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On stable K-theory and topological Hochschild homology

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0. Introduction

Let R be a commutative ring with unit and A be an R-bimodule. It is proved in [S–S–W] that stable K-theory of a ring R with coefficients in A (denoted here $K^s(R, A)$) is equal to topological Hochschild homology of R with the same coefficients (denoted here THH (R, A)). Let $R_{f\text{Mod}}$ denote the category of finitely generated free R-modules and let R_{Mod} stand for the category of R-modules. In [P–W] the authors define the notion of topological Hochschild homology groups with coefficients in any functor $T : R_{f\text{Mod}} \to R_{\text{Mod}}$ (denoted here THH (R, T)) which generalizes THH (R, A) (= THH $(R, \cdot \otimes A)$. On the other hand the similar generalization for stable K-theory is described in [Be3]. Hence there appears a question: are THH (R, T) equal to $K^s(R, T)$ for any functor T? The purpose of this note is to prove that they are equal for functors of finite degree k, away from primes p such $p \leq k$. The proof of the main result goes through by an appropriate reduction from the general case to the case of linear functors (functors of degree 1). This can be used for computations: for the large class of rings the case of linear functors is computable by [B] and [P–W].

The paper is organized as follows: first section is devoted to recalling the basic notion of monoids and their homology groups. We give there also some preliminary, simple lemmas. Section 2 contains results on vanishing of homology groups of monoids with twisted coefficients and can be considered as a direct generalization of [Be1] and [Be2]. By [P–W] and [J–P] we know that THH-theory can be described in terms of the homology groups of monoids of matrices (not necessarily invertible). Hence in section 3 we can use the methods from sections 1 and 2 to get our main result.

1. Homology of monoids

This section is devoted to recalling the basic notion of homology groups of a monoid with twisted coefficients and to proving some special properties of these.

We think that everything that is done in this paragraph can be obtained in greater generality for homology of small categories with coefficients in bifunctors but we shall not need such general statements.

Let us first recall the notion of a monoid. This is a set X with an associative multiplication. We shall always assume that our monoids contain identity. Every monoid X can be viewed as a category consisting of one object $*$ and $\mathrm{Mor}(*, *) = X$ with composition given by multiplication in X. The map of monoids is a map of underlying sets which preserves multiplication and sends identity to identity or equivalently a functor between corresponding categories.

Let k be any commutative ring with unit and X be a monoid. Then we can form a monoid ring $k[X]$ by the formula:

$$k[X] = \bigoplus_{x \in X} k$$

as an abelian group with the multiplication induced from the multiplication in X as in the case of group rings. Let now M be a $k[X]$-bimodule. We define homology groups of X with coefficients in M, which will be denoted by $H_*(X; M)$, as the Hochschild homology groups of the k-algebra $k[X]$ with coefficients in M (see [C–E] or [Mc; X, 5]).

1.1. REMARK [Mc; X, 5.5]. Let X be a monoid which is a group. Then for any $k[X]$-bimodule M the homology groups $H_*(X; M) = h_*(X; M')$ where M' is the left $k[X]$-module obtained from M by putting

$$x \circ m = xmx^{-1}$$

for any $x \in X$ and any $m \in M$ and $h_*(\cdot)$ denotes the ordinary homology groups of a group with twisted coefficients.

1.2. REMARK. Let X be a monoid, M a $k[X]$-bimodule and t an invertible element of X. Let f_t and F_t be maps induced by conjugation by t on X and M correspondingly. Then the pair (f_t, F_t) induces an identity isomorphism on homology groups $H_*(X; M)$.

Proof. First see the proof for the case of groups, for example [Mc; IV, 5]. Then do the same for monoids (see also [McC, example 2.4.2]).

