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Amenable groups and Euler characteristic

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0. Introduction

0.1. We consider an infinite amenable group G and a free cocompact G-space Y, i.e., a connected cell-complex on which G acts freely with Y/G being a finite cell-complex (G is necessarily finited generated). The purpose of this paper is to show that the Euler characteristic $\chi(Y/G)$ has some special properties due to the amenability of G.

In particular, if $H_i Y = 0$ for $0 < i < m =$ dimension of Y then $(-1)^m \chi(Y/G)$ is ≥ 0, and $= 0$ if and only if $H_m Y = 0$. For example, assume that G admits an Eilenberg–MacLane complex $K(G, 1)$ with finite m-skeleton X and take for Y the universal cover $\tilde{X}$ or X; then $(-1)^m \chi(X) \geq 0$, and $= 0$ if and only if Y is contractible (whence X a $K(G, 1)$, and the cohomology dimension cd G is $\leq m$). As a corollary one obtains the fact (cf. Cheeger–Gromov [3] by different methods) that an amenable group admitting a finite $K(G, 1)$ has Euler characteristic $\chi(G) = 0$. Another corollary (case $m = 2$) tells that a finitely presented infinite amenable group G has defect ≤ 1, and $= 1$ if and only if cd $G \leq 2$.

0.2. We first recall that a group G is amenable if it admits an invariant mean for bounded real (or complex) functions. Finite and Abelian groups are amenable, and the class of all amenable groups is closed with respect to subgroups and factor groups, to group extensions, and to increasing unions. These operations applied to finite and Abelian groups yield a big class of groups called "elementary amenable"; all virtually solvable groups (i.e., containing a subgroup of finite index which is solvable) are elementary amenable, but the converse is not true. Moreover there are examples of finitely generated amenable groups which are not "elementary" (cf. Grigorchuk [5]).

0.3. A free group on two generators is easily seen not to be amenable. Thus an amenable group cannot contain a free subgroup of rank 2. As a consequence, an infinite amenable group G has one or two ends, i.e. $H^1(G; \mathbb{Z}G) = 0$ or $\mathbb{Z}$. For otherwise, by virtue of Stallings' structure theorem, G would be either a non-trivial amalgamated free product or an HNN-extension which is not virtually infinite cyclic, and thus contain a free subgroup of rank 2.

0.4. Our approach to the discussion of free cocompact G-spaces Y for infinite amenable G is based on the *Følner criterion* for amenability of discrete groups; this is a combinatiorial characterization which can be translated into a "Følner sequence" in Y, i.e. a sequence of finite subcomplexes with properties described in Section 1.1 below. Elementary arguments involving limits over that sequence yield the statements $(-1)^m\chi(Y/G) \geq 0$, and $=0$ if $H_m Y = 0$, mentioned in 0.1. For the partial converse stating that $\chi(Y/G) = 0$ implies $H_m Y = 0$ (Y contractible) a stronger tool seems necessary, namely ℓ_2-cohomology, in the cellular sense, of Y and a lemma of [3]; this lemma also uses the Følner sequence in Y. The ℓ_2-cohomology method could also be used to yield the above results which we have preferred to present through the more elementary approach.

0.5. In a further section we apply the same procedure as before to get information on $\chi(M)$ where M is a closed 4-manifold with infinite amenable (finitely presented) fundamental group. Namely, $\chi(M)$ is always ≥ 0; this can also be expressed in terms of the group invariant $q(G)$ considered by Hausmann–Weinberger [6].

0.6. Some of the results remain valid for groups G which need not be amenable but are extensions of infinite amenable groups by groups with finite Betti numbers. This is shown in Section 5, where also some applications are discussed.

0.7. In an appendix we mention briefly how most of the results of this paper can be obtained, by an entirely different approach, in the case where G is elementary amenable.

It is a pleasure to thank Ralph Strebel and Ross Geoghegan for many helpful discussions.

