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Rational tori, semisimple orbits and the topology of hyperplane complements

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0. Introduction

Let G be a connected reductive group defined over the finite field $\mathbb{F}_q$ of q elements. Let F be the associated Frobenius endomorphism of G and for any F-stable subgroup H of G write H^F for the fixed-point subgroup of F on H. If $\mathscr{G}$ is the Lie algebra of G, we also write F for the corresponding Frobenius endomorphism of $\mathscr{G}$. It is well known that the G^F-conjugacy classes of F-stable maximal tori of G correspond to F-conjugacy classes in the Weyl group W of G. Thus it might be expected that functions which are defined on F-stable maximal tori (e.g. functions which count rational tori or rational semisimple classes or elements) may be evaluated in terms of the character theory of W. We give two general formulae of this nature and apply them to various situations, among which is the particular case of counting regular semisimple classes in G^F (resp. orbits in the Lie algebra $\mathscr{G}^F$), both with and without sign. A corollary of our results is that if ε is the alternating character of W, and M_W is the complexified hyperplane complement corresponding to W (cf. [L1]) then ε does not occur in the cohomology modules $H^*(M_W)$. This generalises a combinatorial result of Stanley (cf. [S], [LS]), which amounts to this statement for type A. The corresponding combinatorial statements are obtained in the general case by using the work of Orlik-Solomon [OS] to deduce that ε does not occur in $H^*(L(\mathscr{A}_W))$ where $L(\mathscr{A}_W)$ is the lattice of intersections of the hyperplanes in the arrangement $\mathscr{A}_W$ which corresponds to W.

As a corollary of our method, we obtain the curious statement that if G is semisimple, the number of regular semisimple classes of G^F is always odd, regardless of the characteristic. Another consequence of our method is a direct and simple proof that the number of unipotent (resp. nilpotent) elements in G^F (resp. $\mathscr{G}^F$) is equal to the number of F-stable maximal tori of G^F and that these numbers are equal to q^{2N}, where $2N$ is the number of roots of G with respect to a maximal torus. This proof depends on the fact (cf. (1.16) below) that the sum of the cardinalities of the rational points of the F-stable maximal tori of G is equal to $|G^F|$, and the corresponding result for $\mathscr{G}$. Since this treatment is independent of the Steinberg representation, it provides an alternative approach to the Steinberg character through the invariant theory of W.

In the final section (§5), we define maps from the set of rational semisimple classes of G (resp. orbits of $\mathscr{G}$) to the conjugacy classes of W. This amounts to a "rational classification" of semisimple classes. Our earlier results are then applied to give a formula for the number of semisimple classes (resp. orbits) of a given type. We compute this explicitly for split semisimple orbits and Coxeter semisimple orbits (i.e. rational semisimple orbits which correspond respectively to the trivial and Coxeter class of W). The split case leads to a divisibility property for the set $\{m_1, \ldots, m_l\}$ of exponents of a Weyl group which may classify an only slightly larger set of integer sequences. The Coxeter case also leads to some interesting number theoretical observations.

1. Some computations in the ring $A(W)$

Let $C(W)$ be the ring of complex-valued class functions on the finite group W.

(1.1) DEFINITION. Let t be an indeterminate. Then define

$$A(W) = C(W)[t, t^{-1}]$$

and

$$B(W) = C(W)[[t, t^{-1}]].$$

Clearly $A(W)$ is a subring of $B(W)$. In $C(W)$ we have the usual inner product of class functions: for $\phi, \psi \in C(W)$, $\langle \phi, \psi \rangle_W = |W|^{-1} \Sigma_{w \in W} \phi(w)\bar{\psi}(w)$. If $\alpha = \Sigma_{i \in \mathbb{Z}} \alpha_i t^i$ and $\beta = \Sigma_{i \in \mathbb{Z}} \beta_i t^i$ are elements of $B(W)$, we write $\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle_W = \Sigma_{i \in \mathbb{Z}} \langle \alpha_i, \beta_i \rangle_W t^i \in \mathbb{C}[[t, t^{-1}]]$. If $\alpha, \beta \in A(W)$, then clearly $\langle \alpha, \beta \rangle \in \mathbb{C}[t, t^{-1}]$. For $w \in W$ and $f \in B(W)$, we write $f(w) = \Sigma_{i \in \mathbb{Z}} f_i(w) t^i \in \mathbb{C}[[t, t^{-1}]]$, where $f = \Sigma_{i \in \mathbb{Z}} f_i t^i$. Again if $f \in A(W)$, then $f(w) \in \mathbb{C}[t, t^{-1}]$.

Note that $A(W)$ and $B(W)$ may be thought of as the rings of class functions on W with values in $\mathbb{C}[t, t^{-1}]$ and $\mathbb{C}[[t, t^1]]$ respectively, and the inner products above are just the usual ones. Sometimes we shall "evaluate" $f(t) \in A(W)$, by substituting $a \in \mathbb{C}^*$ for t. We then write $f(a; w) = \Sigma_{i=-k}^{k} f_i(w) a^i$, where $f(t) = \Sigma_{i=-k}^{k} f_i t^i$.

The most common source of elements of $B(W)$ is revealed in

(1.2) DEFINITION. Let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a $\mathbb{Z}$-graded W-module, where the M_i are finite dimensional. Define the corresponding element of $B(W)$ by

$$P_M(t) = \sum_{i \in \mathbb{Z}} \mu_i t^i,$$

where μ_i is the trace of W on M_i.

(1.3) LEMMA. Let V be a finite dimensional complex vector space which is a module for the finite group W. Let $\Lambda = \Lambda(V)$ and $S = S(V)$ be the exterior and symmetric algebras on V respectively. Then Λ and S are graded $\mathbb{C}W$-modules and we have the following formula in $B(W)$.

$$P_\Lambda(-t)P_S(t) = 1.$$

Proof. We have, for $w \in W$,

$$P_\Lambda(-t; w)^{-1} = \det(1 - wt)^{-1}$$

$$= \prod_{i=1}^{r} \frac{1}{1-\lambda_i t} \quad \text{where } \lambda_1, \ldots, \lambda_r \text{ are the eigenvalues of } \omega \text{ on } V$$

$$= \prod_{i=1}^{r} (1 + \lambda_i t + \lambda_i^2 t^2 + \cdots)$$

$$= P_S(t, w). \qquad \square$$

The formula of (1.3) clearly holds also when V is a real vector space, or more generally, when V is defined over any subfield of $\mathbb{C}$, since the coefficients of all the power series treated lie in the smaller field.

Now take G to be as in the introduction: G is connected, reductive and defined over $\mathbb{F}_q$. Let T_0 be a fixed maximally split maximal torus of G and let W be the Weyl group of G with respect to T_0, i.e. $W = N_G(T_0)/T_0$. Since F stabilizes T_0, it defines an action on W, which we also denote by F. Now both W and F act on the real vector space $V = Y(T_0) \otimes_{\mathbb{Z}} \mathbb{R}$ where $Y(T_0)$ is the cocharacter group of T_0. Moreover it is known (see, e.g. [C, §2.9] or [St1, §11]) that the F-action on V is given by $F = qF_0$, where F_0 is an automorphism of finite order and q is the prime power introduced above.

If ${}^gT_0 = gT_0g^{-1}$ is F-stable, then $F(g)^{-1}g \in N = N_G(T_0)$. Moreover if $\eta: N \to W$ is the canonical map, the G^F-conjugacy class of ${}^gT_0$ is determined by the F-conjugacy class of $\eta(F(g)^{-1}g)$, where w_1 and w_2 ($\in W$) are said to be F-conjugate if $w_2 = F(x)w_1x^{-1}$ for some $x \in W$. Thus, if $F(g)^{-1}g \in N$ and $\eta(F(g)^{-1}g) = w \in W$, we say that ${}^gT_0$ is "w-twisted", or obtained from T_0 by twisting by $w \in W$. Note that here w is determined only up to F-conjugacy in W, so we sometimes write "(w)-twisted" where (w) denotes the F-conjugacy classes of $w \in W$. In this case, the action of F on ${}^gT_0$ is equivalent to the action of $w^{-1}F$ on T_0 (since $F(\operatorname{Int} g(x)) = \operatorname{Int} g(w^{-1}F(x))$). The facts in the last two paragraphs may all be found in [Sp–St].

