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Autor(en): **Carter, J. Scott**

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Extending immersed circles in the sphere to immersed disks in the ball

J. SCOTT CARTER

Abstract. Consider a general position immersion of a circle into the 2-sphere. Suppose the immersion has an even number of double points. Then there is a proper immersion of the 2-disk that has the given curve as its boundary. Of all such extentions there is one with a minimum number of triple points. This minimum is obtained algorithmically in terms of a number that is associated to the double point set.

1. Introduction

1.1. *History*

The study of immersed curves dates as far back as Gauss [15]. In this work, he defines a word on the crossing points of an immersed curve. This word together with crossing information determines a knot in 3-space. Whitney [21] computes the tangential winding number of a curve in terms of the crossing information. More recent studies ([1], [14], and [18]) give information on extending immersed curves to singular maps of surfaces into the plane.

This paper is motivated by several factors. First, the "kinky box" described in [16] has the peculiarity that it must have a triple point (see also section 1.4). Second, the problem of finding minimal genus immersions that represent mod 2 homology classes in 3-manifolds requires an understanding of how immersed surfaces look when a 3-manifold is decomposed into a Heegaard splitting (See [9]). Fourth, properly immersed disks can be glued together along their boundary to provide projections of immersed spheres in 4-space. There is a relationship between triple points and slice disks that is not completely clear [20]. Finally, there is a nice interplay between the geometry of the curves and the algebra of the Gauss words that I find extremely appealing.

Turning to the mundane, all maps and manifolds are smooth and in general position. Immersed manifolds with boundary are proper in the sense that the boundaries are immersed in the boundaries and tubular neighborhoods of the boundaries inject as subbundles appropriately.

1.2. *The Problem*

Let $f: S^1 \to S^2$ denote a general position immersion. Suppose there is an immersion $F: (D^2, S^1) \to (B^3, S^2)$ such that $\partial F = f$. Among all such extensions there is one with a minimum number of triple points. The problem addressed herein is to compute this minimum as a function of the immersion f. Hence, define the non-negative integer, $T(f)$, to be the minimum number of triple points of all the immersed disks that have (f, S^1) as their boundary. The triple points sets of any two extensions have the same parity by a theorem of Banchoff [2]. The solution given in Theorem 3.5 is based on an invariant that is associated to pairings of letters in the Gauss word of the immersed circle.

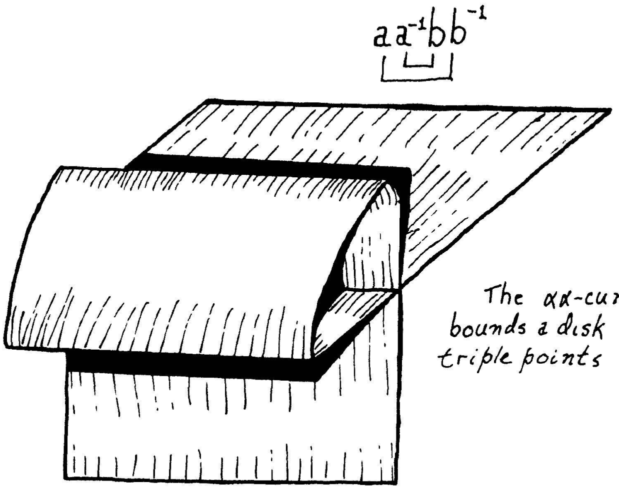
1.3. *Organization*

In Section 1.4 the kinky box of [16] is depicted. Section 2 briefly reviews Gauss words. The Reidemeister moves and (k, r) surgeries change curves and the resulting Gauss words. These changes are illustrated in Section 2.2. Theorem 2.3 states that any immersed circle with an even number of double points bounds an immersed disk in the 3-ball; this disk may be obtained by a sequence of $(1, 0)$ surgeries, $(2, 0)$ surgeries, type II, and type III moves.

