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Eisenstein cocycles for $GL_2\mathbb{Q}$ and values of L-functions in real quadratic fields

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1.1. This is the first part in a series of papers devoted to the explicit construction of Eisenstein cocycles representing certain Eisenstein cohomology classes in $H^{n-1}(G, M)$ with $G = GL_n(\mathbb{Q})$ in the sense of Harder [1]. In the case $n = 2$ (only this case is being considered in the present paper), these cocycles are maps $\Phi : G \to M$ into a G-module M satisfying the homomorphism property $\Phi(AB) = \Phi(A) + A\Phi(B)$. A characteristic property of the values $\Phi(A)$ is that they can be expressed by Dedekind sums. In fact, the classical reciprocity formula for Dedekind sums is only a special case of the homomorphism property of Φ. In general, one is interested in the Eisenstein cohomology for two reasons. First, it represents the most accessible and best understood part of the group cohomology of G. Secondly, but more importantly, the special values of zeta- and L-functions in totally real number fields can be expressed by Eisenstein cocycles. Besides its theoretical importance, this fact has also practical implications as the cocycle property can be used to design an efficient algorithm for calculating the special values in question explicitly. The construction of the Eisenstein cocycles has been accomplished so far only in the case $n = 2$ mainly because of the combinatorial and analytical problems arising in the general case. These difficulties are already visible in the simplest case where $\Phi(A)$ is the period

$$\Phi(A) = \int_{\tau}^{A\tau} \sum_{m,n}{}' (mz + n)^{-2}\, dz, \qquad A \in G \tag{1}$$

of an Eisenstein series of weight 2. As is well known, this series converges only conditionally, so some limiting procedure must be specified. Moreover, since the path of integration must completely belong to the upper (or lower) halfplane, the determinant of A must be positive. In this paper, we present a new approach for dealing with these difficulties (for a different approach, compare the recent paper of Stevens [14]). Instead of (1), we study the series

$$\Phi(A) = \sum_{m,n}{}' \frac{A\tau - \tau}{(mA\tau + n)(m\tau + n)}$$

which arises from (1) by termwise integration. This series has much better analytic properties than (1). For instance, it still converges for rational τ where (1) does not make sense anymore. In this way $\Phi(A)$ is defined for all $A \in G$, not only for A of positive determinant. Although the last series still converges only conditionally, its sum can be defined as the limit of the partial sums taken over all (m, n) with $|Q(m, n)| < t$ as t approaches ∞, where Q is some fixed nondegenerate binary form. The definition depends of course on the choice of Q, but as it turns out, the dependence is a very simple one, c.f. Theorem 1. But the main advantage of defining Φ by the second series is that this construction has a natural generalization for $n > 2$. Let A_i $(i = 1, \ldots, n)$ be n matrices in $G = GL_n\mathbb{Q}$, and let A_{ij} be the j-th column of A_i. Then for every nonzero vector $x \in \mathbb{R}^n$, and every matrix A_i, there is at least one column A_{ij} in A_i such that the standard scalar product $\langle x, A_{ij}\rangle$ does not vanish. For given $x \neq 0$, we denote by A_{ij_i} the first column in A_i with this property, and define

$$\phi(A_1, \ldots, A_n)(x) = \frac{\det(A_{1j_1}, \ldots, A_{nj_n})}{\langle x, A_{1j_1}\rangle \cdots \langle x, A_{nj_n}\rangle}.$$

The map $\phi : G^n \to M$ is a homogenous $(n-1)$ cocycle for G with values in the G-module M of complex valued functions on $\mathbb{R}^n \backslash \{0\}$ with the G-action given by $(Af)(x) = f(xA)$ for $x \in \mathbb{R}^n \backslash \{0\}$, $A \in G$. The Eisenstein group cocycle Φ for G is then constructed by averaging the values of ϕ over a coset $x + \mathbb{Z}^n$ of $\mathbb{Z}^n$ in $\mathbb{R}^n$. This process requires considerable caution as the corresponding infinite series converges only conditionally. It turns out that for a suitable choice of a homogeneous polynomial Q on $\mathbb{R}^n$ (essentially a normform in a totally real number field), the following limit does exist,

$$\Phi(A_1, \ldots, A_n)(Q, x) = \lim_{t \to \infty} \sideset{}{'}\sum_{\substack{m \in \mathbb{Z}^n + x \\ |Q(m)| < t}} \phi(A_1, \ldots, A_n)(m),$$

and defines a homogeneous $(n-1)$ cocycle, called the Eisenstein group cocycle for $GL_n\mathbb{Q}$. As in the classical case $n = 2$, the values of Φ can be expressed in finite (but complicated) terms using higher dimensional Dedekind sums. Moreover, the cocycle Φ is universal in the sense that its values parametrize all special values of Hecke L-functions in totally real number fields which are known to be algebraic numbers or more generally, algebraic numbers times a power of π. Roughly speaking, special values of Hecke L-functions arise when Φ is evaluated on cycles (representing homology classes in $H_{n-1}(GL_n\mathbb{Z})$) which are constructed out of $n-1$ independent units in a totally real number field. The proof of these statements for $n > 2$ will be given in a future paper. In the present paper we wish to treat the case $n = 2$ in great detail as an introduction and motivation for the general case.

The paper is organized as follows. In section 1.3, we present a limit formula which allows us to control the conditional convergence of all series discussed in this paper. In order to illustrate our basic idea in a simple case, we discuss in section 2.1 an example due to Eisenstein himself. The main section is 2.2 where we construct a rational cocycle for G. By summing this cocycle over the lattice $\mathbb{Z}^2$ in two different ways, we construct a trigonometric cocycle in section 2.3 and a Bernoulli cocycle in section 2.4. In 2.5 we show how these cocycles naturally lead to special values of L-functions.

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1.2. Notation

We abbreviate $e(x) = \exp(2\pi i x)$, and denote by c_k, $k = 1, 2, 3, \ldots$, the trigonometric function

$$c_k(x) = \pi^{-k} \sum_{m \in \mathbb{Z}+x}' m^{-k} \tag{2}$$

(with summation according to increasing values of $|m|$ in the case $k = 1$). For $x \in \mathbb{C}\backslash\mathbb{Z}$, we have $c_1(x) = \cot \pi x$, $c_2(x) = \sin^{-2} \pi x$, etc. In addition to the c_k, it's convenient to introduce the functions

$$d_k(x) = (-1)^{k-1}(k-1)!\, c_k(x) \tag{3}$$

which satisfy the relation $\pi d_{k+1}(x) = d_k'(x)$ for non-integral x. Besides the c_k and d_k, we use the periodic Bernoulli functions

$$\mathscr{B}_k(y) = -\frac{k!}{(2\pi i)^k} \sum_{m \in \mathbb{Z}}' \frac{e(my)}{m^k}$$

(again with summation according to increasing values of $|m|$ if $k = 1$) which are defined for real y and coincide with the Bernoulli polynomials $B_k(y)$ for $0 < y < 1$.