The next result is well-known (see, e.g. [C, (3.3)] and [Sp–St, §(2.7)]).

(1.4) LEMMA. (i) *Let* $T = T_w$ *be an* F*-stable maximal torus of* G *which is obtained from* T_0 *by twisting by* $w \in W$. *Then* $|T_w^F| = |T_0^{w^{-1}F}| = \det_V((q - F_0^{-1}w) = q^r P_A(-q^{-1}; F_0^{-1}w)$ *where* $V = Y(T_0) \otimes_{\mathbb{Z}} \mathbb{R}$, $A = A(V)$ *is as in* (1.3) *and we regard* V *as a* $\bar{W}$*-module, where* $\bar{W} = \langle W, F_0 \rangle$.

(ii) *Let* (w) *be an* F*-conjugacy class in* W. *Then the number of* (w)*-twisted* F*-stable maximal tori of* G *is* $|G^F|/(|T_0^{wF}||C_{W,F}(w)|)$ *where* $C_{W,F}(w) = \{x \in W \mid F(x)wx^{-1} = w\}$.

We say that a subalgebra $\mathcal{T}$ of $\mathcal{G}$ is a *toral subalgebra* if it is of the form $\mathcal{T} = \text{Lie}\,(T)$, where T is a maximal torus of G. For any closed subgroup H of G, we have $\text{Lie}\,(gHg^{-1}) = \text{Ad}\, g \cdot \text{Lie}\,(H)$. It follows that all the toral subalgebras of $\mathcal{G}$ are conjugate under $\text{Ad}\, G$; however since $N_G(\text{Lie}\, T_0)$ is in general larger than $N_G(T_0)$, we do not have a bijection between the maximal tori of G and the toral subalgebras of $\mathcal{G} = \text{Lie}\, G$.

Note that since $\text{Lie}\,(T)$ is a vector space (T a maximal torus), it follows from the uniqueness of the $\mathbb{F}_q$-structure on $\text{Lie}\,(T)$ that for T F-stable,

$$|(\text{Lie}\,(T))^F| = q^r \qquad \text{where } r = \dim T = \text{rank}\, G. \tag{1.5}$$

Now let S be the symmetric algebra on $V = Y(T_0) \otimes_{\mathbb{Z}} \mathbb{R}$; let I be the ring of W-invariant elements of S and let J be the ideal of S generated by the W-invariants of positive degree. Since F_0 normalises W (on V), F_0 acts on I and so on S/J. Thus V, S, I and S/J are all $\bar{W}$-modules, where $\bar{W} = \langle W, F_0 \rangle$, and the $\bar{W}$ action preserves the grading on the last three spaces. It is known [Ch] that as W-module, S/J is the regular respresentation. The polynomial $P_{S/J}(t) \in A(\bar{W})$ is important in the representation theory of G^F. If χ is an irreducible character of W, the degree $d_\chi(q)$ of the corresponding principal series representation of G^F is a polynomial in q, called the generic degree of χ. The polynomial $\langle P_{S/J}(q), \chi \rangle_W$ is closely related to $d_\chi(q)$, and is called the fake degree of χ.

(1.6) PROPOSITION. *We have* $(S/J) \otimes I \cong S$ *as graded* $\bar{W}$*-modules.*

Proof. This is essentially a result of Chevalley [Ch, p. 781, in proof of Theorem B] who proves the statement without the $\bar{W}$-equivariance; but equivariance is also immediate from Chevalley's proof. □

(1.7) COROLLARY. *We have, in the ring* $B(\bar{W})$,

$$P_{S/J}(t)P_I(t) = P_S(t).$$

Now F_0 acts in degree-preserving fashion on I. It follows that the basic invariants (homogeneous algebraically independent generators of I) $I_1, \ldots, I_r$ may be chosen to be eigenvectors for the F_0-action. Write $d_j = \text{degree}(I_j)$ and $F_0 I_j = \varepsilon_j I_J$, where ε_j is a root of unity.

(1.8) LEMMA. (i) *For any $w \in W$ we have*

$$P_I(t; F_0^{-1}w) = \prod_{j=1}^{r} (1 - \varepsilon_i t^{d_i})^{-1}.$$

(ii) *With G as above, we have, for any $w \in W$,*

$$|G^F| = q^{2N+r} P_I(q^{-1}; F_0^{-1}w)^{-1}$$

where N is the number of positive roots and r is the rank of G.

Proof. (i) is clear from the above remarks, the fact that w acts trivially on I and the observation that since $P_I(t; F_0^{-1}w)$ is a real polynomial and the ε_i are roots of unity, we may replace ε_i by ε_i^{-1} in the given formula.

(ii) follows from the standard formula for $|G^F|$ (see, e.g. [C, §(2.9)]) together with some rearrangement, using (i) above and the relation $\Sigma_{i=1}^{r} d_i = 2N + r$. □

(1.9) COROLLARY (i) *With the above notation, we have for any $w \in W$*

$$|G^F| = q^{2N} P_{S/J}(q^{-1}; F_0^{-1}w)|T_w^F|$$

for any element $w \in W$.

(ii) *The number $\tau(w)$ of w-twisted maximal tori of G is $q^{2N} P_{S/J}(q^{-1}; F_0^{-1}w)/|C_{W,F}(w)|$.*

Proof. By (1.8)(ii), we have

$$\begin{aligned}
|G^F| &= q^{2N+r} P_I(q^{-1}; F_0^{-1}w)^{-1} \\
&= q^{2N+r} P_{S/J}(q^{-1}; F_0^{-1}w) P_S(q^{-1})^{-1} \quad \text{(by (1.7))} \\
&= q^{2N+r} P_{S/J}(q^{-1}; F_0^{-1}w) P_\Lambda(-q^{-1}; F_0^{-1}w) \quad \text{(by (1.3))} \\
&= q^{2N} P_{S/J}(q^{-1}; F_0^{-1}w)|T_w^F| \quad \text{(by (1.4)(i))}.
\end{aligned}$$

This proves (i), and (ii) follows immediately from (i) and (1.4)(i). □

We remark that since $P_{S/J}(q)$ and $P_\Lambda(q^{-1})$ lie in the ring $B(\bar{W})$, (i) is to be thought of as an equation in $B(\bar{W})$.

Suppose g is any F-class function on W, i.e. g is a function on W which is constant on F-conjugacy classes. Define an associated function $\bar{g}$ on $\bar{W} = \langle W, F_0 \rangle$ by

$$\bar{g}(x) = \begin{cases} g(w) & \text{if } x = F_0^{-1}w \ (w \in W), \\ 0 & \text{otherwise.} \end{cases}$$

It is easily checked that $\bar{g}$ is constant on the W-orbits in $\bar{W}$. Moreover if g is F-invariant then $\bar{g}$ is a class function on $\bar{W}$. In the next statement, we use the notation

$$\langle f, g \rangle_{\bar{W}} = |\bar{W}|^{-1} \sum_{x \in \bar{W}} f(x)g(x) \qquad \text{for functions } f, g \text{ on } \bar{W}.$$

(1.10) THEOREM. *With G, F, W etc. as above, let g be an F-class function on W. Regard g as a function on the F-stable maximal tori of G by defining $F(T) = g(w)$ if T is w-twisted. Then*

$$\sum_T g(t) = |F_0| q^{2N} \langle P_{S/J}(q^{-1}), \bar{g} \rangle_{\bar{W}}$$

where the sum is over the F-stable maximal tori T of G, $2N$ is the number of roots of G, $\bar{W} = \langle W, F_0 \rangle$, F_0 is the periodic automorphism of the cocharacter group induced by F and $\bar{g}$ is the function on $\bar{W}$ associated with g (see preamble above).