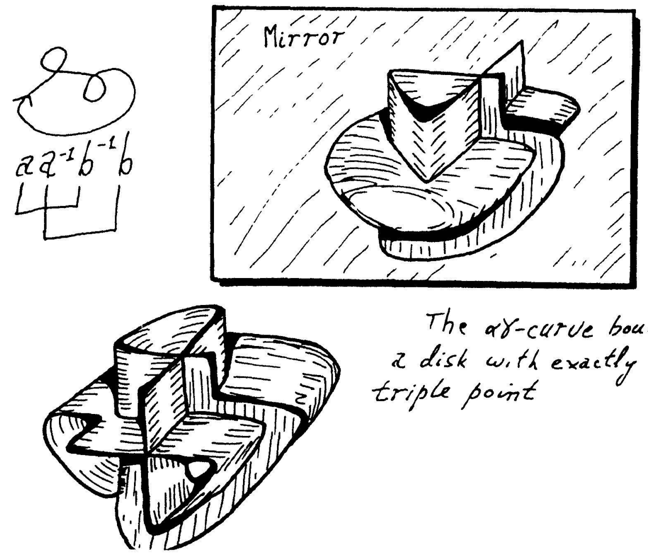
In section 3, an immersed disk is assumed to exist. This disk defines a partition of the double point set of the boundary into two element subsets. Such a partition can be defined in the abstract; it is called a bifilaration. The dictionary definition of *bifilar* is an object with two threads or filaments. The filaments here are arcs that connect letters in the Gauss word; the points at which such arcs intersect correspond roughly to triple points of disks. Hence, a crossing number is associated to a bifilaration. The minimum crossing number of all bifilarations of the given curve is a lower bound for the number of triple points of a disk bounded by the curve. A bifilaration that achieves this minimum is induced by an immersed disk (Lemma 3.4.4). Thus the minimum crossing number is equal to the minimum number of triple points of immersed disks. This, the main result, is stated and proved in Theorem 3.5; the proof depends in a key way on planarity.

1.4. *Two standard immersed disks*

Figure 1 illustrates the standard disk that is bounded by the $(\alpha\alpha)$-immersion. Figure 2 illustrates the "kinky disk" that is bounded by the $(\alpha\gamma)$-immersion. The illustrations clarify these names.



The αα-curve bounds a disk without triple points



The αγ-curve bounds a disk with exactly one triple point

2. Gauss words

2.1. *Definitions*

Let (f, S^1) denote an immersed curve. Assume that f is in general position. Choose an orientation and a base point (other than a double point) for S^1. Label each double point of S^1 in S^2 with a letter from some finite alphabet. The *Gauss word* associated to f is defined as follows. At the base point the Gauss word is the empty word. A particle that starts at the base point and travels in the direction of the orientation of S^1 encounters double points. As each double point is encountered, the letter that corresponds to the double point is juxtaposed with an exponent (± 1) to the right end of the word determined thus far. The exponent is chosen to be positive if and only if the other sheet of the immersed curve crosses from left to right. Each encounter of the double point set is thus recorded for one complete circuit of the curve.

No group theoretic meaning of the words is intended. In particular, the syllable aa^{-1} does not cancel. In general, *a syllable* is a segment of the Gauss word. The letters of the Gauss word and their inverses can be safely confused with the double point set of the immersed circle. This abuse of notation will be used without further ado.

The Gauss word is defined only up to cyclic permutation, permutations of the letters, and involution of the exponents. The papers [17, 12, 19, 10, 11] contain further details. Gauss words classify immersed curves [10].

2.2. *Reidemeister moves and surgery*

There are two types of operations to be performed to immersed curves: the "Reidemeister" moves and surgery on immersions. The parity of the double point set is to be preserved; so a type I move inverts a curl as in Figure 3 rather than eliminates it. Some of the surgery operations can be factored as surgery together with a Reidemeister move, but these will still be codified in their unfactored form as in [4]. A (k, r) surgery uses a k-disk embedded in the r-tuple set, with boundary in the $(r+1)$-tuple set, as the core of a hollow handle. The papers [4], [5], and [6] contain further details.