Finally, it is natural to define

$$c_0(x) = \begin{cases} 0 & \text{for } x \notin \mathbb{Z} \\ -1 & \text{for } x \in \mathbb{Z}. \end{cases}$$

With this definition, we have $c_k(x) = c_k(x, 0)$ for all $k \geq 0$ where $c_k(x, 0)$ is defined by the meromorphic continuation of

$$c_k(x, s) = \pi^{-k} \sum_{m \in \mathbb{Z}+x}' m^{-k}|m|^{-s}, \qquad \operatorname{Re}(s) > 1.$$

1.3. A limit formula

Let $Q(p)$ be a binary form with nonzero coefficients α_j, β_j (real or complex),

$$Q(p) = \prod_{j=1}^{n} (\alpha_j p_1 - \beta_j p_2).$$

We consider for $u \in \mathbb{C}^2$, $v \in \mathbb{R}^2 \backslash \mathbb{Z}^2$ and integers $k, l \geq 1$, the absolutely convergent series

$$S(t) = \sum_{\substack{p \in \mathbb{Z}^2+u \\ |Q(p)|<t}}' \frac{1}{p_1^k p_2^l}, \qquad S^*(t) = \sum_{\substack{p \in \mathbb{Z}^2 \\ |Q(p)|<t}}' \frac{e(pv)}{p_1^k p_2^l}.$$

THEOREM 1. *The limits* $S = \lim_{t\to\infty} S(t)$, $S^* = \lim_{t\to\infty} S^*(t)$ *exist and have the value*

$$S = \pi^r(c_k(u_1)c_l(u_2) + s_r(Q)), \qquad S^* = \frac{(2\pi i)^r}{k!\,l!}\mathscr{B}_k(v_1)\mathscr{B}_l(v_2), \qquad r = k+1$$

with

$$s_2(Q) = 1 - \frac{2}{\pi n}\sum_{j=1}^{n} |\arg(\alpha_j/\beta_j)| \quad \textit{and} \quad s_r(Q) = 0 \text{ for } r > 2.$$

Here, arg denotes the principal branch of the argument function. In particular, for real α_j, β_j, we have

$$s_2(Q) = \frac{1}{n}\sum_{j=1}^{n} \operatorname{sign}(\alpha_j \beta_j).$$

A proof of this theorem was given in [8]. For a generalization, see [9].

2.1. The addition law

In order to illustrate our basic idea in a simple case, we prove in this section the addition law for the cotangent function. Interpreted properly, this simple example already contains all the key ingredients for the following more sophisticated examples. We start with the relation $p+q+r=0$ where p, q, r run over $\mathbb{Z}+u$, $\mathbb{Z}+v$, $\mathbb{Z}+w$ with some fixed complex numbers u, v, w satisfying $u+v+w \in \mathbb{Z}$. For simplicity, we assume $u, v, w \notin \mathbb{Z}$. Then $pqr \neq 0$, and we get

$$\frac{1}{pq}+\frac{1}{qr}+\frac{1}{rp}=0. \tag{4}$$

Let α, β, γ be a tripple of nonzero real numbers such that $\alpha+\beta+\gamma=0$. Then $\alpha p-\beta q=\gamma q-\alpha r=\beta r-\gamma p$. Applying to (4) the summation process

$$\lim_{t\to\infty} \sum_{\substack{p,q \\ |\alpha p-\beta q|<t}} = \lim_{t\to\infty} \sum_{\substack{q,r \\ |\gamma q-\alpha r|<t}} = \lim_{t\to\infty} \sum_{\substack{r,p \\ |\beta r-\gamma p|<t}}$$

gives immediately

$$c_1(u)c_1(v)+c_1(v)c_1(w)+c_1(w)c_1(u)+\operatorname{sign}(\alpha\beta)+\operatorname{sign}(\gamma\alpha)+\operatorname{sign}(\beta\gamma)=0. \tag{5}$$

It is easy to see that the last three signs always add up to -1 (for instance, take $u=v=1/4$). This is the well known addition formula for the cotangent function. It has the following useful extension valid for all complex u, v, w with $u+v+w \in \mathbb{Z}$:

$$c_1(u)c_1(v)+c_1(v)c_1(w)+c_1(w)c_1(u)=1+c_0(u)c_2(v)+c_0(v)c_2(w)+c_0(w)c_2(u).$$

This follows again from (4). For instance, if $u \in \mathbb{Z}$, then summing (4) formally as we did before will lead to an error generated by terms with $p=0$. In order to compensate for this error, we have to add on the right

$$\sum_q \frac{1}{qr} = -\sum_q \frac{1}{q^2} = -\pi^2 c_2(v), \qquad (p=0),$$

which explains the term $c_0(u)c_2(v)$ in the formula above. Finally, we point out that multiplication of (4) by

$$e(pu-qv)=e(qw-ru)=e(rv-pw)$$

and summation over all integral p, q, r with $p+q+r=0$ leads to the "dual" addition formula for the Bernoulli function $\mathscr{B}_1$ in the form

$$2\mathscr{B}_1(u)\mathscr{B}_1(v)+2\mathscr{B}_1(v)\mathscr{B}_1(w)+2\mathscr{B}_1(w)\mathscr{B}_1(u)+\mathscr{B}_2(u)+\mathscr{B}_2(v)+\mathscr{B}_2(w)=0,$$

which is valid for real u,v,w (not all of them integral) such that $u+v+w\in\mathbb{Z}$.

2.2. The Eisenstein cocycle for GL_2

In this section we want to show that the formulas of the previous section are in fact special cases of one universal relation, namely the cocycle relation satisfied by the Eisenstein cocycle for the group $PGL_2\mathbb{Q}$ which we are going to introduce now.

