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Invariants on three-manifolds with spin structure

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Introduction

New invariants on three-manifolds associated to certain roots of unity were recently obtained by Witten and constructed by Reshetikin and Turaev ([W], [RT]). These invariants are combinations of generalised Jones polynomials, calculated on a framed link in S^3, representing a surgery presentation of the manifold.

Following Lickorish, one can construct the invariants corresponding to $SU(2)$ using the one variable Kauffman bracket. See [L1–L2–L3] for the roots $A = e^{i\pi/2r}$ and [BHMV] for a generalisation.

Kirby and Melvin have shown that the invariant corresponding to the root $A = e^{i\pi/8k}$ decomposes as a sum of spin invariants, over all spin structures on a given manifold ([KM2]; see also [T2]). For $A = e^{i\pi/(8k+4)}$, they give an analogous decomposition as a sum over all the modulo 2 cohomology classes on the manifold.

In this article, we use the elementary methods of Lickorish's, as refined in [BHMV], and a spin version of the Kirby calculus, to construct an invariant on three-manifold with spin structure for each root of unity A of order ρ, $\rho \not\equiv 8$ modulo 16, in a convenient ring. Up to normalisation this invariant appears as a generalisation of the spin invariants of [KM2]–[T2]. We show that for a given ρ, the invariant is essentially unique.

In a similar way, one can construct an invariant for three-manifolds M^3 equipped with a cohomology class $c \in H^1(M^3, \mathbb{Z}/2)$, for each root of unity A of order ρ, $\rho \not\equiv 0$ modulo 16.

Different relations between these invariants are given: If $\rho \equiv 0$ modulo 16, we prove that the sum of the spin invariants, for all spin structures on a given manifold, is equal to the invariant of the manifold without structure, which will be called the 'unspun' invariant. (This word was suggested by the referee). If $\rho \equiv 8$ modulo 16, the unspun invariant decomposes as a sum over all the cohomology classes in an analogous way. In all other cases, the invariants defined for a given A can be expressed using only the invariant associated to the zero cohomology class, which we will call the even invariant.

We show that, for A of order 16, the spin invariant is equivalent to the Rochlin invariant (compare with [KM2], theorem 7.1). It enables one to distinguish the two spin structures on the lens space $L(n, 1)$ n even and $n \not\equiv 0$ modulo 16. The two spin structures on $L(16k, 1)$, $k > 0$, are distinguished by the invariant corresponding to a root of unity A of order $32k$.

§I. Kirby calculus and spin structure

It is well known that every oriented three-manifold M^3 can be obtained by surgery on a framed link $L = (L_i, p_i)_{1 \le i \le n}$ in S^3. M^3 is diffeomorphic to $M_L = \partial W_L$, where W_L is the manifold which is obtained by glueing a handle $A_i = D^2 \times D^2$ along a tubular neighbourhood V_i of each component L_i. $(S^1 \times D^2, S^1 \times 1)$ is identified with (V_i, l_i); here l_i is the preferred longitude: $lk(L_i, l_i) = p_i$. To each spin structure s on M_L, is associated an obstruction:

$$w_2(s) \in H^2(W_L, M_L, \mathbb{Z}/2).$$

(See [M] or [Ki2] for generalities about spin structures.) Using Poincaré duality, the class corresponding to $w_2(s)$ can be written in the basis of $H_2(W_L, \mathbb{Z}/2)$ corresponding to the handles:

$$\sum c_i[L_i] = [K(s)] \quad (c_i \in \mathbb{Z}/2).$$

This gives the characteristic sublink $K(s)$, which is known to satisfy:

$$\forall x \in H_2(W_L, \mathbb{Z}/2) \quad x \cdot x = [K(s)] \cdot x \quad (\text{in } \mathbb{Z}/2).$$

This condition is equivalent to the system:

$$B(c_i) = (p_i) \quad (\text{modulo } 2).$$

Here B is the linking matrix:

$$B = (b_{ij}) \qquad \text{with: } b_{ii} = p_i \quad \text{and} \quad b_{ij} = lk(L_i, L_j) \quad \text{for } i \neq j.$$

This gives a one to one correspondence between the spin structures and the solutions of the condition above. We denote by $M_{(L,K)}$ the manifold M_L equipped with the spin structure corresponding to the characteristic sublink K: every closed three-manifold with spin structure admits such a characteristic surgery presentation.