Proof. We have $\Sigma_T g(T) = \Sigma_{(w)} g(w)\tau(w)$, where $\tau(w)$ is the number of (w)-twisted maximal tori of G and the sum is over the F-conjugacy classes in W. The number of elements of W in the F-conjugacy class (w) is $|W| \, |C_{W,F}(w)|^{-1}$. Hence the sum may be written $\Sigma_{w \in W} g(w)\tau(w)|W|^{-1}|C_{W,F}(w)|$. Using (1.9)(ii) it follows that

$$\begin{aligned} \sum_T g(t) &= |W|^{-1} \sum_{w \in W} q^{2N} P_{S/J}(q^{-1}; F_0^{-1}w) g(w) \\ &= q^{2N} |W|^{-1} \sum_{w \in W} P_{S/J}(q^{-1}; F_0^{-1}w) \bar{g}(F_0^{-1}w) \\ &= |F_0| q^{2N} \langle P_{S/J}(q^{-1}), \bar{g} \rangle_{\bar{W}}. \end{aligned}$$

□

Note that if g is F-invariant (i.e. is constant on the F-orbits of W) then $\bar{g}$ is a class function, and $\langle P_{S/J}(q^{-1}), \bar{g} \rangle_{\bar{W}}$ is evaluated in the ring $A(\bar{W})$. Moreover in the split case (i.e. when F_0 is the identity map) the statement of (1.10) simplifies as follows.

(1.10)′ COROLLARY. *Suppose G (in (1.10)) is F-split. Then F-conjugacy coincides with conjugacy and we have*

$$\sum_T g(T) = q^{2N}\langle P_{S/J}(q^{-1}), g\rangle_W$$

where the right hand side is evaluated in $A(W)$.

(1.11) COROLLARY (Steinberg). *The number of F-stable maximal tori of G is q^{2N}.*

Proof. Take $g = 1_W$ in (1.10). Then the required number is, by (1.10),

$$|F_0|q^{2N}|\bar{W}|^{-1}\sum_{w\in W} P_{S/J}(q^{-1}; F_0^{-1}w) = q^{2N}P_{S/J}(q^{-1}; F_0^{-1}e_W)$$

where $e_W = |W|^{-1}\Sigma_{w\in W} w$. But e_W is the projection onto the trivial component of the representation of W on S/J. This is known to be $(S/J)_0$, which has dimension 1, and on which F_0 acts trivially. Thus $P_{S/J}(q^{-1}; F_0^{-1}e_W) = 1$, and the result follows. □

(1.12) COROLLARY. *Let ε be the alternating character of W. Then ε is constant on F-conjuugacy classes and is F-invariant. Say that an F-stable maximal torus of G is positive if it is w-twisted with ε(w) = 1. Otherwise it is negative. Then the number of F-stable positive tori is $\frac{1}{2}(q^{2N} + q^N)$ while the number of negative F-stable maximal tori is $\frac{1}{2}(q^{2N} - q^N)$.*

Proof. Take g to be ε in (1.10). If n_+ and n_- are respectively the numbers of positive and negative rational maximal tori, then by (1.10) we have

$$n_+ - n_- = |F_0|q^{2N}|\bar{W}|^{-1}\sum_{w\in W} P_{S/J}(q^{-1}; F_0^{-1}w)\varepsilon(w) = q^{2N}P_{S/J}(q^{-1}; F_0^{-1}\varepsilon_W)$$

where $\varepsilon_W = |W|^{-1}\Sigma_{w\in W}\varepsilon(w)w$. But ε_W is the projection onto the alternating component of the representation of W on S/J. This is known to be $(S/J)_N$, which again has dimension 1 (its generator is the product of the positive coroots). Moreover since F_0 preserves the set of positive roots it acts trivially on $(S/J)_N$. It follows that $n_+ - n_- = q^{2N}\cdot q^{-N} = q^N$. By (1.11) $n_+ + n_- = q^{2N}$, and the result follows. □

(1.13) COROLLARY (Steinberg [St]). *The number of unipotent elements in G^F is equal to the number of F-stable maximal tori in G, i.e. q^{2N}.*

Proof. Write $\tau(G)$ for the number of F-rational maximal tori of G, and write $\mu(G)$ for the number of unipotent elements in G^F. We show by induction on the semisimple rank $\sigma(G)$ of G that $\tau(G) = \mu(G)$.

When $\sigma(G) = 0$, G is a torus and $\tau(G) = \mu(G) = 1$.

We show first that

$$\sum_T |T^F| = |G^F| \tag{1.13.1}$$

where the sum is over the rational maximal tori of G. To prove (1.13.1) take $g(w)$ in (1.10) to be $|T_w^F|$, where T_w is w-twisted. Then by (1.10) the left hand side of (1.13.1) is equal to $q^{2N}|W|^{-1} \Sigma_{w \in W} P_{S/J}(q^{-1}; F_0^{-1}w)|T_w^F|$. But by (1.9)(i), each summand is equal to $|G^F|$, whence (1.13.1).

Now

$$\sum_T |T^F| = \sum_{\substack{s \in G^F \\ s \text{ semisimple}}} n(s),$$

where $n(s)$ is the number of F-stable maximal tori of G which contain s. But $s \in T$ if and only if $T \leq C_G(s)^o$. Thus we have

$$\sum_T |T^F| = \sum_{\substack{s \in G^F \\ s \text{ semisimple}}} \tau(C_G(s)^o). \tag{1.13.2}$$

On the other hand, from the Jordan decomposition in G^F, we have

$$|G^F| = \sum_{\substack{s \in G^F \\ s \text{ semisimple}}} \mu(C_G(s)^o) \tag{1.13.3}$$

Here we use the fact that all unipotent elements in $C_G(s)$ are in $C_G(s)^o$.

Comparing (1.13.1), (1.13.2) and (1.13.3), we obtain

$$\sum_{\substack{s \in G^F \\ s \text{ semisimple}}} \tau(C_G(s)^o) = \sum_{\substack{s \in G^F \\ s \text{ semisimple}}} \mu(C_G(s)^o). \tag{1.13.4}$$

By induction on semisimple rank, we have $\tau(C_G(s)^o) = \mu(C_G(s)^o)$ unless $s \in Z(G)$. It follows from (1.13.4) that $|Z(G)^F|\tau(G) = |Z(G)^F|\mu(G)$, whence the result. □

(1.14) SCHOLIUM. *Let γ be a class function on G^F satisfying*

$$\gamma(x) = \begin{cases} 0 & \textit{unless } x \textit{ is semisimple}, \\ \pm|C_{G^F}(x)|_p & \textit{if } x \textit{ is semisimple} \end{cases}$$

where p is the characteristic of $\mathbb{F}_q$ *and for any integer k,* k_p *is the highest power of p dividing k. Then* $\langle \gamma, \gamma \rangle_{G^F} = 1$.

Proof. The equation (1.13.2) (or (1.13.3)) amounts (by (1.11)) to $\Sigma\, \gamma(x)^2 = |G^F|$, which is the required statement. □

The Lie algebra analogue of (1.13) can also be dealt with by our method.

(1.15) COROLLARY. *With G as above, let* $\mathscr{G} = \mathrm{Lie}\,(G)$. *Write* $v(\mathscr{G})$ *for the number of nilpotent elements of* $\mathscr{G}^F$. *Then* $v(\mathscr{G}) = \tau(G)$ $(=q^{2N})$.

Proof. This is similar to that of (1.13). First observe that

$$\sum_T |\mathrm{Lie}\ T)^F| = |\mathscr{G}^F| \tag{1.15.1}$$

where the sum is over the F-rational maximal tori of G. For each summand is equal to q^r ($r = \mathrm{rank}\ G$) and there are q^{2N} summands. So the left side is equal to $q^{r+2N} = q^{\dim \mathscr{G}} = |\mathscr{G}^F|$.

Now argue as in (1.13). We have

$$\sum_T |(\mathrm{Lie}\ T)^F| = \sum_{\substack{X \in \mathscr{G}^F \\ X \text{ semisimple}}} n(X),$$

where $n(X)$ is the number of rational maximal tori T such that $X \in \mathrm{Lie}\ T$. But $X \in \mathrm{Lie}\ T$ if and only if $\mathrm{Lie}\ T \subseteq C_{\mathscr{G}}(X) = \{Y \in \mathscr{G} \mid [Y, X] = 0\}$, and by Borel [B, pp. 225 and 321] $C_{\mathscr{G}}(X)$ is the Lie algebra of the reductive group $C_G(X)^o$, where $C_G(X) = \{g \in G \mid \mathrm{Ad}\, g(X) = X\}$. Thus we have

$$\sum_T |(\mathrm{Lie}\ T)^F| = \sum_{\substack{X \in \mathscr{G}^F \\ X \text{ semisimple}}} \tau(C_G(X)^o). \tag{1.15.2}$$

Using the Jordan decompositon in $\mathscr{G}$, we obtain (using Borel's results [loc.cit.])

$$|\mathscr{G}^F| = \sum_{\substack{X \in \mathscr{G}^F \\ X \text{ semisimple}}} v(\mathrm{Lie}\,(C_G(X)^o)). \tag{1.15.3}$$

The proof is now completed as in (1.13) by induction on the semisimple rank of G. □

To close this section we show how our result is used to prove the following result, essentially due to Kawanaka [C, (7.6.8)].