As before, we start with a suitable partial fraction identity. Let $\sigma,\tau\in\mathbb{C}^2$ be two column vectors, neither of them the zero vector. To σ,τ we associate the rational function $f(\sigma,\tau)$ given by

$$f(\sigma,\tau)(m)=\frac{\det(\sigma,\tau)}{\langle m,\sigma\rangle\langle m,\tau\rangle},$$

where $m\in\mathbb{C}^2$ is a row vector, and $\langle m,\sigma\rangle$ resp. $\langle m,\tau\rangle$ denotes the usual scalar product on $\mathbb{C}^2$ given by

$$\langle x,y\rangle=x_1y_1+x_2y_2.$$

Note that $f(\sigma,\tau)$ is defined only on the complement of the lines $\langle m,\sigma\rangle=0$ and $\langle m,\tau\rangle=0$. Since

$$f(\lambda\sigma,\tau)=f(\sigma,\tau)=f(\sigma,\lambda\tau)\qquad\text{for }\lambda\in\mathbb{C}^*,$$

σ and τ may be viewed as points on the projective line $P_1\mathbb{C}$. As a generalization of (4), we have the following identity:

$$f(\rho,\sigma)+f(\sigma,\tau)=f(\rho,\tau).\tag{6}$$

This can be proved either by direct calculation, or by noticing that

$$f(\sigma,\tau)(m)=\int_{\tau_1/\tau_2}^{\sigma_1/\sigma_2}\frac{dz}{(m_1z+m_2)^2},$$

or by writing the identity in a homogeneous form as

$$\det \begin{vmatrix} \langle m, \rho \rangle & \langle m, \sigma \rangle & \langle m, \tau \rangle \\ \rho_1 & \sigma_1 & \tau_1 \\ \rho_2 & \sigma_2 & \tau_2 \end{vmatrix} = 0.$$

The last equation is obvious because the first row is a linear combination of the second and third one. In particular, we have $f(\sigma, \tau) = -f(\tau, \sigma)$. Another obvious property of f which we will use is

$$f(A\sigma, A\tau)(m) = \det(A) f(\sigma, \tau)(mA)$$

for every matrix $A \in G := PGL_2\mathbb{C}$. Let $H \subseteq \mathbb{C}[x, y]$ be the subspace of homogeneous polynomials $P(x, y)$ of degree $k = 0, 1, 2, 3, \ldots$ For every $P(x, y) \in H$, we define the function

$$f(\sigma, \tau)(P, m) = P(\partial_{m_1}, \partial_{m_2}) f(\sigma, \tau)(m) \tag{7}$$

where ∂_{m_1} resp. ∂_{m_2} denotes the partial derivative with respect to m_1 resp. m_2. For fixed P, this is again a rational function of m which clearly satisfies the identity (6). Calculating the right side of (7) explicitly, we get

$$f(\sigma, \tau)(P, m) = (-1)^k \det(\sigma, \tau) \sum_{0 \le j \le k} \frac{j!(k-j)!\, P_j(\sigma, \tau)}{\langle m, \sigma \rangle^{1+j} \langle m, \tau \rangle^{1+k-j}} \tag{8}$$

with

$$P_j(\sigma, \tau) = P_{k-j}(\tau, \sigma) = \sum_{0 \le i \le j} \partial_x^i \partial_y^{j-i} P(\tau_1, \tau_2) \frac{\sigma_1^i}{i!} \frac{\sigma_2^{j-i}}{(j-i)!}.$$

We will use this representation of $f(\sigma, \tau)$ in the next section. Let $A = (a, b, c, d) \in GL_2\mathbb{C}$, and let $n = mA = (am_1 + cm_2, bm_1 + dm_2)$. By the chain rule, we have

$$(\partial_{m_1}, \partial_{m_2}) f(am_1 + cm_2, bm_1 + dm_2) = (a\partial_{n_1} + b\partial_{n_2}, c\partial_{n_1} + d\partial_{n_2}) f(n).$$

Let $AP \in H$ be the homogeneous polynomial defined by

$$AP(x, y) = P(ax + cy, bx + dy).$$

Then $(AB)P = A(BP)$, i.e. the map $P \mapsto AP$ defines a $GL_2\mathbb{C}$ action on H from left.

CLAIM. $f(A\sigma, A\tau)(P, m) = \det(A) f(\sigma, \tau)({}^tAP, mA)$.

Proof. By definition of f, we have

$$\begin{aligned} f(A\sigma, A\tau)(P, m) &= P(\partial_{m_1}, \partial_{m_2}) f(A\sigma, A\tau)(m) \\ &= P(\partial_{m_1}, \partial_{m_2})(\det(A) f(\sigma, \tau)(mA)) \\ &= \det(A)\, {}^tAP(\partial_{n_1}, \partial_{n_2}) f(\sigma, \tau)(n) \\ &= \det(A) f(\sigma, \tau)({}^tAP, mA). \end{aligned}$$

This property of f can be reformulated as follows. Consider the space M of all functions $h : H \times \mathbb{C}^2 \mapsto \mathbb{C}$ satisfying the homogeneity property

$$\lambda^2 h(\lambda^k P, \lambda m) = h(P, m), \qquad k = \deg P$$

for $\lambda \neq 0$. This property implies that $G = PGL_2\mathbb{C}$ acts on M from the left by

$$Ah(P, m) = \det(A) h({}^tAP, mA) \qquad \text{for } h \in M, \quad A \in G.$$

Then, for every fixed τ, we have the map

$$\phi = \phi_\tau : G \mapsto M, \qquad \phi(A) = f(A\tau, \tau) = -f(\tau, A\tau).$$

LEMMA 1. *ϕ is a 1-cocycle for G whose cohomology class does not depend on the choice of τ.*

Proof.

$$\begin{aligned} \phi(AB)(P, m) &= f(AB\tau, \tau)(P, m) \\ &= f(AB\tau, A\tau)(P, m) + f(A\tau, \tau)(P, m) \\ &= \det(A) f(B\tau, \tau)({}^tAP, mA) + \phi(A)(P, m), \end{aligned}$$

i.e., ϕ satisfies the cocycle property $\phi(AB) = A\phi(B) + \phi(A)$. If τ' is another choice of τ, then $\tau' = B\tau$ for some $B \in G$, and we get by repeated application of the cocycle property,

$$\begin{aligned} \phi_{\tau'}(A) &= f(A\tau', \tau') \\ &= f(AB\tau, B\tau) = B\phi(B^{-1}AB) \\ &= \phi_\tau(A) + (A - 1)\phi_\tau(B) \end{aligned}$$

which shows that $\phi_{\tau'}$ differs from ϕ_τ only by a coboundary., In other words, the cohomology class of ϕ is independent of the choice of τ.