The components of a framed link $L = (L_i, p_i)$, with characteristic sublink K are denoted by the triple (L_i, p_i, c_i). Here $c_i \in \mathbb{Z}/2$ is one whenever L_i belongs to K.

The following theorem is a refinement of the well known Kirby calculus.

THEOREM (I.1). *The two spin manifolds $M_{(L,K)}$ and $M_{(L',K')}$ are spin diffeomorphic if and only if (L, K) and $L', K')$ are equivalent under the relation generated by the following moves:*

KS1: Add to (L, K) an unknotted component $(L_i, \varepsilon, 1)$, isolated from the others by an embedded two-sphere, with $\varepsilon = \pm 1$.

KS2: Modify a component (L_i, p_i, c_i) adding (using a band) a push-off of another one (L_j, p_j, c_j). The new component indexed by i, is: (L_i', p_i', c_i'), with $p_i' = p_i + p_j + 2lk(L_i, L_j)$. The linking number is calculated with orientations of L_i and L_j, coherent with an orientation of L_i'; $c_i' = c_i$, and $c_j' = c_j + c_i$ (mod 2).

The only changes needed to be made to the proof given in [Ki1] concerns the characteristic sublink:

The $KS2$ move (handle slide) does not modify W_L; the basis of $H_2(W_L, \mathbb{Z}/2)$ is changed, and it is easy to calculate the new characteristic coefficients.

The $KS1$ move (stabilization) adds $\pm\mathbb{C}P^2$ to W_L. The new coefficient is determined by the characteristic condition; the c_i coefficients, which determine the obstruction on the other handles, are not modified.

Remarks (I.2)

1. Following [FR], one can replace the two moves by:

KS: Add an unknotted component (L_i, ε, c_i) with

$$\varepsilon = \pm 1 \quad \text{and} \quad c_i = 1 + \Sigma_{j \neq i}\, c_j lk(L_i, L_j).$$

One must then add to each p_j the number $\varepsilon(lk(L_i, L_j))^2$.

2. The framed link can be represented by a diagram in which the coefficient p_i is given by the longitude which is parallel to the component L_i in the plane. The positive KS move ($KS+$) is described by the Figure 1. An integer n near a curve means the presence of n parallels in the plane.

3. An analogous statement holds for three-manifolds M^3 equipped with a cohomology class $c \in H^1(M^3, \mathbb{Z}/2)$. If L is a surgery presentation of M^3, then c corresponds to a sublink C such that the modulo 2 homology class $[C]$ is the kernel

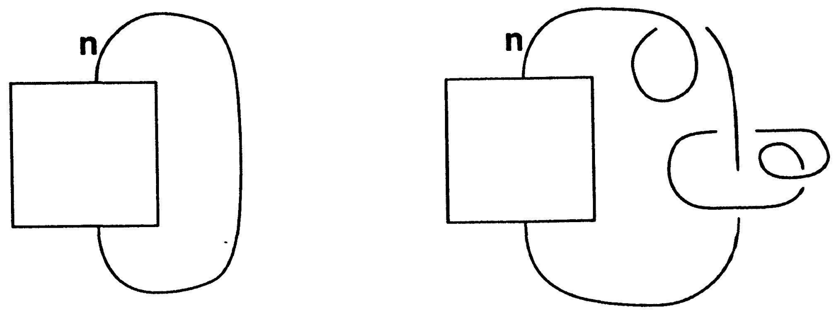


Figure 1

of B (interpreted modulo 2). The move of Remark I.2.1 above becomes:
KC: Add an unknotted component (L_i, ε, c_i) with

$$\varepsilon = \pm 1 \quad \text{and} \quad c_i = \Sigma_{j \neq i} c_j lk(L_i, L_j).$$

Add to each p_j the number $\varepsilon(lk(L_i, L_j))^2$.