1. The Følner sequence

1.1. Let G be an infinite amenable group, and Y a free cocompact G-space; i.e., Y is a connected CW-complex on which G acts freely and cellularly such that $X = Y/G$ is a finite CW-complex (this implies, of course, that G is finitely generated since it is a factor group of $\pi_1 X$). We denote by D a closed cellular fundamental domain for the operation of G on Y.

Using the Følner criterion [4] for amenability of discrete groups one can construct an increasing sequence Y_j, $j = 1, 2, 3, \dots$, of finite subcomplexes of Y with the following properties: (1) Y_j consists of N_j translates xD of D, $x \in G$. (2) $\bigcup_j Y_j = Y$. (3) Let $\dot{N}_j$ be the number of translates of D which meet the topological boundary $\dot{Y}_j$ of Y_j; then

$$\lim_{j \to \infty} \frac{\dot{N}_j}{N_j} = 0.$$

1.2. We now consider the Euler characteristic χ of the finite complexes X and Y_j, i.e. the alternating sum of the numbers of i-cells, $i = 0, 1, \dots, m = \dim X = \dim Y = \dim D = \dim Y_j$. Then

$$\chi(Y_j) = N_j \chi(X) + \Delta_j$$

where Δ_j comes from the topological boundary $\dot{Y}_j$ of Y_j. Clearly $|\Delta_j| \le \dot{N}_j \Delta$ where Δ is the total number of cells of the boundary of D. Thus

$$\chi(X) = \frac{1}{N_j} \chi(Y_j) + \frac{\Delta_j}{N_j}$$

with $|\Delta_j|/N_j \le (\dot{N}_j/N_j)\Delta$ which goes to 0 with $j \to \infty$.

PROPOSITION 1.1. *With assumptions and notations above one has*

$$\chi(X) = \lim_{j \to \infty} \frac{\chi(Y_j)}{N_j}.$$

Expressing the Euler characteristic by the alternating sum of the Betti numbers $\beta_i = \dim_{\mathbb{Q}} H_i(\, ; \mathbb{Q})$ we thus obtain

THEOREM 1.2. *Let G be an infinite amenable group, Y a free cocompact G-space and Y_j a Følner sequence in Y. Then*

$$\chi(Y/G) = \lim_{j \to \infty} \sum_{i=0}^{\dim Y} (-1)^i \frac{\beta_i(Y_j)}{N_j}.$$

2. Finiteness assumptions

2.1. With assumptions and notations as in Section 1 we further assume that, for some i, the Betti number $\beta_i(Y) = \dim_{\mathbb{Q}} H_i(Y; \mathbb{Q})$ is finite. Exactness and excision yield the commutative diagram for homology with $\mathbb{Q}$-coefficients

$$\begin{array}{ccccc} H_{i+1}(Y \backslash Y_j', \dot{Y}_j) & \longrightarrow & H_i(\dot{Y}_j) & \xrightarrow{\varphi} & H_i(Y \backslash Y_j') \\ \downarrow \cong & & \downarrow \psi & & \downarrow \\ H_{i+1}(Y, Y_j) & \longrightarrow & H_i(Y_j) & \xrightarrow{\rho} & H_i(Y) \end{array}$$

(Y_j' = interior of Y_j). Since ψ maps the kernel of φ onto the kernel of ρ we have $\beta_i(Y_j) \le \beta_i(\dot{Y}_j) + \beta_i(Y)$. But $\beta_i(\dot{Y}_j)$ is at most equal to the number of i-cells of $\dot{Y}_j$ which is $\le \dot{N}_j d_i$ where d_i is the number of i-cells of D. Thus

$$\frac{1}{N_j}\beta_i(Y_j) \le \frac{\dot{N}_j}{N_j} d_i + \frac{1}{N_j}\beta_i(Y),$$

and finally

$$\lim_{j\to\infty} \frac{1}{N_j}\beta_i(Y_j) = 0.$$

PROPOSITION 2.1. *If $\beta_i(Y)$ is finite then* $\lim_{j\to\infty}(1/N_j)\beta_i(Y_j) = 0$.