Recall that $\phi(A)(P, m)$ is not defined for $\langle m, \tau \rangle = 0$ or $\langle m, A\tau \rangle = 0$. It will be convenient to introduce an extension $\psi(A) \in M$ of $\phi(A)$ which is well defined for all $m \in \mathbb{C}^2$. To this end let $\bar{\tau} = {}^t(\bar{\tau}_2, -\bar{\tau}_1)$ for $\tau = {}^t(\tau_1, \tau_2)$. We set $\psi(A)(P, 0) = 0$, and define $\psi(A) = \psi(A)(P, m)$ for $m \neq 0$ by

$$\psi(A) = \begin{cases} f(A\tau, \tau) & \text{if } \langle m, A\tau \rangle \neq 0, \quad \langle m, \tau \rangle \neq 0 \\ f(A\tau, \bar{\tau}) & \text{if } \langle m, A\tau \rangle \neq 0, \quad \langle m, \tau \rangle = 0 \\ f(A\bar{\tau}, \tau) & \text{if } \langle m, A\tau \rangle = 0, \quad \langle m, \tau \rangle \neq 0 \\ f(A\bar{\tau}, \bar{\tau}) & \text{if } \langle m, A\tau \rangle = 0, \quad \langle m, \tau \rangle = 0 \end{cases}$$

(the last case arises only if $\det(A\tau, \tau) = 0$). Thus $\psi(A)$ equals $\phi(A)$ whenever the latter is defined. Note that formally, ψ differs from ϕ by a 1-coboundary,

$$\psi(A) = \phi(A) - (A - 1)h$$

where $h = \delta f(\tau, \bar{\tau})$ with $\delta = 1$ or 0 according to $\langle m, \tau \rangle = 0$ or $\neq 0$. In particular, Lemma 1 is also valid for ψ.

2.3. The trigonometric cocycle Ψ

As in section 2.1, we intend to sum the relation

$$\psi(AB)(P, m) = \psi(A)(P, m) + \det(A)\psi(B)({}^tAP, mA)$$

over all $m \in \mathbb{Z}^2 + u$ with some fixed vector $u \in \mathbb{C}^2$. This will lead to an interesting result only if A, B belong to $\Gamma := PGL_2\mathbb{Q}$. Every element in Γ can be represented by an integral matrix. We assume that A, B and AB are represented in this way. Then

$$Z_A := \det(A)\mathbb{Z}^2 \subseteq L_A := \mathbb{Z}^2 A$$

is a sublattice of L_A with index $|\det(A)|$, and we have the finite coset decomposition

$$L_A = \bigcup_r (r + Z_A) \qquad \text{(disjoint union).}$$

Ignoring questions of convergence for a moment, we can write

$$\sum_{m \in \mathbb{Z}^2 + u} \psi(B)({}^tAP, mA) = \sum_{q \in L_A + u'} \psi(B)({}^tAP, q), \qquad u' = uA$$

$$= \sum_r \sum_{q \in Z_A + r + u'} \psi(B)({}^tAP, q)$$

$$= \left(\frac{1}{\det A}\right)^{2+k} \sum_r \sum_{p \in \mathbb{Z}^2 + r'} \psi(B)({}^tAP, p), \qquad r' = \frac{r + uA}{\det A}.$$

(Note that in the last series, every $m \in \mathbb{Z}^2 + u$ can be written as $m = pA^*$ for some p with $AA^* = \det A$.) Hence

$$\sum_{m \in \mathbb{Z}^2 + u} \psi(AB)(P, m) = \sum_{m \in \mathbb{Z}^2 + u} \psi(A)(P, m)$$

$$+ \left(\frac{1}{\det A}\right)^{1+k} \sum_r \left(\sum_{p \in \mathbb{Z}^2 + r'} \psi(B)({}^tAP, p) \right).$$

In order to guarantee the convergence of every individual series in this formal identity, the parameter τ has to be chosen in $P_1\mathbb{Q}$. (The series converge and can be evaluated also for $\tau \in P_1\mathbb{C}\backslash P_1\mathbb{R}$ provided A and B have a positive determinant, c.f. [8]. But it is a subtle question whether the series converge for $\tau \in P_1\mathbb{R}\backslash P_1\mathbb{Q}$; we shall discuss a special case in section 2.5.) Moreover, since convergence is only conditional, the passage to the limit has to be taken over the (absolutely convergent) partial sums with $|Q(m)| < t$ resp. $|Q(pA^*)| < t$, $t \to \infty$ where Q is a product of linear forms whose coefficients are linearly independent over $\mathbb{Q}$. For such a choice of τ and Q, we define

$$\Psi(A)(P, Q, u) = \pi^{-2-k} \lim_{t \to \infty} \sum_{\substack{m \in \mathbb{Z}^2 + u \\ |Q(m)| < t}} \psi(A)(P, m).$$

Furthermore, if we define the action of $A \in \Gamma$ on the space of functions $F(P, Q, u)$ by

$$AF(P, Q, u) = \left(\frac{1}{\det A}\right)^{1+k} \sum_{r \in L_A/Z_A} F\left({}^tAP, A^{-1}Q, \frac{r + uA}{\det A}\right)$$

(where A has been represented by an integral matrix, and $A^{-1}Q$ is defined by $(A^{-1}Q)(p) = Q(pA^{-1})$), then we can summarize our considerations as follows.

THEOREM 2. *For every fixed $\tau \in P_1\mathbb{Q}$, the map Ψ which assigns to $A \in \Gamma$ the function $\Psi(A)$ of the three variables P, Q, u, is a 1-cocycle, i.e. it satisfies the relation $\Psi(AB) = \Psi(A) + A\Psi(B)$.*