§II. The spin invariants

(1) *The bracket and the metabracket*

We first recall a few results about the Jones–Kauffman module, of the solid torus, which is a key ingredient in the construction. Links in the solid torus $S^1 \times D^2 = S^1 \times I \times I$ can be represented by diagrams in the annulus $S^1 \times I$. $K(S^1 \times D^2)$ is the $\mathbb{Z}[A, A^{-1}]$-module freely generated by these link diagrams, quotiented by the relations:

(a) $(D \cup o) = (-A^2 - A^{-2})(D) = \delta(D)$,
(b) $(\diagup\!\!\!\diagdown) = A(\asymp) + A^{-1}()()$.

A product is defined in $K(S^1 \times D^2)$ by the union of two annuli along a component of their boundary. $K(S^1 \times D^2)$ is the polynomial algebra $\mathbb{Z}[A, A^{-1}][z]$, z representing a simple curve, essential in the annulus. The degree gives a $\mathbb{Z}/2$-graduation. $K(S^1 \times D^2) = K^0(S^1 \times D^2) \oplus K^1(S^1 \times D^2)$.

The Kauffman bracket is defined on link diagrams in the plane by:

(i) $\langle \varnothing \rangle = 1$,
(ii) $\langle D \cup o \rangle = (-A^2 - A^{-2})\langle D \rangle = \delta \langle D \rangle$,
(iii) $\langle \diagup\!\!\!\diagdown \rangle = A\langle \asymp \rangle + A^{-1}\langle\,)(\, \rangle$.

This polynomial is an invariant of unoriented framed links in S^3. We recall that on a link diagram the framing of a component is given by the parallel in the plane.

Given a link diagram, one can replace each of the n components L_i by k_i parallels in the plane, and then evaluate the Kauffman bracket of this cabled link, which only depends on the framed link L, represented by the given diagram. This yields a multilinear form:

$$\langle\,,\ldots,\,\rangle_L \quad (K(S^1\times D^2))^{\otimes n}\to\mathbb{Z}[A,A^{-1}],$$

which we call the meta-bracket (cf: [L3], [BHMV]).

We shall simply use $\langle\ \rangle$ for the linear form associated with the unknot with zero framing, and $\langle\ ,\ \rangle_k$ for the bilinear form associated with the Hopf link, with framing coefficient k on each component.

As in [L3] and [BHMV], we will use the Chebyshev polynomials: $(e_n)_{n\in\mathbb{Z}}$ is the family of polynomials in $K(S^1\times D^2)$ defined by

$$e_{-1}=0,\ e_0=1 \text{ and for every } n\text{: } ze_n=e_{n+1}+e_{n-1}.$$

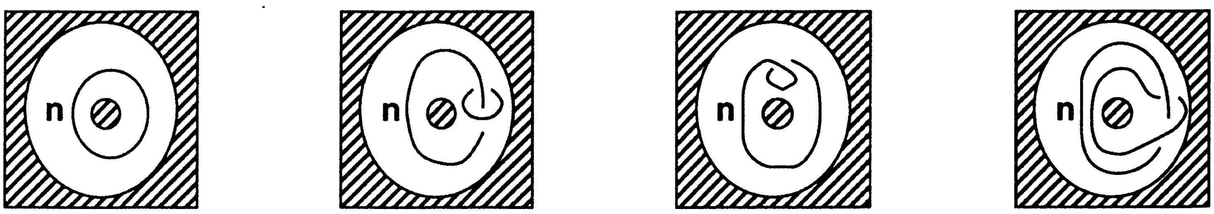
e_n is odd or even depending on the parity of n, and: $e_{-n}=-e_{-2+n}$. It is easily shown that for every integer n:

$$\langle e_n\rangle=\frac{(-1)^n(A^{2n+2}-A^{-2n-2})}{A^2-A^{-2}}.$$

The operators c, t and τ are defined on $K(S^1\times D^2)$ by Figure 2.