2.2. If $\beta_i(Y)$ is finite for all $i < m = \dim Y$ it follows from Theorem 1.2 that

$$\chi(Y/G) = (-1)^m \lim_{j\to\infty} \frac{\beta_m(Y_j)}{N_j}:$$

THEOREM 2.2. *Let G be an infinite amenable group and Y a free cocompact G-space with $\beta_i(Y)$ finite for all $i < m = \dim Y$. Then $(-1)^m\chi(Y/G) \ge 0$. If, moreover, also $\beta_m(Y)$ is finite then $\chi(Y/G) = 0$.*

COROLLARY 2.3. *If G is an infinite amenable group which admits a finite Eilenberg–MacLane space $K(G, 1) = X$ then $\chi(X) = \chi(G) = 0$.*

For in that case we can take above $Y = \tilde{X}$, the universal cover of X.

2.3. We obtain another corollary by assuming that G admits a $K(G, 1)$ which is finite in dimensions $\leq m$ (i.e., G is of type FP_m, and finitely presented if $m \geq 2$). We then take for Y the m-skeleton of the universal cover of $K(G, 1)$. Then $X = Y/G$ is the m-skeleton of $K(G, 1)$. Since Y fulfills the assumptions of Theorem 2.2 it follows that $(-1)^m\chi(X)$ is ≥ 0:

COROLLARY 2.4. *Let G be an infinite amenable group of type FP_m (and finitely presented if $m \geq 2$), and let X be the finite m-skeleton of a (suitable) $K(G, 1)$. Then $(-1)^m\chi(X)$ is ≥ 0.*

COROLLARY 2.5. *Let G be a finitely presented infinite amenable group. Then the defect of any finite presentation of G is ≤ 1.*

Indeed, if α_1 is the number of generators and α_2 of defining relations there is a $K(G, 1)$ with one 0-cell, α_1 1-cells and α_2 2-cells. The case $m = 2$ of Corollary 2.4 tells that $1 - \alpha_1 + \alpha_2 \geq 0$, i.e., $\alpha_1 - \alpha_2 \leq 1$. — We remark that this result also follows from the known fact [1] that if $\alpha_1 - \alpha_2 \geq 2$ then G must contain a free subgroup of rank 2 and thus cannot be amenable.

2.4. It seems convenient to express Corollary 2.4 in terms of *group invariants* which we call $q_m(G)$; they are the geometric counterpart of the "partial Euler characteristics" as considered by Swan and Gruenberg in the context of finite groups (see, e.g., [9]). Namely, for G as in Corollary 2.4, consider all $K(G, 1)$ with finite m-skeleton X and let $q_m(G)$ be the minimum value of $(-1)^m\chi(X)$ for all these $K(G, 1)$; the minimum exists since $(-1)^m\chi(X) = \Sigma_{i=0}^{m-1}(-1)^{i+m}\beta_i(G) + \beta_m(X)$, and $\beta_m(X)$ is bounded below by $\beta_m(G) = \beta_m(K(G, 1))$.

COROLLARY 2.4′. *Let G be an infinite amenable group of type FP_m (and finitely presented if $m \geq 2$). Then $q_m(G)$ is ≥ 0, and of course $q_i(G) \geq 0$ for all $0 \leq i \leq m$.*

We note that $q_0(G) = 1$, $q_1(G) = n(G) - 1$ where $n(G)$ is the minimal number of generators of G, and $q_2(G) = 1 - d(G)$ where $d(G)$ is the defect of G (the maximum of the defects of all finite presentations of G).

COROLLARY 2.5′. *The defect $d(G)$ of a finitely presented infinite amenable group is ≤ 1.*

3. ℓ_2-*cohomology*

3.1. In this section we prove a certain converse of the second statement (vanishing of $\chi(Y/G)$) in Theorem 2.2. The method of applying directly the Følner sequence in Y does not seem to yield the result. However, (reduced) ℓ_2-cohomology of Y in the cellular sense and a lemma of Cheeger–Gromov [3] provide the necessary tools; we note that this lemma too is based on the Følner sequence.