Let now X, Y, Z be monoids and L, M, N be bimodules over $k[X]$, $k[Y]$ and $k[Z]$ respectively. Let $f: X \to Z$ and $g: Y \to Z$ be maps of monoids and $F: L \to N$ and $G: M \to N$ maps of bimodules. Moreover assume that $f(X) \cap g(Y) = 1$, $z \cdot z' = z' \cdot z$ for any $z \in f(X)$ and $z' \in g(Y)$, $f(X)$ acts trivially on $G(M)$ and $g(Y)$ acts trivially on $F(L)$. The abelian group $L \oplus M$ has a structure of

$k[X \times Y]$-bimodule given by $(x, y) \circ (l, m) = (xl, ym)$ for any $(x, y) \in X \times Y$ and $(l, m) \in L \times M$ (and similarly for the right action). Now let $\mathrm{diag} : X \times Y \to Z$ and $\mathrm{Diag} : L \oplus M \to N$ denote maps defined by

$$\mathrm{diag}\,(x, y) = f(x)g(y)$$

and

$$\mathrm{Diag}\,(l, m) = F(l) + G(m).$$

It is obvious that diag is a map of monoids and Diag is equivariant. We will need the following lemma:

1.3. LEMMA. *Let X, Y, Z, L, M, N be as above. Assume that $H_i(X; L) = H_i(Y; M) = 0$ for $i < j$. Then on the j-th homology group*

$$(\mathrm{diag}, \mathrm{Diag})_* = (f, F)_* + (g, G)_*.$$

Proof. The Künneth spectral sequence (see [C–E] or [R]) gives us immediately that

$$H_j(X \times Y; L \oplus M) = H_j(X; L) \oplus H_j(T; M).$$

Moreover it is clear that $(\mathrm{diag}, \mathrm{Diag})_*$ restricted to the first (second) summand is just $(f, F)_*((g, G)_*)$.

The universal coefficients spectral sequence (see [R]) gives us immediately the following:

1.4. LEMMA. *Let $k = \mathbf{Z}$, X be a monoid and M a $\mathbf{Z}[X]$-bimodule which is free as an abelian group. Then the following two conditions are equivalent*:

(i) *$H_*(X; M)$ has no p-torsion (is torsion)*;

(ii) *For any algebraically closed field K of characteristic p ($K = \bar{Q}$) we have $H_*(X; M \otimes_{\mathbf{Z}} K) = 0$.*

2. "Vanishing theorems" revisited

This section can be viewed as a generalization from groups to monoids of the results from [Be1] and [Be2]. The methods are not new and everything relies on Remark 1.1. Throughout the section the ring of coefficients of monoid rings is considered to be equal to $\mathbf{Z}$. We shall give sometimes only sketches of proofs if they are the same as the proofs of corresponding lemmas or theorems from [Be1] or [Be2].

Let R be a commutative ring with unit and let for any n, M_n be a monoid such that $Gl_n(R) \subseteq M_n \subseteq M_{n \times n}$, where $M_{n \times n}$ denotes the monoid of $n \times n$-matrices with entries in R. Moreover assume that the upper inclusion $M_{n \times n} \to M_{(n+1) \times (n+1)}$ given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

gives us the embedding of monoids after restriction to M_n. Let Ab denote the category of abelian groups and $T : R_{f\mathrm{Mod}} \to Ab$ be a functor of finite degree (see [E–M] or [Be2] for the definition and properties). Let T^k denote the k-th tensor power functor (tensoring over $\mathbf{Z}$) and let T^kR be the ring $T^k(R)$. This section is devoted to proving the following two theorems:

2.1.a. THEOREM. *Let T be as above. Then*

$$\lim_{n \to \infty} H_*(M_n; T(R^n)) = 0,$$

where the action of M_n on $T(R^n)$ from one side is trivial.

2.1.b. THEOREM. *Let $T = H_0(\Sigma_k; T^k \otimes_{T^kR} A)$, where A is any $R - T^kR$-bimodule with Σ_k-action, $k > 1$ and Σ_k acts diagonally on $T^k \otimes A$. Then* $\lim_{n \to \infty} H_*(M_n; \mathrm{Hom}\,(R^n; T(R^n)))$ *is torsion and can contain p-torsion only for $p \leq k$.*

2.2. REMARK. Theorem 2.1.b can be easily generalized to the case where coefficients are in any functor $S : R_{f\mathrm{Mod}}{}^{\mathrm{op}} \times R_{f\mathrm{Mod}} \to Ab$ of the type $H_0(\Sigma_i \times \Sigma_j;$ $\mathrm{Hom}\,(T^i; T^j) \otimes A)$, see [Be2] for the case $M_n = Gl_n(R)$.