The Figures 3 through 8 depict the operations. The captions show how the Gauss words are affected by these moves. Syllables are depicted outside the neighborhoods in which the changes occur. Of course, some of these syllables may coincide. Figure 9 illustrates many of these moves.

The traces of the surgeries and of the type II and III moves are easy to understand. The trace of a type I move is to insert the "kinky-box" of Haas and Hughes [16]. This immersed disk is also depicted in Figure 2.

2.3. THEOREM A. *Any immersed circle in the 2-sphere that has an even number of double points is equivalent to an embedded circle by a sequence of separating* $(1, 0)$ *surgeries,* $(2, 0)$ *surgeries, type II, and type III moves.*

B. *Such a sequence defines a properly immersed disk in the 3-ball that has the given curve as its boundary and that has no local maxima.*

C. *Any properly immersed disk in the 3-ball can be factored as a sequence of these moves together with* $(0, 0)$ *and connecting* $(1, 0)$ *surgeries.*

The orientation of any arc may be reversed.

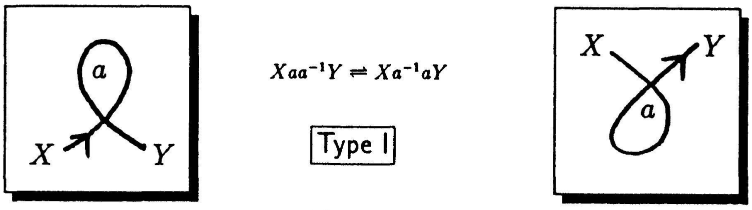


Figure 3

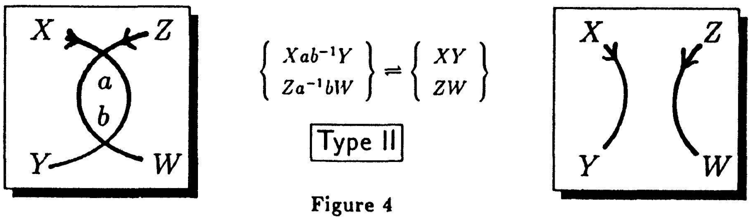


