there exists a $\delta > 0$ so that for any $z \in \tilde{A}$ there is an $N'(z)$ so that

$$\frac{\pi_x(f^n(z)) - \pi_x(z)}{n} > \delta$$

for all $n \geq N'(z)$. Thus using continuity, there exists $\varepsilon(z)$ and $N(z)$ so that $d(y, z) < \varepsilon(z)$ implies

$$\pi_x(f^{N(z)}(y)) - \pi_x(y) \stackrel{\text{def}}{=} D(y, f^{N(x)}) > 2.5.$$

Thus using compactness, there are z_i, ε_i and N_i for $i = 1, \ldots, k$ so that $\bigcup B_{\varepsilon_i}(z_i)$ covers $[0, 1] \times [0, 1] \subseteq \tilde{A}$ and $x \in B_{\varepsilon_i}(z_i)$ implies that $D(z, f^{N_i}) > 2$. Let $B_{\varepsilon_i}(z_i) = B_i$.

Now let $\bar{N} = \max\{N_i\}$ and $C = \sup\{D(z, f^j): 0 \leq j \leq \bar{N}$ and $z \in \tilde{A}\}$. Since $Tf = fT$ and A is compact, C is finite. Pick $m > \max\{C, 2\}$ and let $M = m\bar{N}$.

Given z and $n > M$, let i_0 be so that $z \in B_{i_0}$ and let $z_1 = f^{N_{i_0}}(z)$. Next let i_1 be so that z_1 is contained in an integer translate of B_{i_1} and let $z_2 = f^{N_{i_1}}(z_1)$. Continue until i_k which satisfies

$$\sum_{l=1}^{k-1} N_{i_l} < n \qquad \text{but} \sum_{l=1}^{k} N_{i_l} \geq n.$$

Define j by $j = n - \Sigma_{l=1}^{k-1} N_{i_l}$. We then have

$$D(z, f^n) = D(z, f^{N_{i_0}}) + \cdots + D(z_{k-1}, f^{N_{i_{k-1}}}) + D(z_k, f^j) > 2m - C > 2$$

which proves the claim.

Thus $\tilde{A}/f$ is a compact surface. Its easy to check that its first homology is $\mathbb{Z}$ and so its an annulus. □

Proof of Theorem 1. We prove the second assertion first. By virtue of Proposition 2(1) and (2), β is pseudo-Anosov. Let ϕ and $\tilde{\phi}$ represent β with $bt(x, \phi) = \beta$. If we let $I(p/q) = [m/n, k/l]$ then since $pi(\beta)$ is a closed interval, it suffices to show that $\{m/n, k/l\} \in \rho(\tilde{\phi})$.

We proceed by contradiction and assume that $m/n \notin \rho(\tilde{\phi})$, the case $k/l \notin \rho(\tilde{\phi})$ is similar. By Lemma 1, $\tilde{A}/\tilde{\phi}^n T^{-m}$ is an annulus which we denote B. Let $X \subset \tilde{A}$ be the total lift of $o(x, \phi)$. Because $o(x, \phi)$ is of type (p/q), the elements of X can be labeled as $\{\tilde{x}_i\}$ which satisfy $\tilde{\phi}(\tilde{x}_i) = \tilde{x}_{i+p}$ and $T(\tilde{x}_i) = \tilde{x}_{i+q}$. Since m/n is an endpoint of the Farey interval of p/q, $mq - np = -1$ and so $\tilde{\phi}^n T^{-m}(x_i) = x_{i+1}$. Thus if $\pi: \tilde{A} \to B$ is the projection, $\pi(X)$ is a single point.

Because T and $\tilde{\phi}$ commute, $T^{-p}\tilde{\phi}^q$ induces a map on B, which will be denoted ψ. It is clear that $\psi(\pi(X)) = \pi(X)$. As is well known, since $\pi(X)$ is a single point and ψ is orientation preserving this implies that $\psi \simeq \text{id rel } \pi(X)$ (where, as above, we allow isotopies that may not fix the boundary pointwise). One way to finish the proof is to obtain a contradiction using results of Fried which show that ψ is flow equivalent to ϕ ([Fr3], pp. 561–562) and is thus psuedo-Anosov ([Fr4], Lemma on Page 261). A direct proof can be given as follows.

Let $\tilde{\gamma} : [0, 1] \to \tilde{A}$ be a simple arc connecting $\tilde{x}_1$ to $\tilde{x}_2$ with $\tilde{\gamma}((0, 1)) \cap X = \emptyset$. Since ϕ and thus ϕ^q is pseudo-Anosov, then using the remarks before Proposition 2, $T^{-p}\tilde{\phi}^q(\tilde{\gamma})$ is not isotopic to $\tilde{\gamma}$ rel X. On the other hand, if γ is the projection of $\tilde{\gamma}$ to B, then since $\psi \simeq \text{id}$ rel $\pi(X)$, we have $\psi(\gamma) \simeq \gamma$ rel (X). Lifting this isotopy to $\tilde{A}$ yields a contradiction.

To prove the first assertion of the theorem, we may assume $\beta \neq \alpha_{p/q}$ and thus it is pseudo-Anosov. Let ϕ, $\tilde{\phi}$ and x continue to be as in the first paragraph of the proof. By the second assertion of the theorem which we have just proved, p/q is in the interior of $\rho(\tilde{\phi})$. Further, as noted in Section 2, ϕ has finitely many periodic orbits of each period and each point of A is nonwandering under ϕ. Under these hypotheses, Theorem 3.3 of [F2] implies that ϕ has at least two (p, q)-periodic orbits in the interior of A with nonzero Lerschetz index. One of these orbits might be $o(x)$ (actually not, see remarks after the proof) and we denote the second by $o(y)$.