By the methods from [Be2, section 2] it is enough to prove Theorems 2.1 only for the functors T as in 2.1.b. By similar methods as in [Be2, proof of 4.2 for (a) and 3.1 for (b)] we can go even further and restrict ourselves to

$$T = T^k \otimes_{T^kR} A.$$

By the standard spectral sequence arguments we can work only with the case $T = T^k$. Moreover it is clear that we have to deal only with the case $R = \mathbf{Z}$ and the general case will easily follow from that as in [Be1, 4.2 and 5.2]. Hence the rest of this section will be occupied by the proof of Theorems 2.1 in the case $R = \mathbf{Z}$ and $T = T^k$.

We shall write T_n^k for $T^k(\mathbf{Z}^n)$, t_n^k for Hom $(\mathbf{Z}^n, T^k(\mathbf{Z}^n))$ and t^k for the bifunctor Hom $(\cdot, T^k(\cdot))$. Let d be any natural number. We shall use the following notation:

$\lim_{n\to\infty} M_n = \mathbf{M}$,

$\lim_{n\to\infty} M_{dn} = \tilde{\mathbf{M}}$,

$\lim_{n\to\infty} T_n^k = \mathbf{T}^{\mathbf{k}}$ and similarly for t_n^k,

$\lim_{n\to\infty} T_{dn}^k = \tilde{\mathbf{T}}^{\mathbf{k}}$ and similarly for t_{dn}^k,

i_n denotes the upper inclusion $M_n \to M_{n+1}$.

F_n denotes the inclusion $T_n^k \to T_{n+1}^k$ or $t_n^k \to t_{n+1}^k$ induced by the standard "upper" inclusion $\mathbf{Z}^n \to \mathbf{Z}^{n+1}$ and projection $\mathbf{Z}^{n+1} \to \mathbf{Z}^n$ on the first n coordinates.

2.3. LEMMA. *The natural inclusions* $\tilde{\mathbf{M}} \to \mathbf{M}$, $\tilde{\mathbf{T}}^{\mathbf{k}} \to \mathbf{T}^{\mathbf{k}}$ *and* $\tilde{\mathbf{t}}^{\mathbf{k}} \to \mathbf{t}^{\mathbf{k}}$ *induce an isomorphism of monoids and modules over them.*

In the rest of the paper we shall identify $H_*(\mathbf{M}; \mathbf{T}^{\mathbf{k}}) = H_*(\tilde{\mathbf{M}}; \tilde{\mathbf{T}}^{\mathbf{k}})$ and $H_*(\mathbf{M}; \mathbf{t}^{\mathbf{k}}) = H_*(\tilde{\mathbf{M}}; \tilde{\mathbf{t}}^{\mathbf{k}})$ using the isomorphism from Lemma 2.3. We have well defined map of monoids and modules over their monoid rings

$$(\text{diag}, \text{Diag}) : (M_n, T_n^k) \to (M_{dn}, T_{dn}^k)$$

induced by the diagonal embedding $\mathbf{Z}^n \to \mathbf{Z}^{dn}$. The pair (diag, Diag) can be decomposed into a composition (diag, diag′) $\circ$ (diag, diag″) just like in [Bel, theorem 5.1]. Instead of defining the maps diag′ and diag″ we say only that they make the following diagram commute (notice that all this is true also for t_n^k):

$$\begin{array}{ccccc}
(M_n, T_n^k) & \xrightarrow{(i_n, F_n)} & (M_{n+1}, T_{n+1}^k) & \longrightarrow & \cdots (\mathbf{M}, \mathbf{T}^{\mathbf{k}}) \\
\downarrow{\scriptstyle (\text{diag}, \text{diag}'')} & & \downarrow{\scriptstyle (\text{diag}, \text{diag}'')} & & \\
((M_n)^d, (T_n^k)^d) & \xrightarrow{(\Pi i_n, \oplus F_n)} & ((M_{n+1})^d, (T_{n+1}^k)^d) & \longrightarrow & \cdots (\mathbf{M}^d, (\mathbf{T}^{\mathbf{k}})^d) \\
\downarrow{\scriptstyle (\text{diag}, \text{diag}')} & & \downarrow{\scriptstyle (\text{diag}, \text{diag}')} & & \\
(M_{dn}, T_{dn}^k) & \xrightarrow{(S, T)} & (M_{(n+1)d}, T_{(n+1)d}^k) & \longrightarrow & \cdots (\tilde{\mathbf{M}}, \tilde{\mathbf{T}}^{\mathbf{k}})
\end{array}$$

where $(S, T) = (C_s \circ i_{(n+1)d-1} \circ \cdots \circ i_{dn}, C_s \circ F_{(n+1)d-1} \circ \cdots \circ F_{dn})$ and C_s denotes the conjugation by the matrix s which makes the following diagram commute:

$$\begin{array}{ccc} (M_n)^d & \xrightarrow{\Pi\, i_n} & (M_{n+1})^d \\ \downarrow \text{diag} & & \downarrow \text{diag} \\ M_{dn} & \xrightarrow{C_s} & M_{(n+1)d} \end{array}$$