(1.16) COROLLARY (Kawanaka). *Let ψ be any function on G^F which satisfies $\psi(su) = \psi(s)$ where $su = s \cdot u$ is the Jordan decomposition of su (s semisimple, u unipotent). Then*

$$\sum_T \langle \psi, \psi \rangle_{T^F} |T^F| = |G^F| \langle \psi, \psi \rangle_{G^F}.$$

Proof. We have $\Sigma_T \langle \psi, \psi \rangle_{T^F} |T^F| = \Sigma_T \Sigma_{t \in T^F} |\psi(t)|^2 = \Sigma_{t \in G^F_{ss}} \Sigma_{T \subset C_G(t)^o} |\psi(t)|^2 = \Sigma_{t \in G^F_{ss}} |\psi(t)|^2 \tau(C_G(t)^o) =$ (by (1.13)) $\Sigma_{t \in G^F_{ss}} |\psi(t)|^2 \mu(C_G(t)) = \Sigma_{x \in G^F} |\psi(x)|^2$. □

2. Rational semisimple classes and orbits

Let G, T_0, F etc be as in §1.

(2.1) PROPOSITION. *Suppose f is a function defined on the F-stable semisimple conjugacy classes of G. Then*

$$\sum_{c \in (G_{ss})^F} f(c) = |W|^{-1} \sum_{w \in W} \sum_{t \in T_0^{wF}} f(t)$$

where $(G_{ss})^F$ denotes the set of F-stable semisimple conjugacy classes of G and f(t) is the function f lifted from $(G_{ss})^F$ to the set of elements of G whose conjugacy class is fixed by F.

Proof. The F-stable semisimple conjugacy classes in G are in bijective correspondence with $(T_0/W)^F$ [C, §3.7]. The element $t \in T_0$ lies in an F-stable W-orbit if and only if $F(t) = t^w$ for some $w \in W$, i.e. precisely when $t \in T_0^{wF}$ for some $w \in W$. Write $T_0^{\text{rat}} = \bigcup_{w \in W} T_0^{wF}$; then the size of the W-orbit of $t \in T_0^{\text{rat}}$ is $|W||W(t)|^{-1}$, where $W(t) = \{w \in W \mid {}^w t = t\}$. Thus we have

$$\sum_{c \in (G_{ss})^F} f(c) = \sum_{t \in T_0^{\text{rat}}} f(t)|W(t)||W|^{-1}. \tag{2.1.1}$$

Moreover for $t \in T_0^{\text{rat}}$, $\#\{w \in W \mid t \in T_0^{wF}\} = |W(t)|$, since $t \in T_0^{w_1 F} \cap T_0^{w_2 F} \Leftrightarrow w_1 w_2^{-1} \in W(t)$. Hence the right hand side of (2.1.1) may be written

$$\sum_t \in T_0^{\text{rat}} f(t)|W(t)||W|^{-1} = \sum_{w \in W} \sum_{t \in T_0^{wF}} f(t)|W|^{-1}, \tag{2.1.2}$$

which proves (2.1). □

(2.2) THEOREM. *Let f be a function defined on the F-stable semisimple conjugacy classes of G. Define the associated F-class function $\tilde{f}$ on W by*

$$\tilde{f}(w) = \sum_{t \in T_0^{w^{-1}F}} f(t)$$

(*where $f(t)$ denotes the value of f on the G-class of t, which is fixed by F, since for $t \in T_0^{w^{-1}F}$, $F(t) = {}^w t$*). *Then $\Sigma_{c \in (G_{ss})^F} f(c) = \langle \tilde{f}, 1 \rangle_W$ where $(G_{ss})^F$ is the set of F-stable semisimple classes of G and the notation $\langle \, , \rangle_W$ denotes inner product of complex valued functions on W* (*defined for class functions as in* (1.1); *the general definition is the same*).

This is just a restatement of (2.1).

(2.3) COROLLARY (Steinberg). *The number of F-stable semisimple classes of G is $|(Z^0)^F| q^l$, where $Z = Z(G)$ and l is the semisimple rank of G.*

Proof. Let $S_0 = T_0 \cap G'$. Then $T_0 = S_0 Z^0$ and correspondingly $Y(T_0) \otimes_{\mathbb{Z}} \mathbb{R} = Y(Z^0) \otimes_{\mathbb{Z}} \mathbb{R} \oplus Y(S_0) \otimes_{\mathbb{Z}} \mathbb{R}$. Since $F_0^{-1} w$ fixes the two summands, it follows that $\det_{Y(T_0) \otimes_{\mathbb{Z}} \mathbb{R}} (q - F_0^{-1} w) = \det_{Y(Z^0) \otimes_{\mathbb{Z}} \mathbb{R}} (q - F_0^{-1} w) \det_{Y(S_0) \otimes_{\mathbb{Z}} \mathbb{R}} (q - F_0^{-1} w)$. But W acts trivially on $Y(Z^0)$, so that the first factor is just $|(Z^0)^F|$. Moreover W acts irreducibly on $Y(S_0) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\det_{Y(S_0) \otimes_{\mathbb{Z}} \mathbb{R}} (q - F_0^{-1} w) = \Sigma_{i=0}^{l} (-1)^{l-i} \rho_{l-i}(F_0^{-1} w) q^i$ where ρ_j is the character of the j^{th} exterior power of the $\bar{W}$-module V. The ρ_j are distinct and irreducible as W-modules (see [Bou, p. 127, ex. 3]), and hence as $\bar{W}$-modules.

Now take $f(c) = 1$ in (2.2). Then by (2.2) and (1.4)(i), the required number is

$$\langle \tilde{f}, 1 \rangle_W = |(Z^0)^F| \, |W|^{-1} \sum_{w \in W} \sum_{i=0}^{l} (-1)^{l-i} \rho_{l-i}(F_0^{-1} w) q^i$$

$$= |(Z^0)^F| \sum_{i=0}^{l} (-1)^{l-i} \rho_{l-i}(F_0^{-1} e_W) q^i$$

where $e_W = |W|^{-1} \Sigma_{w \in W} w$. But $\rho_{l-i}|_W$ is an irreducible character of W, distinct from 1_W unless $i = l$. The result follows. □

(2.3)′ REMARK. If G is such that G' is simply connected, then centralizers of semisimple elements of G are connected. Hence in this case the semisimple classes of G^F correspond bijectively to the F-stable semisimple classes of G.

The next result is proved in a similar way to (2.2), using the statement (2.6) which is proved below.

(2.4) THEOREM. *Let f be a function defined on the F-stable semisimple* Ad G-*orbits of the Lie algebra $\mathscr{G}$. Define the associated F-class function $\tilde{f}$ on W by*

$$\tilde{f}(w) = \sum_{X \in \mathrm{Lie}(T_0^{w^{-1}F})} f(X).$$

Then

$$\sum_{C \in (\mathscr{G}_{ss})^F} f(C) = \langle \tilde{f}, 1 \rangle_W$$

where notation is analogous to that in (2.2).

(2.5) COROLLARY. *The number of F-stable semisimple orbits in $\mathscr{G}$ is q^r, where* $r = \dim T_0 = \mathrm{rank}\, \mathscr{G}$.

Proof. Take $f(C) = 1$ in (2.4). The required number is then $|W|^{-1} \cdot \Sigma_{w \in W} |\mathscr{T}_0^{w^{-1}F}|$ and the result follows from the fact that $|\mathscr{T}_0^{wF}| = q^r$ for all $w \in W$ (here $\mathscr{T}_0 = \mathrm{Lie}\,(T_0)$. □

The proofs of (2.4) and (2.5) require the elementary fact that for any closed subgroup H of G, $\mathrm{Lie}\,(gHg^{-1}) = \mathrm{Ad}\, g\, \mathrm{Lie}\,(H)$, and the following result which is (3.16) of [Sp–St]. We include a proof for the reader's convenience.