Our next goal is to evaluate Ψ in finite terms. To this end let $(\tau, A\tau)$ be represented by an integral matrix M with primitive columns $\tau, A\tau$. Let $c = \det M$. We consider only the case $c \neq 0$. Then for $u \notin \mathbb{Q}^2$, we have

$$\pi^{2+k}\Psi(A)(P, Q, u) = -c \sum_{0 \le j \le k} P_j(\tau, A\tau)T_j,$$

$$T_j = (-1)^k \lim_{t\to\infty} \sum_{\substack{m \in \mathbb{Z}^2+u \\ |Q(m)|<t}} \frac{j!(k-j)!}{p_1^{1+j}p_2^{1+k-j}}, \qquad p = mM$$

$$= \frac{(-1)^k}{c^{2+k}} \sum_{r \in L_M/Z_M} \lim_{t\to\infty} \sum_{\substack{p \in \mathbb{Z}^2+r' \\ |Q(pM^*)|<t}} \frac{j!(k-j)!}{p_1^{1+j}p_2^{1+k-j}}, \qquad r' = \frac{r+uM}{c}$$

$$= \left(\frac{\pi}{c}\right)^{2+k} \sum_{r \in L_M/Z_M} (d_{i+j}(r_1')d_{1+k-j}(r_2') + s_{2+k}(M^{-1}Q))$$

where the trigonometric function d_j is given by (3) and $s_{2+k}(M^{-1}Q)$ denotes the correction term according to Theorem 1 (note that $s_{2+k}(M^{-1}Q) = 0$ for $k > 0$). If u or $uA \in \mathbb{Q}^2$, then we have to consider the additional tems coming from those m with $\langle m, \tau\rangle = 0$ or $\langle m, A\tau\rangle = 0$. This can happen only if $\langle u, \tau\rangle \in \mathbb{Z}$ or $\langle u, A\tau\rangle \in \mathbb{Z}$. Let $\langle u, \tau\rangle \in \mathbb{Z}$. Then $\langle m, \tau\rangle = 0$ iff $m \in T$,

$$T = \mathbb{Z}^2 + u \cap \mathbb{Q}\tau' = (\mathbb{Z} + w)\tau', \qquad \tau' = {}^t(\bar\tau), \quad w = \frac{\langle u, \bar\tau\rangle}{\langle \tau', \bar\tau\rangle}.$$

Moreover, if $\langle m, \tau\rangle = 0$ and $m \neq 0$, then writing $A\tau$ as a linear combination of τ and $\bar\tau$, we get

$$\langle m, A\tau\rangle = \frac{\langle \tau', A\tau\rangle}{\langle \tau', \bar\tau\rangle}\langle m, \bar\tau\rangle,$$

therefore

$$\psi(A)(P, m) = (-1)^k \det(A\tau, \bar\tau) \sum_{0 \le j \le k} \frac{j!(k-j)!\,P_j(A\tau, \bar\tau)}{\langle m, A\tau\rangle^{1+j}\langle m, \bar\tau\rangle^{1+k-j}}$$

$$= (-1)^k \det(A\tau, \bar\tau) \sum_{0 \le j \le k} \frac{j!(k-j)!\,P_j(A\tau, \bar\tau)}{\langle \tau', A\tau\rangle^{1+j}\langle \tau', \bar\tau\rangle^{1+k-j}} \left(\frac{\langle \tau', \bar\tau\rangle}{\langle m, \bar\tau\rangle}\right)^{2+k}$$

$$= \psi(A)(P, \tau')\left(\frac{\langle \tau', \bar\tau\rangle}{\langle m, \bar\tau\rangle}\right)^{2+k}.$$

A similar calculation in the case $\langle u, A\tau\rangle \in \mathbb{Z}$ resp. $c = 0$ leads to the following result.

THEOREM 3. *Let $(\tau, A\tau)$ be represented by an integral matrix M with primitive columns. If $c = \det M \neq 0$, then*

$$\begin{aligned}\Psi(A)(P, Q, u) = &-\frac{1}{c^{1+k}} \sum_{0 \le j \le k} P_j(M) \sum_{r \in L_M/Z_M} d_{1+j}(r'_1) d_{1+k-j}(r'_2), \qquad r' = \frac{r + uM}{\det M}\\ &- \psi(A)(P, \tau')c_0(\langle u, \tau\rangle) c_{2+k}\left(\frac{\langle u, \bar{\tau}\rangle}{\langle \tau', \bar{\tau}\rangle}\right)\\ &- \psi(A)(P, (A\tau)')c_0(\langle u, A\tau\rangle) c_{2+k}\left(\frac{\langle uA, \bar{\tau}\rangle}{\langle \tau', \bar{\tau}\rangle}\right)\end{aligned}$$

provided the correction term sign $(c)P(M)s_2(M^{-1}Q)$ *is subtracted from the right side in the case $k = 0$. If $c = 0$, then*

$$\Psi(A)(P, Q, u) = -\psi(A)(P, \tau')c_0(\langle u, \tau\rangle) c_{2+k}\left(\frac{(u, \bar{\tau}\rangle}{\langle \tau', \bar{\tau}\rangle}\right).$$

We remark that for every fixed j, the sum over $r \in L_M/Z_M$ is a Dedekind sum. This can be seen as follows. The Euclidean algorithm applied to the columns of M gives the decomposition

$$M = \begin{pmatrix} 1 & 0 \\ * & c \end{pmatrix} R \qquad \text{with } R \in SL_2\mathbb{Z}.$$

Then, as a set of representatives for L_M/Z_M, we can take $r = (i, 0)R$, $1 \le i \le |c|$. In particular, if $\tau = {}^t(1, 0)$, then

$$M = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} R \qquad \text{with } R = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

thus $r = (i, ai)$, $1 \le i \le |c|$ gives the classical Dedekind sum

$$\sum_{r \in L_M/Z_M} d_{1+j}(r_1) d_{1+k-j}(r'_2) = \sum_{1 \le i \le |c|} d_{1+j}\left(\frac{i + u_1}{c}\right) d_{1+k-j}\left(\frac{a(i + u_1)}{c} + u_2\right).$$

As a closing remark, we wish to point out that the addition formula for the cotangent function as discussed in section 2.1, can be rewritten as $\Psi(AB) - \Psi(A) - A\Psi(B) = 0$, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad AB = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and $P = 1$, $Q(x, y) = \alpha x - \beta y$. This can be easily verified using the above theorem.