2.4. LEMMA. *Let K be any algebraically closed field and d be a natural number which is bigger than $k+1$ and prime to the characteristic of K (prime to 0 is an empty condition). Then*:

(a) $(\text{Diag}, \text{diag})_* : H_i(M_n; T_n^k \otimes K) \to H_i(M_{dn}; T_{dn}^k \otimes K)$ *is a trivial map*;

(b) $(\text{Diag}, \text{diag})_* : H_i(M_n; t_n^k \otimes K) \to H_i(M_{dn}; t_{dn}^k \otimes K)$ *is trivial provided characteristic of K is 0 or is equal to p and $p-1$ does not divide $k-1$, $k>1$.*

First we shall show how we can obtain our Theorems 2.1 from the lemma above. We shall proceed by induction with respect to the dimension of the homology group. Assume that our theorem is true for any i such that $i<j$. Then passing to the limit and using 1.2 and 1.3 we get that (Diag, diag) induces multiplication by d on the j-th homology group of $\mathbf{M}$ with coefficients in $\mathbf{T}^k$ or $\mathbf{t}^k$ (see the proof of [Be1] lemma 5.2). On the other hand we know that the limit of trivial maps is a trivial map. That fact, Lemma 1.4 and Lemma 2.4 give us immediately Theorems 2.1.

Now we can sketch the proof of Lemma 2.4. Notice that the map (Diag, diag) can be decomposed into a composition of two maps for any submonoid H contained in $Gl_d(\mathbf{Z})$ (we shall write everything only for T_n^k but the same is true for t_n^k):

1. $(i, \text{diag}) : (M_n, T_n^k \otimes K) \to (M_n \times H, T_{dn}^k \otimes K)$ where i is just inclusion on the first factor;
2. $(j, \text{id}) : (M_n \times H, T_{dn}^k \otimes K) \to (M_{dn}; T_{dn}^k \otimes K)$ where j is a map given by the tensor product.

By the spectral sequence argument it is enough to find H such that $H_*(H; T_{dn}^k \otimes K) = 0$. But correct choice of H is given in [Be1, lemma 3.7] for the part (a) of the lemma and in [Be2, theorem 3.4] for the part (b). This finishes the proof of Lemma 2.4 and hence also the proof of Theorem 2.1.

2.5. COROLLARY. *Let $T : R_{f\text{Mod}} \to Ab$ be a functor of finite degree k. Then up to p-torsion for $p \le k$*

$$\lim_{n\to\infty} H_*(M_n; \text{Hom}(R^n, T(R^n))) = \lim_{n\to\infty} H_*(M_n; \text{Hom}(R^n, A^n)),$$

where A is some R-bimodule.

We shall finish this section by showing that for some rings we can strengthen our results. Let us introduce the following definition:

2.6. DEFINITION. A ring R is called *large* if for any natural number k there is an element $r \in R^*$ such that $r^k - 1 \in R^*$.

For example any algebraically closed field, any field of characteristic 0 or any local ring which has one of those fields as a quotient is *large*. For such rings we can prove:

2.7. THEOREM. *Let R be a* large *ring. Let T be as in* 2.1.b. *Then*

$$\lim_{n\to\infty} H_*(M_n, \mathrm{Hom}\,(R^n, T(R^n))) = 0.$$

Proof. We shall proceed as in the proof of Theorems 2.1, but we shall work directly with R without the middle step over $\mathbf{Z}$. As previously it is enough to prove our theorem only for $T = T^k(\cdot)$ with an arbitrary k. Further observe that the only problem which can occur is to find the correct choice of a submonoid $H \in Gl_d(R)$. But the choice is obvious: it is enough to take $H = \langle x \rangle$ where x is an element of R^* of order prime to $k-1$ and $\langle x \rangle$ denotes the subgroup of $Gl_d(R)$ generated by the diagonal element with x on the diagonal. Obviously for such H we have $H_*(H; \mathrm{Hom}\,(R^{nd}, T^k(R^{nd}))) = 0$ and the rest of the proof is obvious.