We recall that (reduced) cellular ℓ_2-cohomology $\bar{H}^iY$ is defined by means of ℓ_2-cochains $f \in \bar{C}^iY$ with real coefficients (i.e., $\Sigma_\sigma f(\sigma)^2 < \infty$ where σ ranges over all i-cells of Y) and that $\bar{H}^iY = (\ker \delta_i : \bar{C}^iY \to \bar{C}^{i+1}Y)/\text{closure of } \delta\bar{C}^{i-1}Y$. If Y is a G-space the von Neumann dimension $\dim_G \bar{H}^iY$ will be denoted by $\bar{\beta}_i(Y \text{ rel. } G)$. In the case where Y is a free cocompact G-space one has $\dim_G \bar{C}^iY = \alpha_i$ = number of i-cells of Y/G. The standard argument applied to the von Neumann dimensions shows that the analogue of the classical "Euler-Poincaré formula" holds for the (fake) Betti numbers $\bar{\beta}_i$:

$$\chi(Y/G) = \sum_{i=0}^{\dim Y} (-1)^i\alpha_i = \sum_{i=0}^{\dim Y} (-1)^i\bar{\beta}_i(Y \text{ rel. } G).$$

3.2. If G is infinite amenable and Y a free cocompact G-space the Cheeger–Gromov lemma [3] tells that the natural map $\bar{H}^iY \to H^i(Y; \mathbb{R})$ is *injective*. For connected Y it is clear that $\bar{H}^0Y = 0$ since Y is infinite. If we further assume that $H_iY = 0$ for $0 < i < m = \dim Y$ then $H^i(Y; \mathbb{R}) = 0$ for these i and thus $\bar{H}^iY = 0, 0 \le i < m$. It follows that

$$\chi(Y/G) = (-1)^m\bar{\beta}_m(Y \text{ rel. G}).$$

If also $H_mY = 0$ then $\bar{H}^mY = 0$, $\bar{\beta}_m(Y \text{ rel. } G) = 0$ and thus $\chi(Y/G) = 0$—as we have already shown by a different, more elementary, method. But conversely $\chi(Y/G) = 0$ implies $\bar{\beta}_m(Y \text{ rel. } G) = 0$ and this in turn $\bar{H}^mY = 0$. Now for any finite subcomplex Y_j of Y the exact ℓ_2-cohomology sequence

$$\bar{H}^mY \to \bar{H}^mY_j \to \bar{H}^{m+1}(Y, Y_j) = 0$$

shows that $\bar{H}^mY_j = H^m(Y_j; \mathbb{R}) = 0$, i.e., $\beta_m(Y_j) = 0$ which implies $H_mY_j = 0$ since this group is $\mathbb{Z}$-free. Therefore $H_mY = 0$:

THEOREM 3.1. *Let G be infinite amenable and Y a connected free cocompact G-space with $H_iY = 0$, $0 < i < m = \dim Y$. Then $\chi(Y/G) = 0$ if and only if $H_mY = 0$.*

COROLLARY 3.2. *Let G be infinite amenable of type* FP_m *(finitely presented if* $m \geq 2$*) and let X be the finite m-skeleton of a suitable* $K(G, 1)$. *If* $\chi(X) = 0$ *then X is itself a* $K(G, 1)$ *and the cohomology dimension* cd G *of G* is $\leq m$.

Indeed $\chi(X) = 0$ implies $H_i Y = 0$, $0 < i \leq m$, where Y is the m-skeleton of the universal cover of $K(G, 1)$. This means that Y is contractible, i.e., that X is an Eilenberg–MacLane complex for G.

COROLLARY 3.3. *Let G be a finitely presented infinite amenable group. If* $d(G) = 1$ *then* cd $G \leq 2$; *i.e., either G is infinite cyclic or* cd $G = 2$.