Figure 4

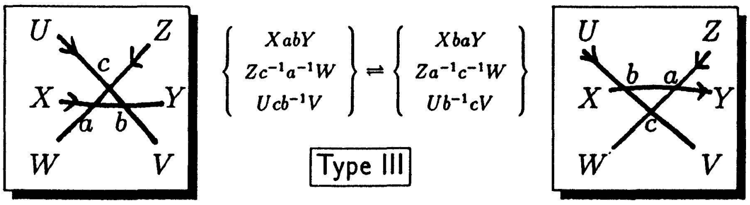


Figure 5

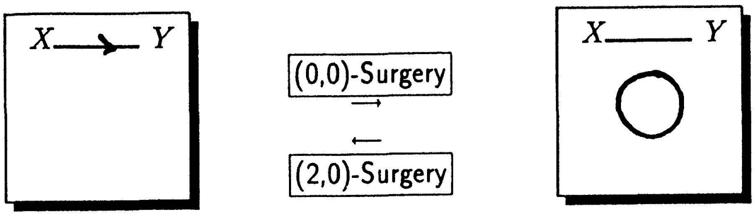


Figure 6

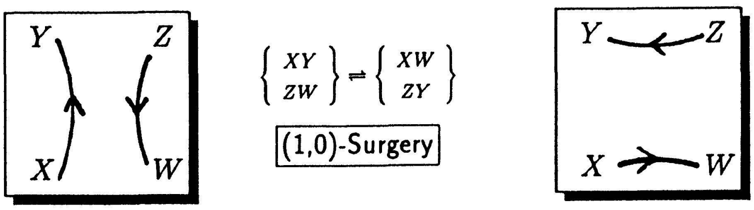


Figure 7

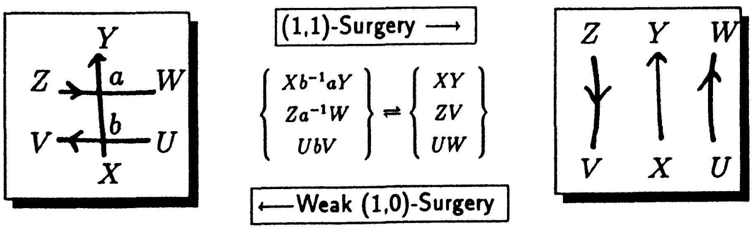


Figure 8

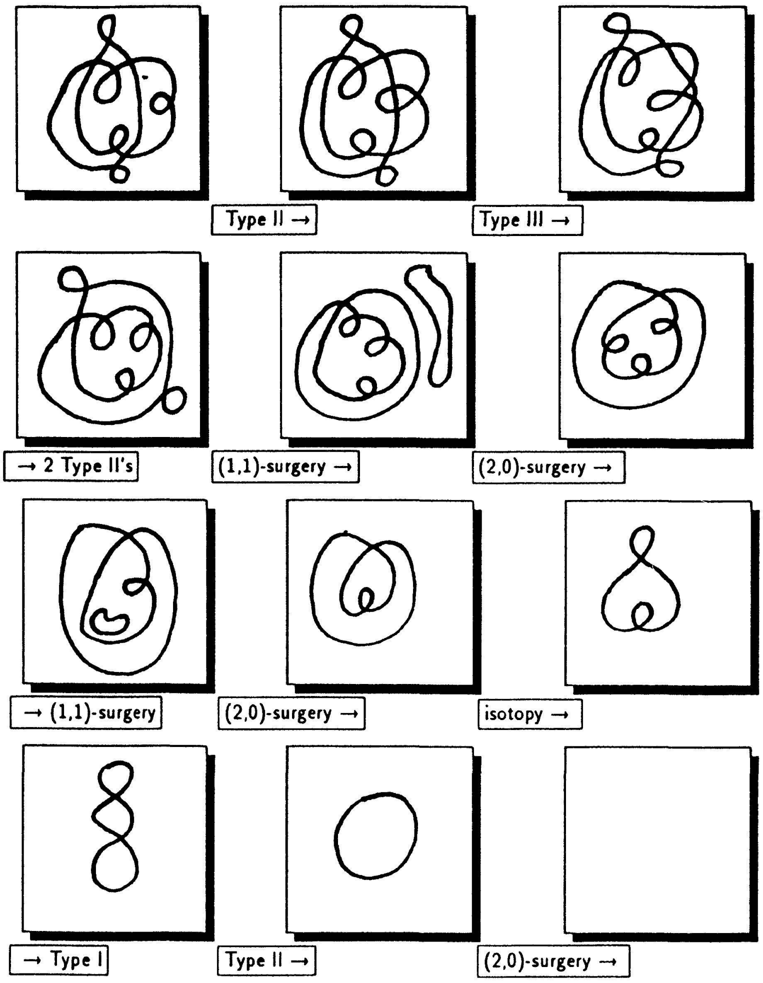


Figure 9

Proof. Haas and Hughes [16], for example, illustrate that a "kinky-box" can be factored as a sequence of type II and III moves. Thus type I moves are not necessary. A $(1, 1)$ surgery can be factored as a $(1, 0)$ surgery and a type II move.

Regular homotopy in the plane is generated by type II and type III moves. Thus the given curve can be regularly homotoped in the sphere to an immersed curve that has Gauss word of the form $aa^{-1}bb^{-1}\ldots$. A sequence of $(1, 0)$-surgeries will separate the curve into a collection of curves each of which is essentially the curve illustrated in Figure 1. These $(1, 0)$ separating surgeries form critical levels for an immersed planar surface. Figure 1 can be rearranged so that the double points vanish by means of a type II move. A local minimum in a Morse decomposition corresponds to a type $(2, 0)$ surgery. This completes the proof of A and B.