2.8. REMARK. By the same method as in 2.7 we can show that

$$\lim_{n\to\infty} H_*(M_n; \mathrm{Hom}\,(R^n, T(R^n))) = 0$$

when R is a finite field with p^m elements, T is as in 2.1.b and k is less than p^m.

3. Comparison between THH and K^S

The topological Hochschild homology groups of a ring R with coefficients in an R-bimodule A, denoted here $\mathrm{THH}\,(R, A)$, were defined in [B]. In [P–W] the notion of topological Hochschild homology groups with coefficients in any functor $T: R_{f\mathrm{Mod}} \to R_{\mathrm{Mod}}$ is introduced and it agrees with Bökstedt's definition in the sense that $\mathrm{THH}\,(R, A) = \mathrm{THH}\,(R, \cdot \otimes A)$. It is also proved in [P–W] that $\mathrm{THH}_*\,(R, T)$ is equal to $H_*(R_{f\mathrm{Mod}}; \mathrm{Hom}\,(\cdot, T(\cdot)))$ where these latter groups denote the Baues–Wirsching homology groups of the category $R_{f\mathrm{Mod}}$ with coefficients in a natural system given by a bifunctor Hom $(\cdot, T(\cdot))$. We also have the following stability result (which is stated in [P] and proved in [J–P] in cohomological setting):

3.1. PROPOSITION. *Let $R_{f\mathrm{Mod},m}$ denote the full subcategory of $R_{f\mathrm{Mod}}$ consisting of modules of rank bounded by m. Then for i small with respect to m the inclusion map $R_{f\mathrm{Mod},m} \to R_{f\mathrm{Mod}}$ induces an isomorphism*

$$H_i(R_{f\mathrm{Mod},m}; \mathrm{Hom}(\cdot, T(\cdot))) = H_i(R_{f\mathrm{Mod}}; \mathrm{Hom}(\cdot, T(\cdot))).$$

3.2. LEMMA. *Let* $\mathbf{C}$ *denote the category consisting of one object* $*$ *and* $\mathrm{Mor}(*, *) = \mathrm{Hom}(R^m, R^m)$. *Then the inclusion* $i : \mathbf{C} \to R_{f\mathrm{Mod},m}$ *taking* $*$ *to* R^m *induces an isomorphism of Baues–Wirsching groups*:

$$H_*(\mathbf{C}; i^* \mathrm{Hom}(\cdot, \mathrm{T}(\cdot))) = \mathrm{H}_*(R_{f\mathrm{Mod},m}; \mathrm{Hom}(\cdot, T(\cdot))).$$

Moreover the Baues–Wirsching groups $H_*(\mathbf{C}; i^* \mathrm{Hom}(\cdot, T(\cdot)))$ *are equal to*

$$H_*(M_{m\times m}; \mathrm{Hom}(R^m, T(R^m)));$$

Proof. The second statement is obvious. For the first one we shall modify slightly the proof of [McC, proposition B.1.4]. For the convenience of the readers we shall give some more details on this subject. Let $\mathbf{D}$ be a small category and let $F : \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to Ab$ be a functor with values in abelian groups. Then the Baues–Wirsching homology groups of $\mathbf{D}$ with coefficients in F (see [B–W]) are defined as homology groups of the complex $(G_n(\mathbf{D}), \delta_n = \Sigma_{i=1}^{n} (-1)^i d_i)$ where

$$G_n(\mathbf{D}) = \bigoplus_{D_0 \leftarrow D_1 \leftarrow \cdots \leftarrow D_n} F(D_0, D_n)$$

and the sum is taken over all composable n-touples of morphisms in $\mathbf{D}$. Write an element of G_n as

$$(f; D_0 \xleftarrow{\alpha_1} D_1 \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_n} D_n),$$

or in shorter version as

$$(f; \alpha_1, \ldots, \alpha_n),$$

where $f \in F(D_0, D_n)$. Then

$$d_i(f; \alpha_1, \ldots, \alpha_n) = (f; \alpha_1, \ldots, \alpha_i \circ \alpha_{i+1}, \ldots, \alpha_n) \qquad \text{for } 0 < i < n,$$
$$d_0(f; \alpha_1, \ldots, \alpha_n) = ((\alpha_1)^*(f); \alpha_2, \ldots, \alpha_n),$$
$$d_n(f; \alpha_1, \ldots, \alpha_n) = ((\alpha_n)_*(f); \alpha_1, \ldots, \alpha_{n-x1}),$$