2.4. The Bernoulli cocycle Φ

We constructed the cocycle Ψ by summing the values $\psi(A)(P, m)$ for all $m \in \mathbb{Z}^2 + u$. Alternatively, we can multiply $\psi(A)(P, m)$ by the additive character

$$e(mv) := \exp(2\pi i \langle m, v \rangle),$$

and sum $e(mv)\psi(A)(P, m)$ over all lattice points m in $\mathbb{Z}^2$. But there is an important difference between these two approaches. Whereas in the former case u could range in $\mathbb{C}^2$, for reasons of convergence we are now forced to assume that v is a point of the real torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. This restriction is compensated by two advantages. First, for v in $\mathbb{Q}^2/\mathbb{Z}^2$ the values of the resulting cocycle Φ are rational (assuming P has rational cofficients). Secondly, if $v \neq 0$ or $k = \deg(P) > 0$ then Φ does not depend on the auxiliary form Q chosen for the limit defining the sum. In the following discussion, points in T^2 will be represented by column vectors. For $v \in T^2$, and $A \in \Gamma = PGL_2\mathbb{Q}$, let

$$\Phi(A)(P, Q, v) = (2\pi i)^{-2-k} \lim_{t \to \infty} \sum_{\substack{m \in \mathbb{Z}^2 \\ |Q(m)| < t}} e(mv)\psi(A)(P, m).$$

It follows from Theorem 1 that the limit on the right depends on the chosen form Q only in the case $k = 0$ and $v = 0$. As in the previous section, one can show that Φ is a 1-cocycle for Γ, and derive a finite formula for Γ. There is no need to repeat all the details except one general remark. In order to evaluate the sum $\Sigma\, \psi(A)(P, m)$, we used the coset decomposition

$$\sum_{q \in L_M} = \sum_{r \in L_M/Z_M} \sum_{q \in r + Z_m} .$$

By duality, this translates into the character relation for the finite group L_M/Z_M in the form

$$\sum_{q \in L_M} |\det M| = \sum_{r \in \mathbb{Z}^2/M\mathbb{Z}^2} \sum_{q \in \mathbb{Z}^2} e(qM^{-1}r).$$

Applying this observation systematically, we get the following result.

THEOREM 4. *The map Φ which assigns to $A \in \Gamma$ the function $\Phi(A)$ of the three variables P, Q, v, is a 1-cocycle, i.e. it satisfies the relation $\Phi(AB) = \Phi(A) + A\Phi(B)$. The action of $A \in \Gamma$ is given by*

$$A\Phi(B)(P, Q, v) = s_A \sum_{r \in \mathbb{Z}^2/A\mathbb{Z}^2} \Phi(B)({}^tAP, A^{-1}Q, A^{-1}(r+v)),$$

where $s_A = \operatorname{sign}(\det A)$, and A is represented by an integral matrix. Let $M = (\tau, A\tau)$ be an integral matrix with primitive columns $\tau, A\tau$. If $\det M \neq 0$, then

$$\begin{aligned}\Phi(A)(P, Q, v) = (-)^{1+k} s_M &\sum_{0 \le j \le k} P_j(M) \\ &\times \sum_{r \in \mathbb{Z}^2/M\mathbb{Z}^2} \frac{\mathcal{B}_{1+j}(r'_1)}{1+j} \frac{\mathcal{B}_{1+k-j}(r'_2)}{1+k-j}, \qquad r' = M^{-1}(r+v) \\ &- \psi(A)(P, \tau') \frac{\mathcal{B}_{2+k}(\langle \tau', v \rangle)}{(2+k)!} - \psi(A)(P, (A\tau)') \frac{\mathcal{B}_{2+k}(\langle \tau', Av \rangle)}{(2+k)!},\end{aligned}$$

except the case $v = 0$ and $k = 0$ where the additional term $s_M P(M) s_2(M^{-1}Q)/4$ has to be added to the right side. If $\det M = 0$, then

$$\Phi(A)(P, Q, v) = -\psi(A)(P, \tau') \frac{\mathcal{B}_{2+k}(\langle \tau', v \rangle)}{(2+k)!}.$$

It should be noted that this theorem gives a rational expression for $\Psi(A)(P, Q, 0) = (2i)^{2+k}\Phi(A)(P, Q, 0)$. As in the case of Theorem 3, the sum over $r \in \mathbb{Z}^2/M\mathbb{Z}^2$ is again a Dedekind sum: Writing

$$M = R\begin{pmatrix} 1 & * \\ 0 & c \end{pmatrix} \qquad \text{with } R \in SL_2\mathbb{Z},$$

we can take $r = R\,{}^t(0, i)$ with $1 \le i \le |c|$ as a set of representatives for $\mathbb{Z}^2/M\mathbb{Z}^2$. In particular, if $\tau = {}^t(1, 0)$, then we can choose $R = 1$ since in this special case

$$M = \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} \quad \text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

thus

$$\sum_{r \in \mathbb{Z}^2/M\mathbb{Z}^2} \mathscr{B}_{1+j}(r'_1)\mathscr{B}_{1+k-j}(r'_2) = \sum_{1 \le i \le |c|} \mathscr{B}_{1+j}\left(v_1 - \frac{a(i+v_2)}{c}\right)\mathscr{B}_{1+k-j}\left(\frac{i+v_2}{c}\right).$$

The right side is the familar Dedekind sum. Except for small values of $|c|$, this expression is not very suitable for explicit calculations. A much more efficient way to calculate $\Phi(A)$ in the case $\tau = {}^t(1, 0)$, is to apply the Euclidean algorithm to the rows of A and use the cocycle property of Φ as described in the following algorithm.

1. Set $\Phi = 0$.
2. If $c = 0$, then output $\Phi + \Phi(A)$ and stop.
3. Set $T = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $x = [a/c]$. Replace Φ by $\Phi + \Phi(T)$, and (A, P, Q, v) by $(T^{-1}A, {}^tTP, T^{-1}Q, T^{-1}v)$.
4. Set $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Replace Φ by $\Phi + \Phi(S)$, (A, P, Q, v) by $(S^{-1}A, {}^tSP, S^{-1}Q, S^{-1}v)$.
5. Go to step 2.

Except $\Phi(S)$, only the values of Φ on upper triangular matrices (for which $\det M = 0$) are required in this algorithm. Note that for these values, Theorem 4 provides a very simple expression not involving any Dedekind sums. For $\Phi(S)$, the corresponding Dedekind sum has exactly one term as M is the identity matrix in this case. Of course, the same algorithm can be used to calculate the values of the trigonometric cocycle Ψ. It is also worth pointing out that the calculation of $A\Phi(B)$ in the case $|\det A| > 1$ is best accomplished by calculating the right side in $A\Phi(B) = \Phi(AB) - \Phi(A)$.