Let a properly immersed disk (F, D^2) be given. The distance from the center of the 3-ball to a point $F(x)$ in the image of F can be approximated by a Morse function on D^2. The double point curves in D^2 are the image of an immersed

1-manifold. The distance function may be further assumed to have non-degenerate critical points on this 1-manifold. Surgery of type (1, 1) is avoided by making the critical points of the 2-disk distinct from those of the double points. Isotope the image of the disk so there is a family of concentric 2-spheres such that between any two there is at most one critical point (of the disk or of the double point set) or one triple point. The intersection of (F, D^2) with any one these spheres is a collection of immersed curves. The difference between any consecutive spheres is a move of the prescribed type. This completes the proof.

3. The double point set of an immersed disk

3.1. *Filaments and bifilarations*

Let $w(f)$ denote the Gauss word of an immersed curve $f: S^1 \to S^2$ that has an even number of double points. Let $L = \{a_1^{\pm 1}, \ldots, a_n^{\pm 1}\}$ denote the letter set of the Gauss word. Thoughout this section if a is an element of the set L, then b^{-1} is a letter with different base and opposite exponent.

A *bifilaration* of $w(f)$ is a partition of the letter set of $w(f)$ into two element subsets, called *filaments*, that satisfy the following conditions:

1. No filament contains both a and a^{-1};
2. The two letters in a filament have opposite exponents;
3. If $\{a, b^{-1}\}$ is a filament, then so is $\{b, a^{-1}\}$.

Given a bifilaration in which $\{a, b^{-1}\}$ is a filament, the pair $\{a, b^{-1}\}$, $\{b, a^{-1}\}$ is called a *bifilar*, and these filaments are called *companions* of each other. A filament is depicted as an embedded arc joining the letters in the Gauss word that define the filament. As such a filament is naturally oriented: it proceeds from the point of negative exponent to the point of positive exponent. This orientation can be specified by denoting the filament as an ordered pair: $[a^{-1}, b]$.

3.2. *Arcs of double points*

Let $F: (D^2, S^1) \to (B^3, S^2)$ denote a properly immersed disk in general position. Then the restriction of F to the boundary has an even number of double points. The preimage under F of the double point set consists of immersed arcs with their end points on the boundary of the disk and closed curves in the interior.

The inverse image of double points of F come in pairs. For example, if the double points a and b^{-1} are joined by an immersed arc, then the double points a^{-1} and b are also joined by an arc and these two arcs have the same image under F. Moreover, orientation considerations force an arc to end in points with opposite exponents.

An immersed disk (F, D^2) that minimizes that number of triple points among all disks with a given boundary may be assumed to have no closed double point curves in its interior. This follows from Dehn's Lemma and from an inner-most curve argument.

Using the language of section 3.1, the following observation has been established.

3.3. OBSERVATION. *A properly immersed disk in the* 3*-ball induces a bifilaration on the double point set of the boundary.*

3.4. *The intersection form of a bifilaration*

Two filaments are said to *cross* if the letters that describe the filaments alternate in the Gauss word. For example, the filaments $[b^{-1}, a]$ and $[d^{-1}, c]$ cross in the word: $w = aXcYb^{-1}Zd^{-1}U$. When filaments are depicted as arcs below the Gauss word they may be assumed to intersect at most once.

3.4.1. LEMMA. *Given a bifilaration, $B(w(f))$, of a Gauss word of a spherical curve f, there is a skew-symmetric intersection form defined on the free $\mathbf{Z}$-module that is generated by the filaments.*

Proof. If ϕ and ψ are oriented filaments, then $\langle \phi, \psi \rangle$ is defined to be the signed intersection number between ϕ and ψ, where the sign is positive if and only if ψ crosses ϕ from left to right when viewed from ϕ. This is the same sign convention given for Gauss words. This completes the proof.