where if $\alpha : M' \to M$ and $\beta : N \to N'$ are morphisms in $\mathbf{D}$ and $f \in F(M, N)$ then

$$\alpha^*(f) = (F(\alpha, \mathrm{id}))(f) \in F(M', N)$$

and

$$\beta_*(f) = (F(\mathrm{id}, \beta))(f) \in F(M, N').$$

Now we have to come back to our situation. We can obviously (canonically) identify $H_*(\mathbf{C}; i^* \operatorname{Hom}(\cdot, T(\cdot)))$ with $H_*(\mathbf{R^m}; \operatorname{Hom}(\cdot, T(\cdot)))$ where $\mathbf{R^m}$ is a full subcategory of $R_{f\mathrm{Mod},m}$ consisting of one object R^m. Let us abbreviate i for the inclusion of $\mathbf{R^m}$ into $R_{f\mathrm{Mod},m}$. Let i_* denote the map induced by i on chain complexes for B–W homology. For any M in $R_{f\mathrm{Mod},m}$ let M' denote some chosen object of $R_{f\mathrm{Mod},m}$ such that $M \oplus M' = R^m$. Then we can define a chain map $j_* : G_*(R_{f\mathrm{Mod},m}) \to G_*(\mathbf{R^m})$ by

$$j_*(f; M_0 \xleftarrow{\gamma_1} M_1 \longleftarrow \cdots \xleftarrow{\gamma_n} M_n)$$
$$= (\alpha(M_n)_* \circ \beta(M_0)^*(f); M_0 \oplus M_0' \xleftarrow{\alpha(M_0) \circ \gamma_1 \circ \beta(M_1) \oplus \mathrm{id}_{(0,1)}} \cdots$$
$$\cdots \xleftarrow{\alpha(M_{n-1}) \circ \gamma_n \circ \beta(M_n) \oplus \mathrm{id}_{(n-1,n)}} M_n \oplus M_n'),$$

where

$$\alpha(M) = (\mathrm{id}_M, 0) : M \to M \oplus M',$$
$$\beta(M) = \mathrm{proj} : M \oplus M' \to M$$

and $\mathrm{id}_{(i,i+1)}$ denotes the identity matrix of the size $m = \max(\dim(M_i), \dim(M_{i+1}))$ placed in the lower right corner of the matrix. Obviously $j_* \circ i_* = \mathrm{id}$. The chain homotopy h joining $i_* \circ j_*$ with the identity is given by the formula (compare [McC, Prop. B.1.4]):

$$h = \sum_{i=0}^{n} (-1)^i h_i,$$

where

$$h_i(f; \gamma_1, \ldots, \gamma_n) = (\beta(M_0)^*(f); \alpha(M_0) \circ \gamma_1 \circ \beta(M_1) \oplus \mathrm{id}_{(0,1)}, \ldots,$$
$$\alpha(M_{i-1}) \circ \gamma_i \circ \beta(M_i) \oplus \mathrm{id}_{(i-1,i)}, \alpha(M_i), \gamma_{i+1}, \ldots, \gamma_n).$$

3.3. COROLLARY. $\mathrm{THH}_*(R, T) = \lim_{n\to\infty} H_*(M_{n\times n}; \mathrm{Hom}\,(R^n, T(R^n)))$ *where the limit for monoids is taken with respect to the upper inclusions with* 1 *in the right lower corner. On the level of coefficients the maps*

$$\mathrm{Hom}\,(R^n, T(R^n)) \to \mathrm{Hom}\,(R^{n+1}, T(R^{n+1}))$$

are induced by the standard linear inclusions $R^n \to R^{n+1}$ and projections $R^{n+1} \to R^n$.