(2.6) PROPOSITION. *With notation as in* (2.4), *the semisimple* Ad G-*orbits in $\mathscr{G}$ are parametrised by $\mathscr{T}_0/W$ (where W acts in $\mathscr{T}_0$ via* Ad). *The F-stable orbits are parametrised by $(\mathscr{T}_0/W)^F$.*

Proof. It is a result of Chevalley and Borel [B, (11.8)] that $X \in \mathscr{G}$ is semisimple if and only if $X \in \mathrm{Ad}\, g(\mathscr{T}_0)$ for some $g \in G$. Thus we have only to show that for X_1 and $X_2 \in \mathscr{T}_0$, if $X_2 = \mathrm{Ad}\, g(X_1)$ for some $g \in G$, then $X_2 = \mathrm{Ad}\, n(X_1)$ for some $n \in N = N_G(T_0)$.

For this, observe that if $X_2 = \mathrm{Ad}\, g(X_1)$, then T_0 and ${}^g(T_0)$ are both maximal tori of $C_G(X_1)^0$, whose Lie algebra is $C_{\mathscr{G}}(X_2) = \{X \in \mathscr{G} \mid [X, X_2] = 0\}$, by [B, (9.1) and (13.19)]. Hence there exists $x \in C_G(X_2)^0$ such that $T_0 = {}^{xg}T_0$. Hence $xg \in N_G(T_0) = N$, and clearly $X_2 = \mathrm{Ad}\, g(X_1) = \mathrm{Ad}\,(xg)X_1$. The statement about F-stable orbits is clear. □

3. Regular semisimple classes and orbits

In this section we apply (2.2) and (2.4) to count the rational regular semisimple orbits.

Let $T_{ors}^{wF} = \{t \in T_0^{wF} |$ is regular semisimple$\}$ and define $\mathscr{T}_{ors}^{wF}$ similarly.

(3.1) DEFINITION. Define the F-class functions N_{rs} and n_{rs} on W by

$$\left.\begin{aligned} N_{rs}(w) &= |T_{ors}^{w^{-1}F}| \\ n_{rs}(w) &= |\mathcal{T}_{ors}^{w^{-1}F}| \end{aligned}\right\}, \qquad w \in W.$$

When G is F-split, n_{rs} and N_{rs} are class functions on W.

In the next section we shall see that the function n_{rs} is related to the cohomology of the hyperplane complement which corresponds to W when G is F-split, and therefore is in some sense explictly known. The function N_{rs} has been investigated by Deriziotis [D] and recently by Fleischmann and Janiszczak [FJ] in the split classical cases and likewise has a combinatorial description.

(3.2) THEOREM.

(i) *The number of F-stable regular semisimple conjugacy classes in G is equal to* $\langle N_{rs}, 1\rangle_W$.

(ii) *The number of F-stable regular semisimple* Ad *G-orbits in* $\mathcal{G}$ *is equal to* $\langle n_{rs}, 1\rangle_W$.

Both statements follow immediately from (2.2) and (2.4) by taking the function f to be 1 on the regular semisimple orbits and zero elsewhere. □

For the rest of this section we assume that G' is simply connected. This implies that centralisers of semisimple elements are connected (see (2.3)$'$ above).

A regular semisimple element x of G^F lies in a unique maximal torus. Say that x is *positive* if the corresponding torus is positive (see (1.12) above) and negative otherwise. Any G^F-conjugacy class of regular semisimple elements consists entirely of either positive or negative elements and we speak of it as *positive or negative* accordingly.

(3.3) THEOREM. *Suppose G is such that G′ is simply connected. The number of regular semisimple classes of* G^F *counted with sign (i.e. the number of positive classes minus the number of negative classes) is equal to* $(-1)^l\,|(Z(G)^0)^F|$ where l *is the semisimple rank of G.*

Proof. In Theorem (2.2) take the function f to be

$$f(c) = \begin{cases} 0 & \text{unless } c \text{ is regular,} \\ \varepsilon(c) & \text{if } c \text{ is regular} \end{cases}$$

for any F-stable semisimple conjugacy class of G.

Then the required number is $\Sigma_{c \in (G_{ss})^F} f(c)$, which we shall denote by E for the rest of this proof. The associated class function $\tilde{f}$ (see (2.2)) on W is clearly given by

$$\begin{aligned}\tilde{f}(w) &= \varepsilon(w) \mid T_{ors}^{w^{-1}F} \mid \\ &= \varepsilon(w) N_{rs}(w) \quad \text{(c.f. (3.1))}.\end{aligned} \tag{3.3.1}$$

It follows from (2.2) that

$$\begin{aligned}E = \langle \varepsilon N_{rs}, 1 \rangle_W = \langle N_{rs}, \varepsilon \rangle_W &= |W|^{-1} \sum_{w \in W} |T_{ors}^{wF}| \varepsilon(w) \\ &= |W|^{-1} \sum_{w \in W} \sum_{t \in T_{ors}^{wF}} \varepsilon(w).\end{aligned} \tag{3.3.2}$$

Recall the notation of the proof of (2.1): $T_0^{\mathrm{rat}} = \bigcup_{w \in W} T_0^{wF}$. Extending this, we write

$$T_{ors}^{\mathrm{rat}} = \bigcup_{w \in W} T_{ors}^{wF} = \{t \in T_0 \mid t \text{ is regular and } F(t) = t^w, \text{ some } w \in W\}. \tag{3.3.3}$$

We then have

$$\sum_{w \in W} |T_0^{wF}| \varepsilon(w) = \sum_{w \in W} \sum_{t \in T_0^{wF}} \varepsilon(w) = \sum_{t \in T_0^{\mathrm{rat}}} \sum_{\substack{w \in W \\ t \in T_0^{wF}}} \varepsilon(w). \tag{3.3.4}$$

But for a fixed element $t \in T_0^{\mathrm{rat}}$, $\{w \in W \mid t \in T_0^{wF}\}$ is of the form $W(t)w_t$ for some element $w_t \in W$, where $W(t) = \{w \in W \mid {}^w t = t\}$ (see (2.1)). Hence the inner sum in (3.3.4) may be written

$$\sum_{\substack{w \in W \\ t \in T_0^{wF}}} \varepsilon(w) = \sum_{w \in W(t)w_t} \varepsilon(w) = \varepsilon(w_t) \sum_{w \in W(t)} \varepsilon(w). \tag{3.3.5}$$

Moreover it follows from Steinberg's characterisation of regular semisimple elements [St2, Prop. 3, p. 96] that if t is not regular, then $W(t)$ contains a reflection, whence the restriction of ε to $W(t)$ is not equal to the identity representation. It follows that the sum in (3.3.5) is zero unless t is regular. Taking this into account, (3.3.4) becomes

$$\sum_{w \in W} |T_0^{wF}| \varepsilon(w) = \sum_{t \in T_{ors}^{\mathrm{rat}}} \sum_{\substack{w \in W \\ t \in T_0^{wF}}} \varepsilon(w) = \sum_{w \in W} |T_{ors}^{wF}| \varepsilon(w). \tag{3.3.6}$$

Hence from (3.3.2) we deduce that

$$E = |W|^{-1} \sum_{w \in W} |T_0^{wF}| \varepsilon(w) = \langle q^r P_A(-q^{-1}; F_0^{-1} w), \varepsilon \rangle_W \quad \text{(cf. (1.4)(i))}. \tag{3.3.7}$$

In the proof of (2.3) it was shown that in the ring $B(W)$

$$q^r P_A(-q^{-1}) = |(Z(G)^0)^F| \sum_{i=0}^{l} (-1)^{l-i} \rho_{l-i} q^i \tag{3.3.8}$$

where $|(Z(G)^0)^F|$ is regarded as a polynomial in q, l is the semisimple rank of G and ρ_j is the jth exterior power of the reflection representation of W.

Thus $\rho_l = \varepsilon$, and for $j \neq l$ we have $(\rho_j, \varepsilon)_W = 0$. Hence combining (3.3.7) and (3.3.8) we obtain

$$E = |(Z(G)^0)^F| \sum_{i=0}^{l} (-1)^{l-i} \langle \rho_{l-i}, \varepsilon \rangle q^i = (-1)^l |(Z(G)^0)^F|. \qquad \square$$

The next result is a restatement of (3.3.2) above. It does not require the extra hypothesis of (3.3).