3.4.2. *Definition of crossing numbers.* Choose an ordering of the filaments of a bifilaration such that companion filaments are adjacent in the ordering. Consider the matrix of the form $\langle *, * \rangle$ with respect to this ordered basis. A *crossing number*, $c(B)$, is associated to the given bifilaration, B. It is defined by the following procedure:

0. Before the procedure begins the crossing number is defined to be the number 0. Let ϕ, ψ, and η denote oriented filaments, and let $\bar{\phi}$, $\bar{\psi}$ and $\bar{\eta}$ denote their respective companions.
1. For each pair of companion filaments that intersect the integer 1 is added to the crossing number. The corresponding two non-zero entries of the intersection matrix are each replaced with 0.

2. If $\langle\phi, \psi\rangle = 1$, and $\langle\phi, \bar{\psi}\rangle = -1$, then the integer 1 is added to the crossing number, and these four non-zero entries in the intersection matrix are replaced with 0.
3. If $\langle\phi, \psi\rangle = 1$, $\langle\bar{\phi}, \eta\rangle = -1$, and $\langle\bar{\psi}, \bar{v}\rangle = 1$, then the integer 1 is added to be crossing number, and these six non-zero entries are replaced by 0.
4. For each of the remaining non-zero entires in the upper triangular block of the matrix in question, add 1 to the crossing number, and replace these entries, and their counterparts below the diagonal, in turn with zeros.

The procedure terminates when the zero matrix is engendered. Thus *the crossing number*, $c(B)$, *of a bifilaration* is defined. The crossing number is a method for computing the minimum number of triple points of an immersed disk with the given curve as boundary (Theorem 3.5).

Define *a crossing number of the curve*, (f, S^1), *by* $c(f) = \min c(B)$ where the minimum is taken over all bifilarations B of f.

3.4.3. *Lengths and complexities.* The *length of a filament*, $\{a, b^{-1}\}$, is the shortest syllable in the cyclic Gauss word that is flanked by the letters a and b^{-1}. The *length of a bifilar* is the sum of the lengths of its companion filaments.

Given an immersed curve f, with Gauss word $w(f)$ consider the pair of letters of the Gauss word that yield the shortest possible bifilar. If there are two such pairs, then consider a pair for which one of the companions is shorter. For example, the bifilar defined by (a, b) in the word $aa^{-1}bb^{-1}$ is better than (c, d) in the word $cc^{-1}d^{-1}d$. The length of the shortest possible bifilar is an intrinsic invariant of the given curve. Therefore, define the *complexity* of the curve f to be the ordered pair (number of double points, length of the shortest possible bifilar). Complexities are given the lexicographical ordering.

3.4.4. LEMMA. *Let an immersed curve* (f, S^1) *that has an even number of double points be given. Then there is a properly immersed disk,* (F, D^2), *in the* 3-*ball with* $\partial F = f$ *such that* $T(F) = c(f)$.

Proof. The proof is dependent on the planarity of the curve and will follow by induction on the complexity of the curve (f, S^1). Certainly, the result is true in case the immersion f is an embedding. If f has two double points, then either f is the $\alpha\alpha$-curve or the $\alpha\gamma$-curve; these are depicted in Figures 1 and 2, respectively. In these figures, immersed disks with the correct number of triple points are also depicted. In the case of the $\alpha\gamma$-curve, there is at least one triple point by Banchoff'sTheorem [2]. Suppose the result is true for all curves of smaller complexity than the given curve. Construct a bifilaration by successively choosing the shortest possible bifilars. This bifilaration achieves the smallest crossing number.

Consider the shortest possible bifilar. The following cases will be examined: Either, (1) this bifilar crosses no other filaments, or (2) there is a sequence of Reidemeister moves, in which exactly one type III move occurs, that shortens the length of this bifilar and that reduces the crossing number by one. In the latter case, the argument is roughly an inner-most curve argument: That is, by choosing the shortest bifilar, not many double points will be seen in the disk bounded by the bifilar, or else that bifilar is not short.