Using the corollary above and the results of section 2 we get:

3.4. COROLLARY. *Let k be any natural number bigger than* 1. *Then the groups* $\mathrm{THH}\,(R, T^k)$ *are torsion and can contain p-torsion only for $p \le k$. Moreover if R is* large *then these groups are trivial.*

Now we must turn towards stable K-theory. The easiest definition of it (but it does not explain the name "stable K-theory") is following: let Ψ denote the homotopy fiber of the natural map $BGl(R) \to BGl(R)^+$ then $K^s_*(R, T) = H_*(\Psi; \lim_{n\to\infty} \mathrm{Hom}\,(R^n, T(R^n)))$ where the action of $\Pi_1(\Psi)$ on the coefficients is induced from the conjugation action of $Gl(R)$ on them. In the remaining part of the paper we shall write $H_*(Gl(R); \mathrm{Hom}\,(\cdot\,, T(\cdot)))$ for the groups $H_*(Gl(R); \lim_{n\to\infty} \mathrm{Hom}\,(R^n, T(R^n)))$. Looking at the Hochschild–Serre spectral sequence of the fibration

$$\Psi \to BGl(R) \to BGl(R)^+$$

we can immediately get the following lemma:

3.5. LEMMA. *Assume that for given T the groups* $H_*(Gl(R), \mathrm{Hom}\,(\cdot, T(\cdot)))$ *are trivial. Then* $K^s_*(R, T) = 0$.

3.6. REMARK. The results of [Be2] imply that $K^s_*(R, T^k)$ are torsion and can contain p-torsion only for $p \le k$. Moreover if R is *large* then these groups are trivial.

We define now the map $\Theta : K^s(R, T) \to \mathrm{THH}(R, T)$. The definition is very natural and the reader can find it also in [P]. Namely observe that the natural embedding of monoids $Gl_n(R) \to M_{n \times n}$ induces a map on homology

$$H_*(Gl_n(R); \mathrm{Hom}(R^n, T(R^n))) \to H_*(M_{n \times n}; \mathrm{Hom}(R^n, T(R^n))).$$

The stabilization processes in both cases are the same and hence we have a map

$$H_*(Gl(R); \mathrm{Hom}(\cdot, T(\cdot))) \to H_*(\mathrm{M}; \mathrm{Hom}(\cdot, T(\cdot))) = \mathrm{THH}(R, T).$$

When we compose this map with the natural map

$$H_*(\Psi; \mathrm{Hom}(\cdot, T(\cdot))) \to H_*(Gl(R); \mathrm{Hom}(\cdot, T(\cdot)))$$

then we obtain our map Θ.

3.7. THEOREM. *Let T be a functor of degree k. Assume that Θ is an isomorphism for all linear functors. Then Θ is an isomorphism up to p-torsion for $p \leq k$.*

Remark. Our assumption that 3.7 is true for linear functors is satisfied by the main theorem of [S–S–W]. We have put it here as a hypothesis because the paper [S–S–W] does not exist yet even in a preprint version so we are not able to give any written source of the proof of that result.

Proof of 3.7. We shall proceed by induction with respect to the degree of T. Assume that our theorem is true for any functor of degree less than k and that T is of degree k. By [Be2, section 2] we know that there is a natural map of functors

$$\psi : H_0(\Sigma_k; T^k \otimes_R A) \to T$$

with the property that the kernel and cokernel of ψ are of lower degree, where A is some R-module. On the other hand observe that if

$$0 \to T_1 \to T_2 \to T_3 \to 0$$

is a short exact sequence of functors then the sequence

$$0 \to \mathrm{Hom}(\cdot, T_1(\cdot)) \to \mathrm{Hom}(\cdot, T_2(\cdot)) \to \mathrm{Hom}(\cdot, T_3(\cdot)) \to 0$$

is also exact. The short exact sequence of functors gives us in a natural way long exact sequences of THH- and K^s-groups. Thse observations plus naturality of Θ give us immediately the fact that if $F: T_1 \to T_2$ is a natural map of functors and our theorem is true for T_1, ker (F) and coker (F) then it is true also for T_2. But our theorem is true for $H_0(\Sigma_k; T^k \otimes_R A)$ by vanishing results and is true for ker (ψ) and coker (ψ) by the inductive hypothesis. Hence our theorem is true for any functor T of degree k.

We want to finish this section with two corollaries which strengthen Theorem 3.7 for special kinds of rings.

3.8. COROLLARY. *If R is a* large *ring then the map Θ is an isomorphism for any functor T of finite degree.*

3.9. COROLLARY. *If R is a field with $q = p^m$ elements then Θ is an isomorphism for functors of degree less than q.*

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