Note that, in Corollary 3.3, $\chi(G) = 1 - \beta_1 G + \beta_2 G = 0$, $\beta_1 G = 1 + \beta_2 G \geq 1$. Thus (see Bieri–Strebel [2]) G is an HNN-extension $G = H*_{K,p}$ with K finitely generated. But K must be equal to H since otherwise G would contain a free subgroup of rank 2. So $G = H*_{H,p}$, H finitely generated, and the possibilities for H can be further discussed. If G is not infinite cyclic then $H^1(G; \mathbb{Z}G) = 0$ by virtue of Stallings' structure theorem; thus G is a duality group of dimension 2.

4. Four-manifolds with amenable fundamental group

4.1. Given any finitely presented group G there exists a (smooth) closed orientable 4-manifold M with $\pi_1 M = G$. For an infinite amenable group G the methods of the previous sections give information on $\chi(M)$.

The universal cover $\tilde{M} = Y$ of M is a free cocompact G-space; clearly $H_1 Y = H_4 Y = 0$. Moreover $H_3 Y = H_3(M; \mathbb{Z}G) = H^1(M; \mathbb{Z}G)$ by Poincaré duality, $= H^1(G; \mathbb{Z}G)$. But as noted in the Introduction (Section 0.3) G has one or two ends, i.e., $H^1(G; \mathbb{Z}G) = 0$ or $\mathbb{Z}$, whence $H_3 Y = 0$ or $\mathbb{Z}$. Thus the only Betti number of Y which is possibly non-finite is $\beta_2(Y)$. The method of Section 2 yields, in terms of a Følner sequence Y_j in Y,

$$\lim_{j \to \infty} \frac{\beta_i(Y_j)}{N_j} = 0 \qquad \text{for } i = 0, 1, 3, 4$$

and thus

$$\chi(M) = \sum_{i=0}^{4} (-1)^i \lim_{j \to \infty} \frac{\beta_i(Y_j)}{N_j} = \lim_{j \to \infty} \frac{\beta_2(Y_j)}{N_j}$$

which is ≥ 0.

THEOREM 4.1. *If M is a closed 4-manifold with infinite amenable fundamental group G then $\chi(M)$ is ≥ 0.*

REMARKS. (1) If above $\beta_2(Y)$ is finite, e.g., if Y has the homotopy type of a finite cell-complex, then $\chi(M) = 0$.

(2) In the context discussed above $\beta_3(Y) = 1$ if and only if G is virtually infinite cyclic (i.e., G has 2 ends, $H^1(G; \mathbb{Z}G) = H_3 Y = \mathbb{Z}$). If such groups are excluded then $H_3 Y = 0$; if moreover $H_2 Y = 0$ then Y is contractible, M is a $K(G, 1)$, and G is a Poincaré duality group of dimension 4 (with $\chi(G) = 0$). We note that thus $H^1(G, \mathbb{Z}G) = 0$ and $H_2 Y = 0$ imply $H^2(G; \mathbb{Z}G) = H^3(G; \mathbb{Z}G) = 0$, $H^4(G; \mathbb{Z}G) = \mathbb{Z}$.

(3) Since $X(M) = 2 - 2\beta_1(M) + \beta_2(M)$, Theorem 4.1 tells that $\beta_2(M) \geq 2\beta_1(M) - 2$. We note that, quite generally, $\beta_1(M) = \beta_1(G)$ and $\beta_2(M) \geq \beta_2(G)$ since a $K(G, 1)$ can be obtained from M by adding cells of dimension ≥ 3.

4.2. As a corollary of Theorem 4.1 we obtain information on the group invariant $q(G)$ for finitely presented groups G considered by Haussman and Weinberger [6]: $q(G)$ is the smallest value of $\chi(M)$ for all closed orientable 4-manifolds M with $\pi_1(M) = G$. Clearly, see Remark (3) above,

$$q(G) \geq 2 - 2\beta_1(G) + \beta_2(G).$$

On the other hand, as shown in [6],

$$q(G) \leq 2(1 - d(G))$$

where $d(G)$ is the defect of G.