Using relations (a) and (b) one can check the following lemma:

LEMMA (II.1). *For every x:*
(a) $tzx=-A^3\tau tx$,
(b) $\tau zx=A^2z\tau x+(1-A^{-4})cx$,
(c) $czx=A^{-2}zcx+(1-A^4)\tau x$.



z^n $\quad$ cz^n $\quad$ $t\,z^n$ $\quad$ $\tau\,z^n$

Figure 2

The proposition below follows by induction (see also [L3] and [BHMV]).

PROPOSITION (II.2). *The basis $(e_n)_{n\geq 0}$ diagonalizes the operators c and t:*

$$ce_n = \lambda_n e_n, \qquad \text{with } \lambda_n = -A^{2n+2} - A^{-2n-2},$$

and

$$te_n = \mu_n e_n, \qquad \text{with } \mu_n = (-1)^n A^{n^2+2n}.$$

(2) *How to construct invariants?*

The following proposition gives a way to construct invariants on three-manifolds with spin structure. The indeterminate A is evaluated in a ring Λ, and we set:

$$K_\Lambda(S^1 \times D^2) = K(S^1 \times D^2) \otimes \Lambda.$$

(A must be invertible and will be a root of unity.)

PROPOSITION (II.3). *If $\omega_0 \in K_\Lambda^0(S^1 \times D^2)$ and $\omega_1 \in K_\Lambda^1(S^1 \times D^2)$ satisfy the following conditions:*

(a) $\forall x_0 \in K_\Lambda^0(S^1 \times D^2)$ $\langle x_0, \omega_1 \rangle_\varepsilon = \langle x_0 \rangle \langle t^\varepsilon \omega_1 \rangle$, *for* $\varepsilon = \pm 1$;

(b) $\forall x_1 \in K_\Lambda^1(S^1 \times D^2)$ $\langle x_1, \omega_0 \rangle_\varepsilon = \langle x_1 \rangle \langle t^\varepsilon \omega_1 \rangle$, *for* $\varepsilon = \pm 1$;

(c) $\langle t^\varepsilon \omega_1 \rangle$ *is invertible in* Λ, *for* $\varepsilon = \pm 1$;

then there exists an invariant θ_Λ of three-manifolds with spin structure such that for any characteristic surgery presentation $(L, K) = (L_i, p_i, c_i)_{1\leq i\leq n}$, one has the following expression:

$$\theta_\Lambda(M_{(L,K)}) = \frac{\langle \omega_{c_1}, \ldots, \omega_{c_n} \rangle_L}{\langle t\omega_1 \rangle^{b_+} \langle t^{-1}\omega_1 \rangle^{b_-}}.$$

Here (b_+, b_-) is the signature of the quadratic form associated with the linking matrix B, which is the intersection form on $H_2(W_L, \mathbb{Z})$.

Proof. The only thing to show is that the expression given above is not changed by a KS-move. If we add a new unknotted component, either $(L_{n+1}, \varepsilon, 1)$ or $(L_{n+1}, \varepsilon, 0)$, we make use of (a) or (b) respectively. In each case we get a multiplicative factor $\langle t^\varepsilon \omega_1 \rangle$ for the meta-bracket, which is canceled because either b_+ or b_- is modified.

Remarks (II.4)

1. Multiplication by z is a symmetric operator for $\langle\ ,\ \rangle_\varepsilon$, $\varepsilon = \pm 1$. It follows that if one requires that $z\omega_0 = \delta\omega_1$, then conditions (a) and (b) are equivalent.