(3.4) COROLLARY. *Suppose G is arbitrary. If* N_{rs} *is the function* (3.1), *we have* $\langle N_{rs}, \varepsilon \rangle_W = |(Z(G)^0)^F|(-1)^l$.

(3.5) COROLLARY. *A semisimple simply connected group G has an odd number of rational regular semisimple conjugacy classes.*

Proof. If G is semisimple then $(Z(G)^0)^F = 1$, whence the number E of (3.3) is $(-1)^l$. But the number of regular semisimple conjugacy classes is $E + 2E^-$ where E^- is the number of negative classes. $\square$

It seems a little curious that (3.5) is independent of the characteristic.

The Lie algebra analogue of (3.4) is

(3.6) THEOREM. *Suppose G is arbitrary, has non-zero semisimple rank, and that for* $w \in W$, $n_{rs}(w)$ *is as defined in* (3.1). *Then we have*

$$\langle n_{rs}, \varepsilon \rangle_W = 0.$$

Proof. Since this is analogous to the computation in the proof of (3.3), we given only a sketch. The proof proceeds by showing that

$$|W|^{-1} \sum_{w \in W} |\mathcal{T}_0^{wF}| \varepsilon(w) = |W|^{-1} \sum_{w \in W} |\mathcal{T}_{ors}^{wF}| \varepsilon(w). \tag{3.6.1}$$

This is the Lie algebra analogue of (3.3.6) and is proved in the same way (note that the definition of regular used here is that no root annihilates the element concerned).

The right hand side of (3.6.1) is $\langle n_{rs}, \varepsilon \rangle_W$, while the left hand side is clearly equal to zero unless $W = 1$, since $|\mathscr{T}_0^{wF}| = q^r$ for all $w \in W$. □

4. Hyperplane complements

In this section we relate the function n_{rs} of (3.1) to the topology of the complex hyperplane complement M_W (see [L1] for notation) corresponding to W, and deduce from (3.7) a result for $H^*(M_W)$. As general references for the facts about l-adic cohomology which we use here we cite [C, §7.1 and Appendix], Milne [M, Ch. VI] and SGA4½ [Springer Lecture Notes in Maths, §569]. For this section, we take G to be F-split; i.e. the automorphism F_0 of §1 in trivial, or equivalently, F acts on V via multiplication by q.

Let $\mathscr{T}_0$ be the toral subalgebra Lie T_0 of $\mathscr{G}$. Then $\mathscr{T}_0 \cong \mathbb{A}^r$ (affine space of dimension r), where r is the rank of $\mathscr{G}$. Moreover the set $\mathscr{T}_{ors}$ of regular semisimple elements of $\mathscr{T}_0$ is just the hyperplane complement over $\bar{\mathbb{F}}_q$ corresponding to W. Following [L1], we write M_W for the corresponding complex hyperplane complement.

(4.1) LEMMA. *The function n_{rs} of (3.1) is given by $n_{rs}(w) = \Sigma_{i=0}^{2r}(-1)^i \operatorname{tr}(wF, H_c^i(\mathscr{T}_{ors}, \bar{\mathbb{Q}}_l))$, where $H_c^i(-, \bar{\mathbb{Q}}_l)$ denotes l-adic cohomology with compact supports.*

This is just Grothendieck's trace formula [C, p. 504], [M, Ch. VI] applied to the computation of $|\mathscr{T}_{ors}^{wF}|$.

(4.2) LEMMA. *Let $\mathscr{A}$ be a finite collection of F-stable hyperplanes in $\mathbb{A}^r$, Then*
(i) $H_c^{2r-i}(\mathbb{A}^r - \bigcup_{H \in \mathscr{A}} H, \bar{\mathbb{Q}}_l) = 0$ *unless* $i = 0, 1, \ldots, r$.
(ii) *All eigenvalues of F on $H_c^{2r-i}(\mathbb{A}^r - \bigcup_{H \in \mathscr{A}} H, \bar{\mathbb{Q}}_l)$ are equal to q^{r-i}.*

The proof may be found in [L3, (2, 4)].

Combining (4.1) and (4.2) we obtain

(4.3) COROLLARY. *We have (if $r =$* rank $\mathscr{G}$) for $w \in W$,

$$n_{rs}(w) = q^{-r} \sum_{i=0}^{2r} (-q)^i \operatorname{tr}(w, H_c^i(\mathscr{T}_{ors}, \bar{\mathbb{Q}}_l))$$

$$= q^r \sum_{i=0}^{r} (-q)^{-i} \operatorname{tr}(w, H_c^{2r-i}(\mathscr{T}_{ors}, \bar{\mathbb{Q}}_l)).$$

This follows immediately from (4.2) and (4.1) since F commutes with w on $\mathscr{T}_0$ and since $\mathscr{T}_0 \cong \mathbb{A}^r$ ($r = \text{rank}\,\mathscr{G}$) and $\mathscr{T}_{ors}$ is the complement in $\mathscr{T}_0$ of the hyperplanes defined by the roots of $\mathscr{G}$ with respect to $\mathscr{T}_0$, which are all F-stable by our assumption that G is F-split.

We say (cf. [L3]) that the characteristic p of $\bar{\mathbb{F}}_q$ is *regular* if the lattice of reflecting hyperplane intersections of W remains the same on reduction mod p (over $\bar{\mathbb{F}}_p$).

(4.4) PROPOSITION. *Given W and the group scheme G, if p is a regular prime for W, we have*

(*i*) $\dim_{\bar{\mathbb{Q}}_l}(H_c^{2r-i}(\mathscr{T}_{ors}, \bar{\mathbb{Q}}_l)) = \dim_{\mathbb{C}}(H^i(M_W, \mathbb{C}))$ *where $H^i(M_W, \mathbb{C})$ denotes ordinary complex cohomology.*

(ii) *For $w \in W$, we have*

$$\operatorname{tr}(w, H_c^{2r-i}(\mathscr{T}_{ors}, \bar{\mathbb{Q}}_l)) = \operatorname{tr}(w, H_c^i(M_W, \mathbb{C})) \in \mathbb{Z}.$$

This is Theorem (1.5) of [L3].

Combining (4.4) and (4.3) we obtain

(4.5) COROLLARY. *In the notation of* (4.1) *we have, if the characteristic is regular,*

$$n_{rs}(w) = q^r \sum_{i=0}^{2r} (-q)^{-i} \operatorname{tr}(w, H^i(M_W, \mathbb{C})).$$

Following the notation of [L1] we make the

(4.6) DEFINITION. If M_W is the complex hyperplane complement corresponding to W acting on $\mathbb{C}^r$, define the Poincaré series

$$P_{M_W}(t; w) = \sum_{i=0}^{r} \operatorname{tr}(w, H^i(M_W, \mathbb{C}))t^i \qquad (w \in W)$$

where $H^i(-, \mathbb{C})$ denotes complex deRham cohomology.

Thus $P_{M_W} \in A(W)$ (the ring introduced in §1 above).

It is now clear that in case the characteristic is regular,

$$n_{rs}(w) = q^r P_{M_W}(-q^{-1}; w) \qquad (w \in W). \tag{4.7}$$

The polynomials $P_{M_W}(t, w)$ have all been computed in the classical cases ([L1], [L2], [FJ]).

As an immediate corollary of (4.7), we obtain

(4.8) THEOREM. *If W is the Weyl group of a simple complex* Lie *algebra and M_W is the corresponding complex hyperplane complement, then*

$$\langle H^i(M_W, \mathbb{C}), \varepsilon \rangle_W = 0$$

where ε is the alternating representation of W and $H^i(M_W, \mathbb{C})$ is the ordinary cohomology group with complex coefficients.

Proof. By (3.7) we have $\langle n_{rs}, \varepsilon \rangle = 0$; it follows from (4.7) that

$$\langle P_{M_W}, \varepsilon \rangle = 0 \quad \text{in the ring } A(W).$$

But the coefficients of the polynomial $\langle P_{M_W}, \varepsilon \rangle$ are just $\langle H^i(M_W, \mathbb{C}), \varepsilon \rangle$, whence the result. □

(4.9) REMARK. Although our proof of (4.8) is essentially by a "reduction modulo p" argument, it may also be possible to give a purely combinatorial proof, which presumably would to some extent be case by case.