From Theorem 4.1 we immediately obtain

COROLLARY 4.2. *For a finitely presented infinite amenable group G the invariant $q(G)$ is always ≥ 0.*

We recall (Corollary 2.5) that for groups G as above the defect $d(G)$ is ≤ 1; the upper bound above for $q(G)$ is thus ≥ 0; and $= 0$ if and only if $d(G) = 1$ (whence $\operatorname{cd} G \leq 2$).

EXAMPLES. (1) [cf. 6] $q(\mathbb{Z}^m) \geq 0$ for all $m \geq 1$. Clearly $q(\mathbb{Z}) = 0$.

(2) Virtually solvable groups G are amenable. Thus if they are infinite and finitely presented then $q(G) \geq 0$. This applies in particular to virtually infinite cyclic groups.

4.3. Under suitable further assumptions Theorem 4.1 (or Corollary 4.2) admits a converse, stating that if $\chi(M) = q(G) = 0$ then M is a $K(G, 1)$ and G is a PD^4-group. We will return to this and similar aspects in a separate paper.

5. Group extensions

5.1. Corollary 2.4 concerning the finite m-skeleton of a $K(G, 1)$ for a group G of suitable type can easily be generalized to groups which need not be amenable but are certain extensions of infinite amenable groups. This generalization has been suggested by Ross Geoghegan.

We consider a group G of type FP_m (and finitely presented if $m \geq 2$) and assume that G contains a normal subgroup N with Betti numbers $\beta_i(N)$ finite for $0 \leq i < m$ and such that $G/N = A$ is infinite amenable. Let X be the finite m-skeleton of a $K(G, 1)$, and Y the m-skeleton of the cover corresponding to the subgroup N of G; this cover is a $K(N, 1)$. Then Y is a free cocompact A-space with $Y/A = X$, and the Betti numbers $\beta_i(Y) = \beta_i(N)$ are finite for $0 \leq i < m$. Thus by Theorem 2.2 one has $(-1)^m\chi(X) \geq 0$.

THEOREM 5.1. *Let G be a group of type* FP_m *(and finitely presented if* $m \geq 2$*), and assume that G is an extension of an infinite amenable group A by a group N with finite Betti numbers* $\beta_i(N)$, $0 \leq i < m$. *If X is the finite m-skeleton of a* $K(G, 1)$ *then* $(-1)^m\chi(X)$ *is* ≥ 0.

Or, in terms of the group invariant $q_m(G)$ introduced in 2.5:

THEOREM 5.1′. *For a group extension G as in Theorem* 5.1 *the invariants* $q_i(G)$, $0 \leq i \leq m$, *are all* ≥ 0.

5.2. The case $m = 2$ presents some special interest.

COROLLARY 5.2. *If G is finitely presented and contains a normal subgroup N with* $\beta_1 N$ *finite and such G/N is infinite amenable then the defect* $d(G)$ *is* ≤ 1.

For example, assume that $d(G) \geq 2$ and that $G/[G, G]$ is infinite; then $[G, G]$ cannot be finitely generated. This particular case can also be proved directly by the group-theoretic methods as used, e.g., in Bieri–Strebel [2].

REMARKS. (1) It should be emphasized that the group G in this section is, in general, not amenable. (2) All the above can be viewed as generalizations of the elementary fact that for a free group of rank ≥ 2 the commutator group is not finitely generated.

5.3. There is a similar generalization of Theorem 4.1 in Section 4 concerning a closed 4-manifold M with fundamental group G. We again assume that $G/N = A$ is infinite amenable; here the only assumption on N is that $\beta_1(N)$ be finite. We recall that $\beta_1(N) = \dim_{\mathbb{Q}} H_1(N; \mathbb{Q}) = \dim_{\mathbb{Q}} (N_{ab} \otimes \mathbb{Q})$ where $N_{ab} = N/[N, N]$.