2. Suppose Λ is equipped with a conjugation which send A on A^{-1}, one has on $K_A(S^1 \times D^2)$ a compatible involution defined by taking the mirror image of a diagram. If ω_0 and ω_1 are fixed by this involution, then the conditions corresponding to $\varepsilon = 1$ and $\varepsilon = -1$ are equivalent.

3. In a similar way, one can define an invariant on three-manifolds with cohomology class by the formula:

$$\theta_A(M_{(L,C)}) = \frac{\langle \omega_{c_1}, \ldots, \omega_{c_n} \rangle_L}{\langle t\omega_0 \rangle^{b_+} \langle t^{-1}\omega_0 \rangle^{b_-}}$$

if:

(a) $\forall x_0 \in K_A^0(S^1 \times D^2)$ $\langle x_0, \omega_0 \rangle_\varepsilon = \langle x_0 \rangle \langle t^\varepsilon \omega_0 \rangle$, for $\varepsilon = \pm 1$;
(b) $\forall x_1 \in K_A^1(S^1 \times D^2)$ $\langle x_1, \omega_1 \rangle_\varepsilon = \langle x_1 \rangle \langle t^\varepsilon \omega_0 \rangle$, for $\varepsilon = \pm 1$;
(c) $\langle t^\varepsilon \omega_0 \rangle$ is invertible in Λ, for $\varepsilon = \pm 1$.

4. An 'unspun' invariant is defined ([BHMV]) by the formula:

$$\theta_A(M_L) = \frac{\langle \omega, \ldots, \omega \rangle_L}{\langle t\omega \rangle^{b_+} \langle t^{-1}\omega \rangle^{b_-}}$$

if $\forall x$ $\langle x, \omega \rangle_\varepsilon = \langle x \rangle \langle t^\varepsilon \omega \rangle$ and $\langle t^\varepsilon \omega \rangle$ is invertible, $\varepsilon = \pm 1$.

In §IV we will systematically study the conditions (a), (b) and (c) of the proposition above, and the use of roots of unity will be justified there. However, in order to construct invariants we only need to exhibit solutions to (a), (b) and (c). We will now do so.

(3) *Existence of the invariants*

Let A be a root of unity in an integral domain Λ, for which the order r of $q = A^4$ is not congruent to 2, modulo 4.

We will assume that

if $r = 1$, then 2 is invertible, and
if $r > 1$, then r is invertible.

Let $\omega = \omega_0 + \omega_1$ be the decomposition corresponding to the $\mathbb{Z}/2$-graduation of the element ω defined as follows:

(a) If $r = 1$, then $\omega = 1 + \delta^{-1} z$.
(b) If $r > 1$ is odd, then $\omega = \Sigma_{i=0}^{r-2} \langle e_i \rangle e_i$.
(c) If $r = 4k$, then $\omega = \frac{1}{2} \Sigma_{i=0}^{r-2} \langle e_i \rangle e_i$.

We shall see in §IV that in the case $r = 4k$, the same invariant is obtained, with ω_0 and ω_1 replaced by the reduced elements $\tilde{\omega}_0$ and $\tilde{\omega}_1$, defined by: $\tilde{\omega}_0 = \Sigma_{i=0}^{k-1} \langle e_{2i} \rangle e_{2i}$, $z\tilde{\omega}_0 = \delta \tilde{\omega}_1$.

Remark. When the order of A is even, the element ω satisfy the condition of Remark II.4.4, and is (up to a coefficient 2 in cases (a) and (b)) the one which is used in [BHMV].