In the former case, there are two possibilities: the bifilar intersects itself, or the bifilar is embedded. When the bifilar is embedded, there is a (1, 1)-surgery which eliminates the pair of double points on the boundary that define the bifilar. A pair of mutually disjoint immersed curves results: one of these is disjointly embedded; the other has smaller complexity. When the bifilar intersects itself, the curve contains an $\alpha\gamma$ arc as a "subword". This arc can be eliminated by introducing a single triple point, and the number of double points has been reduced. In this case the result follows by induction.

The drawings of Figure 10 depict schematically the possibilities that must be analyzed in case possibility (1) does not occur. The question marks indicate that there may be a variety of intersections. The thickened lines indicate that more than one arc may intersect. For each point of intersection that contributes a term to the crossing number, a sequence of Reidemeister moves will be performed that reduce the crossing number by exactly one, and that adds exactly one triple point at the stage of a type III move.

If either companion of the shortest bifilar contains an α-type kink, then a sequence of a type II, type III, and then some type II moves moves the kink out of the realm of the bifilars in question. See for example the top of Figure 9.

In case an embedded arc crosses both of the bifilars as in the top illustration of 10, then there is a triangle found in the picture over which a type III move can be made. This reduces the crossing number of the bifilaration by exactly one, for the relative order of the intersection points of this bad arc is interchanged elsewhere in the Gauss word according to Figure 5.

In case an embedded arc crosses both of the filaments as in the bottom illustration of 10, one of the loops may be slid past this arc as was the case two paragraphs above.

In case this arc that crosses the shortest bifilar is not embedded, then it contains at most one α-kink (or else there is a shorter bifilar). And the companion points for this kink are outside the realm of the figure, and induce further crossing. The kink can be moved as in case considered in the paragraph above, thereby reducing the crossing number.

Finally, in case several arcs cross the shortest bifilar, the intersection points among them induce terms in the crossing number, and these can be slid awayby means of appropriate type II moves. The case of a single embedded arc applies.

In conclusion, by consistently choosing the shortest possible bifilars, a bifilaration that achieves the minimal crossing number is found. This bifilaration is induced by an immersed disk whose triple point number agrees with the crossing number. This completes the proof.

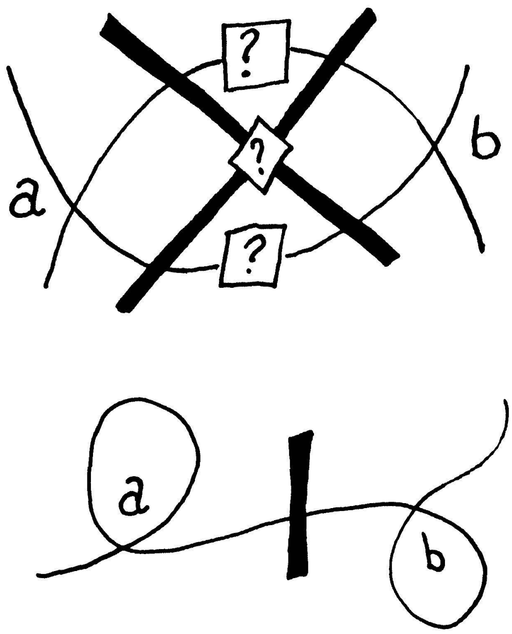


Figure 10

3.5. THEOREM. *Let an immersed curve $f: S^1 \to S^2$ with an even number of double points be given. Then*

$$T(f) = c(f).$$

Proof. Let $F:(D^2, S^1) \to (B^3, S^2)$ be an immersion of the 2-disk that achieves the minimum number of triple points, $T(f)$, among all extensions of (f, S^1). Then by 3.3 there is a bifilaration of the Gauss word of f that is induced by F. The number of triple points of F is greater than or equal to the crossing number of the bifilaration by 3.4.4. Thus $c(f) \leq T(f)$.

On the other hand there is an immersed disk that has $c(f)$ triple points by 3.4.4. Since $T(f)$ is the minimum number of triple points among all extensions of f, $T(f) \leq c(f)$. This completes the proof.

3.6. *Acknowledgements*

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Department of Mathematics
University of South Alabama
Mobile, Alabama 36688

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