Since $o(y)$ is a periodic orbit of a pseudo-Anosov map by Proposition 2(4), $bt(y, \psi) \leq bt(x\ \psi)$. On the other hand, an argument using Brunovsky's theorem similar to the proof of Proposition 1 shows that for each braid type γ, there is a diffeomorphism of the annulus with only one periodic orbit of braid type γ. Thus, in particular, $bt(x, \phi) \neq bt(y, \phi)$. Summarizing, we have shown that if β is a pseudo-Anosov (p, q)-braid type then there is another (p, q)-braid type $\gamma \neq \beta$ with $\gamma \leq \beta$.

Now let $S = \{\sigma \in BT\colon \sigma \text{ is of type } (p, q) \text{ and } \sigma \leq \beta\}$. As noted in Section 2, pseudo-Anosov maps have only finitely many periodic orbits of each period. Thus using Proposition 2(4), S is finite. Let $\alpha \in S$ be minimal in S, i.e. it satisfies $\gamma \leq \alpha$ implies $\gamma = \alpha$ for any $\gamma \in S$. Using the result of the last paragraph, α is not pseudo-Anosov so by Proposition 2(1) and (2), it is finite order and $\alpha = \alpha_{p/q}$. □

Remarks

(1) With reference to the 5th paragraph of the proof, we can apply the Euler–Poincaré formula to the invariant foliation of ϕ (after blowing down the boundary circles to points). This yields $\Sigma\, 2 - \bar{p} = 2\chi(S^2) = 4$ where the sum is over the singularities of the foliation and $\bar{p}$ is the number of prongs at the singularity ([FLP], pg. 75). Since $o(x)$ is a periodic orbit, all the singularities at points of $o(x)$ must have the same number of prongs. There must be only one prong to obtain $\Sigma\, 2 - \bar{p} = 4$. On the other hand, the Lefschetz index of a one-pronged singularity that is fixed by a pseudo-Anosov map must be zero. Thus both the periodic orbits given by Franks theorem are different from $o(x)$.

(2) The proof given above of the second assertion of Theorem 1 was inspired by the alternative characterization of flow equivalence given on page 561 of [Fr3]. The proof given has the merit of being somewhat self-contained. Also, Lemma 1 perhaps has some independent interest.

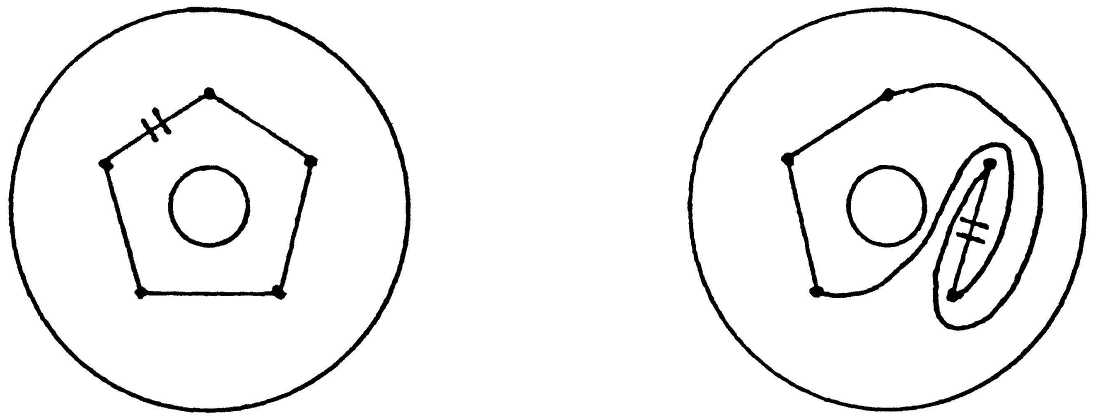

Figure 2. The braid type $\beta^+_{2/5}$.

One can also obtain the result directly using flow equivalence. Briefly, the fact that $m/n \notin \rho(\tilde{\phi})$ implies that the cohomology class $u \in H^1(M_\phi)$ given by $u(av_1 + bv_2) = na - mb$ is strictly positive on the homology directions of ϕ. (Here M_ϕ is the suspension manifold of ϕ and $H_1(M_\phi)$ has generators v_1 in the annulus direction and v_2 in the flow direction.)

This implies (Theorem D of [Fr1]) the existence of a cross section to the suspension flow that represents u. The value of the cohomology class just represents the number of intersections of a homology class with the cross section. Thus, the fact that $mq - np = 1$ means that the suspension of $o(x)$ hits the cross section in just one point. The return map to this cross section is conjugate to the map ψ in the proof.

(3) The inclusion $I(p/q) \subset \rho i(\beta)$ is, in general, the best one can do. There is a braid type with $I(p/q) = \rho i(\beta)$. Define $\beta^+_{p/q}$ to be the braid type obtained by rigidly rotating by p/q and then doing a Dehn twist around two adjacent points on an orbit. (See figure 2, the arcs between points are included to indicate the action of the complement of the orbit). The braid type $\beta^+_{p/q}$ is the same as $ot(m/n, k/l)$ given in [Bd2] (where $I(p/q) = [m/n, k/l]$). For this braid type one can compute an invariant train track and compute that $\rho(\phi) = I(p/q)$ directly.

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