2.5. L-functions in real quadratic fields

Let $A \in SL_2\mathbb{Z}$ be a hyperbolic matrix (i.e. $\operatorname{tr}(A)^2 > 4$). Then we can write

$$A = WEW^{-1} \quad \text{with } E = \begin{pmatrix} e & 0 \\ 0 & e' \end{pmatrix}, \quad W = \begin{pmatrix} \omega_1 & \omega'_1 \\ \omega_2 & \omega'_2 \end{pmatrix}$$

where e is a unit in the real quadratic field $K = \mathbb{Q}(\sqrt{d})$, $d = \operatorname{tr}(A)^2 - 4$, and ω_j, ω_j' are some conjugated numbers in K with the property that $\omega = \omega_1/\omega_2$ and $\omega' = \omega_1'/\omega_2'$ are the two fixpoints of A acting via fractional linear transformation. Applying the non-trivial automorphism of $K/\mathbb{Q}$ to this equation interchanges e, e' and ω_j, ω_j', while replacing A by A^{-1} interchanges e and e' only. Thus there is no loss of generality in assuming $\det(W) > 0$ and $|e| > 1$. Moreover, replacing A by $-A$ if necessary, we can even assume $e > 1$. For the quadratic forms Q and Q^*,

$$Q(m) = N(m_1\omega_1 + m_2\omega_2), \qquad Q^*(m) = -N(m_1\omega_2 - m_2\omega_1)/\det(W)^2,$$

we have in our earlier notation $AQ = Q$ and ${}^tAQ^* = Q^*$. Note that replacing A by the transpose matrix ${}^tA$ interchanges Q with Q^*. Let $u \in \mathbb{Q}^2$, and suppose A acts on $\mathbb{Z}^2 + u$ (if not, then some power A^k will do). Since $e \neq \pm 1$, every orbit of A contains infinitely many elements in $\mathbb{Z}^2 + u$. In other words, we have the formal identity

$$\sum_{m \in \mathbb{Z}^2 + u} \psi(A)(P, m) = \sum_n \sum_{k \in \mathbb{Z}} \psi(A)(P, nA^k)$$

where $n \neq 0$ runs over a set of representatives for the orbits of A. For $P = (Q^*)^{s-1}$, $s = 1, 2, 3, \ldots$, we can use the cocycle property of ψ to evaluate the (absolutely convergent) inner series over k as follows.

$$\begin{aligned}\sum_{k \in \mathbb{Z}} \psi(A)(P, nA^k) &= \sum_{k \in \mathbb{Z}} \psi(A)({}^tA^kP, nA^k) = \sum_{k \in \mathbb{Z}} [\psi(A^{k+1}) - \psi(A^k)](P, n) \\ &= \lim_{k \to +\infty} f(A^k\tau, A^{-k}\tau)(P, n).\end{aligned}$$

If $\tau = \omega$ or ω' (one of the fixpoints of A), then every term vanishes in this identity. But for all other $\tau \in P_1\mathbb{C}$, the sequence $(A^k\tau, A^{-k}\tau)$, $k = 1, 2, 3, \ldots$, converges exponentially fast to the limit (ω, ω'). Therefore, for $\tau \neq \omega, \omega'$ we always have

$$\sum_{k \in \mathbb{Z}} \psi(A)(P, nA^k) = f(\omega, \omega')(P, n). \tag{9}$$

The expression on the right can be simplified considerably. A straightforward calculation using (8) leads to the result

$$f(\omega, \omega')(P, n) = \frac{((s-1)!)^2 d(W)}{Q(n)^s}. \tag{10}$$

In this way we get the identity

$$\lim_{t\to\infty} \sum_{\substack{m\in\mathbb{Z}^2+u \\ |Q(m)|<t}} \psi(A)(P,m) = ((s-1)!)^2\, d(W) L(A,u,s),$$

$$L(A,u,s) = \lim_{t\to\infty} \sum_{\substack{n \\ |Q(n)|<t}} Q(n)^{-s},$$

where n runs over the (nonzero) orbits of A in $\mathbb{Z}^2+u$. For $\tau \in P_1\mathbb{Q}$ we know that the left side converges. Since the right side does not depend on τ, it follows that the left side converges for all $\tau \in P_1\mathbb{C}$ different from ω, ω' to the limit $\pi^{2s}\Psi(A)(P,Q,u)$, but vanishes identically for $\tau = \omega, \omega'$. We note that the series defining $L(A,u,s)$ converges absolutely for $s > 1$, i.e. the limiting condition $|Q(n)| < t$ is only necessary for $s = 1$. In order to identify $L(A,u,s)$ as a value of a Hecke L-function, let

$$\mathfrak{R} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \qquad \rho = u_1\omega_1 + u_2\omega_2, \qquad \mathfrak{U} = \{e^k \mid k \in \mathbb{Z}\}.$$

Then $\mathfrak{R}$ is a fractional ideal with respect to an order in K, and $\mathfrak{U}$ is a group of units which acts on $\mathfrak{R}+\rho$. To the orbit space $\mathfrak{R}+\rho/\mathfrak{U}$ belong the two Hecke L-series $(j = 0, 1)$,

$$L_j(\mathfrak{R}+\rho, s) := \sum_{\nu\in\mathfrak{R}+\rho/\mathfrak{U}}{}' \frac{\chi_j(\nu)}{|N(\nu)|^s}, \qquad \chi_j(\omega) = (\operatorname{sign} N(\nu))^j,$$

whose halfplane of absolute convergence is at $\operatorname{Re}(s) > 1$. The considerations above lead to a closed formula for the values of $L_j(\mathfrak{R}+\rho, s)$ at positive integral values of $s \equiv j(2)$ in terms of the cocycle Ψ.

THEOREM 5.

$$\frac{((s-1)!)^2 d(W)}{\pi^{2s}} L_j(\mathfrak{R}+\rho, s) = \Psi(A)((Q^*)^{s-1}, Q, u) \qquad \textit{for } s \equiv j(2), \quad s > 0.$$

As we have seen, the right side is independent of the choice of $\tau \in P_1\mathbb{Q}$ in the definition of Ψ. Another non-obvious corollary to this theorem is the observation that the left side is a cyclotomic number.