(4.10) COROLLARY. *Let $\mathscr{L} = \mathscr{L}(W)$ be the lattice of hyperplane intersections corresponding to W; this is a geometric lattice whose reduced homology is therefore zero except in the top dimension. Then $\langle \tilde{H}_{\mathrm{top}}(\mathscr{L}), \varepsilon \rangle = 0$.*

This is because by the results of Orlik and Solomon ([OS], see also [L3]), $\tilde{H}_{\mathrm{top}}(\mathscr{L}) \cong H^r(M_W)$.

(4.11) THEOREM. (i) *The number of regular semisimple G^F-orbits in $\mathscr{G}^F$ is equal to*

$$q^{-r} \sum_{i=0}^{r} (-q)^i \dim H_c^i(\mathscr{T}_{ors}/W, \bar{\mathbb{Q}}_l).$$

(ii) *If the characteristic is regular, this number is equal to*

$$q^r P_{M_W/W}(-q^{-1})$$

where $P_{M_W/W}(t)$ is the Poincaré series of the orbit space M_W/W.

Proof. By (3.2)(ii) the required number is $\langle n_{rs}, 1\rangle_w$, and by (4.3) this is equal to

$$q^{-r}\sum_{i=0}^{r}(-q)^i\langle H_c^i(\mathcal{T}_{ors}, \bar{\mathbb{Q}}_l), 1\rangle_W = q^{-r}\sum_{i=0}^{r}(-q)^i \dim H_c^i(\mathcal{T}_{ors}, \bar{\mathbb{Q}}_l)^W$$

$$= q^{-r}\sum_{i=0}^{r}(-q)^i \dim H_c^i(\mathcal{T}_{ors}/W, \bar{\mathbb{Q}}_l)$$

by the transfer theorem for étale cohomology ([Sr, p. 53], [SGA4, XVII]).

If p is regular, then the results of [L3] apply and by (4.3) and (4.7) we have

$$\langle n_{rs}, 1\rangle_W = q^r\sum_{i=0}^{r}(-q)^{-i}\langle H^{2r-i}(M_W, \mathbb{C}), 1\rangle_W.$$

Applying the usual transfer theorem for complex cohomology yields the required result. □

The polynomials $P_{M_W/W}(t)$ have been computed by Brieskorn ([Br], see also [L1] and [L2]). They give the following formulae for the number of regular semisimple orbits when G is semisimple:

Type	*Number of regular semisimple orbits* (*for regular p*)
A_l	$q^l - q^{l-1}$
B_l, C_l	$q^l - 2q^{l-1} + 2q^{l-2} - \cdots + (-1)^l$
D_l	$q^l - q^{l-1}$ (l odd) $q^l - q^{l-1} + (-1)^{l-1}(q-1)$ (l even)
G_2	$q^2 - 2q + 1$
F_4	$q^4 - 2q^3 + 2q^2 + 1$
E_6	$q^6 - q^5$
E_7	$q^7 - q^6 + q - 1$
E_8	$q^8 - q^7 - q + 1.$

(4.12) COROLLARY. *If q is as in* (4.11), *the number of regular semisimple elements of $\mathcal{G}^F$ is*

$$q^{\dim \mathcal{G}}\langle P_{S/J}(q^{-1}), P_{M_W}(-q^{-1})\rangle_W = |\mathcal{G}^F|\langle P_{S/J}(q^{-1}), P_{M_W}(-q^{-1})\rangle_W.$$

This follows easily from (4.7) and (1.10) using the fact that each regular semisimple element of $\mathscr{G}^F$ lies in a unique F-stable toral subalgebra of $\mathscr{G}$ and for regular p, the number of such elements in Lie $(T_w)^F$ is $n_{rs}(w)$.

We close this section with a remark about regular primes. Suppose $\phi = \{\omega_1, \ldots, \omega_l\}$ is a set of fundamental weights for W and for each positive root $\beta \in \Phi$ (the root system corresponding to W) we have an explicit expression $\beta = \Sigma_{i=1}^{l} a_{\beta i}\omega_i$. The condition that p be regular is equivalent to the following:

(4.13). *For any subset of the rows of the integer matrix* $M = (a_{\beta i})$, *the rank of the corresponding matrix is the same over* $\mathbb{F}_p$ *as over* $\mathbb{Q}$.

(4.14) REMARK. If the characteristic is regular for $\mathscr{G}$, the formulae in the above table prove that regular semisimple elements exist in $\mathscr{G}$.

5. A rational classification of conjugacy classes

In this section we assume that G has simply connected derived group. This implies by Steinberg's theorem (cf. [C, Th. 3.5.6]) that the centralisers of semisimple elements of G are connected. In the Lie algebra case we may replace this restriction on G by stipulating that the characteristic is not a torsion prime for G. This also ensures ([Sp–St, 3.19, p. 201]) that the centralisers of semisimple elements of $\mathscr{G}$ are connected. We also assume that G is F-split.

We begin by defining a map

$$\omega\colon (G_{ss})^F \to (W) \tag{5.1}$$

where $(G_{ss})^F$ denotes the set of F-stable semisimple conjugacy classes of G and (W) denotes the set of conjugacy classes of W, as follows. For $c \in (G_{ss})^F$ there is an element $x \in G^F \cap c$, and by our assumption concerning the connected nature of centralizers, x is determined up to conjugacy in G^F. Thus $C_G(x)$ is determined up to G^F-conjugacy. Let T be a maximally split torus in $C_G(x)$. This is an F-stable maximal torus of G, whose G^F conjugacy class is determined by c. But this G^F-conjugacy class corresponds to a unique conjugacy class of W. We define $\omega(c)$ to be the conjugacy class determined by T.

(5.2) EXAMPLE. In type A (say GL_n), a rational semisimple class c has characteristic polynomial $f(c)$ over $\mathbb{F}_q$. We have a factorisation $f(c) = \Pi_i f_i^{e_i}$ over $\mathbb{F}_q$, where the f_i are irreducible over $\mathbb{F}_q$. Then $\omega(c)$ is the partition $(d_i^{e_i})$ where $d_i = \deg(f_i)$.

(5.3) REMARK. The same construction yields maps (also denoted ω) on the set of all rational classes of G (by composing with the map from a class to its semisimple part) and from $(\mathscr{G}_{ss})^F$ to (W). We shall be using these maps without further comment.

Now let χ be any class function on W. By composition with ω, we may lift χ to the rational semisimple classes of G (or $\mathscr{G}$) and using (2.2) we obtain immediately

(5.4) PROPOSITION. *For any class function* χ on W, we have

$$\sum_{c \in (G_{ss})^F} \chi(\omega(c)) = \langle \tilde{\chi}, 1 \rangle_W$$

where $\tilde{\chi}(w) = \sum_{x \in T_0^{wF}} \chi(\omega(x))$, *and similarly in the Lie algebra* $\mathscr{G}$.

To compute $\tilde{\chi}(w)$ we shall require

(5.5) LEMMA. (i) *The* $\mathbb{F}_q$*-rank of* T_w *is* $r - n(w)$ *where* $r = \dim T_0 = \mathbb{F}_q$*-rank of* G, *and* $n(w)$ *is the smallest integer* n *such that* w *is a product of* n *reflections in* W ($n(w) = \dim(\mathrm{im}\,(w-1))$ *in the reflection representation*).
(ii) *For* $x \in \mathscr{T}_0^{wF}$, *we have* $\{w' \in \mathscr{T}_0^{w'F}\} = C_W(x)w$.

These facts are well-known.
We shall carry out the computation of the right hand side of (5.4) in the Lie algebra case, in the case where the prime p is regular. The notation below is that of [L3].