The cover Y of M with fundamental group N is a free cocompact A-space, $Y/A = M$. Clearly $\beta_1(Y) = \beta_1(N)$ is finite and $\beta_4(Y) = 0$. As for $\beta_3(Y)$, we consider $H_3(Y; \mathbb{Q}) = H_3(M; \mathbb{Q}A) = H^1(M; \mathbb{Q}A) = H^1(G; \mathbb{Q}A)$ and the first terms of the "Five-term exact sequence" for $G/N = A$:

$$0 \longrightarrow H^1(A; \mathbb{Q}A) \longrightarrow H^1(G; \mathbb{Q}A) \longrightarrow \mathrm{Hom}_A (N_{ab}, \mathbb{Q}A) \longrightarrow \cdots.$$

Now $H^1(A; \mathbb{Q}A) = H^1(A; \mathbb{Z}A) \otimes \mathbb{Q}$ is 0 or $\mathbb{Q}$. Moreover $\mathrm{Hom}_A (N_{ab}, \mathbb{Q}A) = \mathrm{Hom}_A (N_{ab} \otimes \mathbb{Q}, \mathbb{Q}A) = 0$; indeed, for any $f \in \mathrm{Hom}_A (N_{ab} \otimes \mathbb{Q}, \mathbb{Q}A)$ the image is a $\mathbb{Q}A$-submodule of $\mathbb{Q}A$ whose dimension over $\mathbb{Q}$ is finite, and A being infinite this is possible for $f = 0$ only. Thus $H^1(G; \mathbb{Q}A) = 0$ or $\mathbb{Q}$, and $\beta_3(Y) = 0$ or 1. As in Section 4.1 the Følner sequence argument for Y then yields $\chi(M) \geq 0$:

THEOREM 5.3. *Let M be a closed 4-manifold whose fundamental group G is an extension of an infinite amenable group by a group N with $\beta_1(N)$ finite. Then $\chi(M)$ is ≥ 0.*

Or, in terms of the group invariant $q(G)$, see 4.2,

THEOREM 5.3′. *For a finitely presented group G as in Theorem 5.3 the invariant $q(G)$ is ≥ 0.*

6. Appendix: The "elementary amenable" case

6.1. For elementary amenable (cf. Section 0.2) groups G one can obtain, under mild restrictions, the main results by an entirely different approach. It is based on the classical ring of fractions $R = (\mathbb{Q}G \backslash 0)^{-1}\mathbb{Q}G$ of the group algebra $\mathbb{Q}G$. In [8] Kropholler, Linnell and Moody have shown that if G is elementary amenable, does not contain finite normal subgroups $\neq 1$, and has bounded torsion orders, then the ring of fractions R exists and is a matrix ring $M_\ell(D)$ over a division ring D (in particular, if G is torsion-free then $R = D$). Finitely generated R-modules have a well-defined rank over R since they are D-vector spaces.

6.2. If G is such a group and Y a free cocompact G-space one applies to the cellular $\mathbb{Q}$-chain complex $\underline{C}$ of Y, which is a free $\mathbb{Q}G$-complex, the "localization method"; one passes from $\underline{C}$ to the free R-complex $R \otimes_{\mathbb{Q}G} \underline{C}$. The rank of C_i over $\mathbb{Q}G$ is α_i, the number of i-cells of Y/G, and so is the rank of $R \otimes_{\mathbb{Q}G} C_i$ over R. Moreover $R \otimes_{\mathbb{Q}G} \mathbb{Q} = 0$ so that the augmentation $\underline{C} \to \mathbb{Q}$ becomes 0 after localizing. This procedure yields the results of Sections 2, 3 and 4 concerning the Euler characteristic of Y/G. The details, which I had carried through before dealing with the *general* amenable case as described in the present paper, will not be given here—not because they are uninteresting, in the contrary: I have learnt, in the meanwhile, of a forthcoming paper of Hillman [7] which gives a very complete and interesting treatment of elementary amenable groups and of the localization method in all their aspects.

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