The L-function $L_j(\mathfrak{R}+\rho, s)$ has an analytic continuation to the whole complex plane. Let

$$\mathfrak{R}^* = \{\mu \in K \mid \operatorname{tr}(\mu\nu) \in \mathbb{Z} \text{ for } \nu \in \mathfrak{R}\}$$

be the dual (complementary) ideal of $\mathfrak{N}$, and let for $j = 0, 1$,

$$L_j^*(\mathfrak{N}+\rho, s) = \sum_{\nu \in \mathfrak{N}^*/\mathfrak{U}}' \frac{\chi_j(\nu)}{|N(\nu)|^s} e(\operatorname{tr}(\nu\rho)), \qquad \operatorname{Re}(s) > 1.$$

According to a well known result of Hecke, c.f. [4, 10], we have the relation

$$\epsilon G(s) L_j(\mathfrak{N}+\rho, s) = G(1-s) L_j^*(\mathfrak{N}+\rho, 1-s),$$

$$\epsilon = (-1)^j |\det(W)|, \qquad G(s) = \pi^{-s}\Gamma\left(\frac{j+s}{2}\right)^2,$$

which gives the analytic continuation of L_j and L_j^* to the halfplane $\operatorname{Re}(s) < 0$. In particular, for $s = 1, 2, 3, \ldots$, we have

$$L_j(\mathfrak{N}+\rho, 1-s) = \frac{4\Gamma(s)^2}{\epsilon(2\pi)^{2s}} L_j^*(\mathfrak{N}+\rho, s), \qquad s \equiv j(2)$$

$$= \frac{4\Gamma(s)^2}{\epsilon(2\pi)^{2s}} \lim_{t\to\infty} \sum_{\substack{\nu \in \mathfrak{N}^*/\mathfrak{U} \\ |N(\nu)| < t}}' \frac{e(\operatorname{tr}(\nu\rho))}{N(\nu)^s}.$$

The last expression is equal to

$$\frac{4}{(2\pi i)^{2s}} \lim_{t\to\infty} \sum_{\substack{m \in \mathbb{Z}^2 \\ |Q^*(m)| < t}} e(m\,{}^t u)\psi({}^t A)(Q^{s-1}, m).$$

To prove this identity, it's enough to replace (A, Q, Q^*, W) in (9) and (10) by $({}^t A, Q^*, Q, {}^t W^{-1})$.

THEOREM 6. *$L_j(\mathfrak{N}+\rho, 1-s) = 4\Phi({}^t A)(Q^{s-1}, Q^*, {}^t u)$ for $s = 1, 2, 3, \ldots$ with $s \equiv j(2)$.*

The last equation can be brought into a somewhat nicer shape by introducing the partial zeta function

$$\zeta(\mathfrak{N}+\rho, s) := \sum_{\substack{\nu \in \mathfrak{N}+\rho/\mathfrak{U} \\ \nu \gg 0}} N(\nu)^{-s}, \qquad \operatorname{Re}(s) > 1.$$

Then as a corollary from the Hecke functional equation [10], we have for $s = 1, 2, 3, \ldots$ with $j \equiv s(2)$,

$$\zeta(\mathfrak{N}+\rho, 1-s) = \tfrac{1}{4} L_j(\mathfrak{N}+\rho, 1-s) = \Phi({}^t A)(Q^{s-1}, Q^*, {}^t u).$$

This equation together with the algorithm discussed at the end of the previous section provide an efficient method to calculate the rational numbers $\zeta(\mathfrak{N}+\rho, 1-s)$, $s = 1, 2, 3, \ldots$. In applications, very often only $\mathfrak{N}+\rho$ is given, and one needs to determine a generator e of the unit group $\mathfrak{U}$ first. The classical solution to this problem involves the reduction theory (continued fraction method) which is essentially another version of the algorithm described at the end of section 2.5. For that reason, the calculation of e as well as Φ can be combined into one algorithm which requires as input only a $\mathbb{Z}$-basis for $\mathfrak{N}$ and the coordinates u of ρ with respect to this basis. One advantage of this procedure is that it avoids the calculation of the matrix A whose entries can become very large even if all other numbers involved are small. In any case, the amount of work necessary to calculate the numbers $\zeta(\mathfrak{N} + \rho, 1 - s)$ is directly proportional to the amount of work involved in calculating a generator e for $\mathfrak{U}$.

The results of this section can be used to calculate (for integral s) the values of a L-funtion $L(\chi, s)$ attached to a ray class character χ in K which is either totally ramified ($j = 1$) or totally unramified ($j = 0$) at all infinite places (i.e. $\chi((v)) = \chi_j(v)$ for $v \equiv 1(f)$, f the conductor of χ), and satisfying the parity condition $j \equiv s(2)$ if $s > 0$. We have for $s = 1, 2, 3, \ldots,$

$$L(\chi, s) = \frac{1}{4} \sum_b \chi(b) N(b)^{-s} L_j(fb^{-1} + 1, s),$$

$$L(\chi, 1 - s) = \sum_b \chi(b) N(b)^{s-1} \zeta(fb^{-1} + 1, 1 - s),$$

where b runs over a set of integral representatives for the narrow ray classes mod f and the unit group $\mathfrak{U}$ is generated by the smallest totally positive unit $e > 1$ satisfying the congruence $e \equiv 1(f)$.

2.6. Historical remarks

The proof of the addition formula for the cotangent function in section 2.1 goes back to Eisenstein, c.f. [16]. The function $f(\sigma, \tau)$, which is at the center of our considerations, was already studied by Hurwitz [5] in connection with a class number formula for quadratic forms. Dedekind sums were introduced for the first time by Dedekind in connection with the logarithm of the Dedekind-eta-function $\eta(\tau)$. That he was in fact studying a special case of Eisenstein cohomology, follows from the identity

$$\log \eta(\tau) - \log \eta(A\tau) = \frac{1}{4\pi i} \int_{\tau}^{A\tau} \sum_m \sum_n{}' (mz + n)^{-2} \, dz,$$

valid for $A \in SL_2\mathbb{Z}$ and $\mathrm{Im}(\tau) > 0$. Periods of Eisenstein series of weight 2 were investigated systematically for the first time by Hecke in [3] where he identifies these periods with values of L-functions in real quadratic fields at $s = 1$. An explicit formula for the cocycle Φ in terms of Dedekind sums was established by Schoeneberg [13], who considered the case $\Gamma = SL_2\mathbb{Z}$. His result was extended to $\Gamma = GL_2\mathbb{Q}^+$ in a recent paper of Stevens [14, 15]. The trigonometric cocycle Ψ was introduced in [7] in connection with a topological invariant arising from the Atiyah–Singer index theorem. The calculation of L-values began with Hecke [2], and was completed by Meyer [6] and Siegel [11]. Our approach in 2.5 solves a problem raised by Siegel in his last paper [12], namely to sum the L-series (at $s = 1, 2, 3, \ldots$) using only methods of real analysis. In some sense, our paper is a natural extension of the ideas Siegel introduced in this paper. However, the ultimate credit should be given to Eisenstein who used this circle of ideas for the first time in his classical paper "Genaue Untersuchung der unedlichen Doppelprodukte . . .", c.f. [16].

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