(5.6) THEOREM. *Let* χ *be a class function on* W, *and denote by* χ *also the induced function on the rational semisimple orbits of* $\mathscr{G}$. *Then*

$$\sum_{c \in (\mathscr{G}_{ss})^F} \chi(c) = |W|^{-1} \sum_{X \in L} \sum_{w \in S_X} |W_X|\,|M_X^{wF}|\chi(w)$$

where L *is the lattice of hyperplane intersections corresponding to* W *and for* $X \in L$, W_X *is the corresponding parabolic subgroup of* W, N_X *is its normaliser*, S_X *is a set of coset representatives for* W_X *in* N_X *which have minimal* n*-value in their coset* (*cf.* [LS]) *and* $M_X = X - \bigcup_{Y \in L,\, Y \subsetneq X} Y$.

Proof. In view of (5.4), we compute $\tilde{\chi}(w) = \sum_{x \in \mathscr{T}_0^{wF}} \chi(\omega(x))$. Now by regularity, we have, for $w \in W$,

$$\mathscr{T}_0^{wF} = \coprod_{X \in L^w} M_X^{wF} \tag{5.6.1}$$

where M_X is $\{x \in X \mid x \notin Y \text{ for } Y \subsetneq X\}$. Moreover for $x \in M_X$, $C_W(x) = W_X$, and hence $\{w' \mid x \in \mathcal{T}_0^{w'F}\} = W_X w$. It follows that $\omega(x)$ is the class in W, represented by $S_X \cap W_X w$. Hence

$$\tilde{\chi}(w) = \sum_{X \in L^w} \sum_{x \in M_X^{wF}} \chi(S_X \cap W_X w).$$

Thus

$$\begin{aligned}\langle \tilde{\chi}, 1 \rangle_W &= |W|^{-1} \sum_{w \in W} \sum_{X \in L^w} \sum_{x \in M_X^{wF}} \chi(S_X \cap W_X w) \\ &= |W|^{-1} \sum_{X \in L} \sum_{w \in N_X} |M_X^{wF}| \chi(S_X \cap W_X w) \\ &= |W|^{-1} \sum_{X \in L} \sum_{w \in S_X} |W_X| |M_X^{wF}| \chi(w),\end{aligned}$$

the last step following because $N_X = W_X S_X$ and W_X acts trivially on M_X. □

The above formula may be further refined using the fact ([L3]) that

(5.7) LEMMA. *With notation as in* (5.6), *we have*

$$|M_X^{wF}| = \sum_{\substack{Y \in L^w \\ Y \geq X}} \mu_w(X, Y) q^{\dim Y}$$

where μ_m is the Möbius function on the lattice L^w.

(5.8) COROLLARY. *We have, in the notation of* (5.6)

$$\sum_{c \in (\mathcal{G}_{ss})^F} \chi(c) = |W|^{-1} \sum_{X \in L} \sum_{w \in S_X} \sum_{\substack{Y \in L^w \\ Y \geq X}} |W_X| \mu_w(X, Y) \chi(w) q^{\dim Y}.$$

We shall now give some examples of this computation for particular class functions χ.

(5.9) PROPOSITION. *The number of F-rational semisimple classes c of $\mathcal{G}$ which are split (i.e. such that $\omega(c) = 1$) is equal to*

$$q^{r-l} \prod_{i=1}^{l} \frac{q + m_i}{1 + m_i}$$

provided that the characteristic is regular (where $m_1, \ldots, m_l$ are the exponents of the Weyl group W).

Proof. Take $\chi(\omega)$ (in (5.4)) to be 1 if $\omega = 1$ and 0 otherwise. To compute $\chi(\omega(x))$ for $x \in \mathcal{T}_0^{wF}$, observe that $\chi(\omega(x)) = 0$ unless $w(x) = 1$, i.e. unless $x = \operatorname{Ad} gx'$ for some $x' \in \mathcal{T}_0^F$. But $x = \operatorname{Ad} gx'$ implies that $x = \operatorname{Ad} w'x'$ for some $w' \in W$, whence $x \in \mathcal{T}_0^F$. Hence $\omega(x) = 1$ (for $x \in \mathcal{T}_0^{wF}$) if and only if $x \in \mathcal{T}_0^F \cap \mathcal{T}_0^{wF}$, i.e. if and only if w centralises x. Thus

$$\sum_{x \in \mathcal{T}_0^{wF}} \chi(w(x)) = |(\mathcal{T}_0^w)^F| = |\ker_{\mathcal{T}_0}(w-1)^F|.$$

But $\ker_{\mathcal{T}_0}(w-1)$ is an affine space of dimension $r - n(w)$, so that $|(\mathcal{T}_0^w)^F| = q^{r-n(w)}$. Hence the required number is $|W|^{-1} \Sigma_{w \in W} q^{r-n(w)}$, which by the formula of Shephard and Todd (proved by Solomon) gives the required formulae. □

Note that (5.9) may also be proved from (5.6) or (5.8), which yield the formulae

$$|W|^{-1} \sum_{X \in L} |W_X||M_X^F| \tag{5.9.1}$$

and

$$|W|^{-1} \sum_{X \in L} \sum_{Y \geq X} \mu(X, Y)|W_X|q^{r-r(Y)} \tag{5.9.2}$$

for the required number. The equivalence of all these formulae is easily established.

(5.10) COROLLARY. *Let $m_1, \ldots, m_l$ be the set of exponents of a Weyl group. Then for almost all primes q, the number*

$$\frac{(q+m_1)\cdots(q+m_l)}{(1+m_1)\cdots(1+m_l)}$$

is an integer.

(5.11) QUESTION. To what extent does the property (5.10) characterise the sets of exponents of Weyl groups?

For $l \leqslant 3$, one obtains the Weyl group exponents, and one additional sequence, (1, 11). Although there are infinite sequences other than those which correspond to types A, B and D, one might ask whether these sequences classify some geometric objects.

(5.12) COROLLARY. *We have the following polynomial identity*

$$\sum_{\lambda \vdash n} f_\lambda(q) = \frac{q(q+1)\cdots(q+n-1)}{n!}$$

where, for a partition $\lambda = 1^{r_1}2^{r_2}\cdots k^{r_k}$,

$$f_\lambda(q) = \frac{q(q-1)\cdots(q-p(\lambda)+1)}{r_1!\cdots r_k!}$$

and $p(\lambda) = \Sigma_{i=1}^k r_i$.

This follows immediately from (5.9) by applying it to the case of type A, since $f_\lambda(q)$ is the number of rational split semisimple classes of "type λ".

We conclude with a discussion of the "Coxeter case".

(5.13) THEOREM. *The number of rational semisimple orbits c in* $\mathscr{G}$ *which are of Coxeter type (i.e. such that* $\omega(c)$ *is the Coxeter class of* W*) is equal to*

$$q^r|C_W(\gamma)|^{-1}P_{M_W}(-q^{-1};\gamma)$$

where γ *is a Coxeter element of* W *(assuming regular characteristic) and* $r = \text{rank}\ (\mathscr{G})$.

Proof. Take χ in (5.6) to be the function

$$\chi(w) = \begin{cases} 1 & \text{if } w \text{ is a Coxeter element,} \\ 0 & \text{otherwise.} \end{cases}$$

Now a Coxeter element γ is in S_X only when $W_X = 1$, since otherwise W_X contains a reflection r and $n(\gamma r) < n(\gamma)$. Thus by (5.6), the required number is

$$|W|^{-1}\#\{\text{Coxeter elements of } W\}|\mathscr{T}_{ors}^{\gamma F}|.$$

The result follows from [L3] and (4.7) above. □

(5.14) EXAMPLES. (i) In type A_{l-1} the number of (5.13) is ([L1])

$$\frac{1}{l}\left(q^{r-l}\sum_{d\mid l}\mu(l/d)q^d\right).$$

Note that this implies that $(1/l)\ \Sigma_{d\mid l}\ \mu(d)q^{l/d}$ is an integer for all q, which follows form Macmahon's theorem.

(ii) In type B_l the number is given (see [L2, (4,5)]) as follows. Write $l=2^k l_1$, with l_1 odd. Then the required number is

$$\begin{cases} \dfrac{1}{2l}\sum_{d\mid l}\mu(d)(q)^{l/d} & \text{if } l \text{ is odd,} \\ \dfrac{q^{2k}-1}{2^{k+1}} & \text{if } l=2^k, k>0, \\ \dfrac{1}{2l}\sum_{d\mid l_1}\mu(d)q^{l/d} & \text{if } l \text{ is even, } l_1 \neq 1. \end{cases}$$

Again this implies divisibility properties for the number theoretic functions concerned.

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