THEOREM (II.5). *There exists an invariant of three-manifolds with spin structure such that for any characteristic surgery presentation* $(L, K) = (L_i, p_i, c_i)_{1 \le i \le n}$, *one has the following expression*:

$$\theta_A(M_{(L,K)}) = \frac{\langle \omega_{c_1}, \ldots, \omega_{c_n} \rangle_L}{\langle t\omega_1 \rangle^{b_+} \langle t^{-1} \omega_1 \rangle^{b_-}}.$$

This invariant is 1 *for* S^3. *It is multiplicative for connected sum. And if* A *has a conjugation sending* A *on* A^{-1}, *changing orientation corresponds to this conjugation.*

Remark (II.6). Let:

$$\Lambda_\rho = (\mathbb{Z}[A, A^{-1}]/\varphi_\rho)\left[\frac{1}{r}\right], \quad \text{if } r > 1 \text{ is odd and } \rho \in \{r, 2r, 4r\},$$

$$\Lambda_\rho = (\mathbb{Z}[A, A^{-1}]/\varphi_\rho)\left[\frac{1}{r}\right], \quad \text{if } r = 4k = 16\rho,$$

$$\Lambda_1 = \Lambda_2 = \mathbb{Z}\left[\frac{1}{2}\right] \quad \text{and} \quad \Lambda_4 = (\mathbb{Z}[A, A^{-1}]/\varphi_4)\left[\frac{1}{2}\right].$$

Here φ_ρ is the ρ-th cyclotomic polynomial. The invariant above is defined in Λ_ρ, it is denoted by θ_ρ and we shall make it clear in §IV in which sense θ_ρ is universal.

The case $r = 1$ will be studied at the end of §II. To prove existence of this invariant for $r > 1$, we are going to show that ω_0 and ω_1 satisfy the conditions of

Proposition II.1. In order to do that we need to define a quotient of the algebra $K_A(S^1 \times D^2)$: $V_A = K_A(S^1 \times D^2)/(e_{r-1})$. Here (e_{r-1}) denote the ideal generated by e_{r-1}. We observe that:

$$(e_{r-1}) = \text{Span}\,\{e_{r-1-k} + e_{r-1+k}, k \in \mathbb{N}\}.$$

LEMMA (II.7). (a) *In the quotient algebra* V_A:

$$e_{n+2r} = e_n, \qquad \textit{for every } n.$$

(b) *The operators c and t, the linear form* $\langle\ \rangle$, *and the scalar products* $\langle\ ,\ \rangle_i$, *are well defined on the quotient* V_A.

The proof is straightforward. For (b), observe that:

$$\langle z^k, z^{k'}\rangle_0 = \langle c^k z^{k'}\rangle, \quad \text{and} \quad \langle x, y\rangle_i = \langle t^i x, t^i y\rangle_0.$$

Let G be defined on V_A by:

$$G(x) = \sum_{j=0}^{r-1} e_{2j+1} t(xe_{2j+1}).$$

PROPOSITION (II.8). *For every* x_0 *in* V_A^0: $G(x_0) = G(1)x_0$.

Proof. It follows from: $e_2 e_n = e_{n-2} + e_n + e_{n+2}$, that for every x:

$$G(xe_2) = \sum_{j=0}^{r-1} e_{2j+1} t(xe_{2j-1}) + \sum_{j=0}^{r-1} e_{2j+1} t(xe_{2j+1}) + \sum_{j=0}^{r-1} e_{2j+1} t(xe_{2j+3}).$$

Using Lemma II.7.(a):

$$G(xe_2) = \sum_{j=0}^{r-1} e_{2j+3} t(xe_{2j+1}) + \sum_{j=0}^{r-1} e_{2j+1} t(xe_{2j+1}) + \sum_{j=0}^{r-1} e_{2j-1} t(xe_{2j+1}),$$

$$G(xe_2) = \sum_{j=0}^{r-1} e_2 e_{2j+1} t(xe_{2j+1}) = G(x)e_2.$$

One can deduce that

(1) $G(e_2) = G(1)e_2$.

(2) for every i, $G(e_2 e_{2i}) = e_2 G(e_{2i})$.

Using this, the proposition can be shown, by induction, for all e_{2i}, which generate V_A^0.

Proof of Theorem II.5

Let:

$$\Omega_1 = \sum_{j=0}^{r-1} \langle e_{2j+1} \rangle e_{2j+1}.$$

In V_A: $\Omega_1 = 2\omega_1$, if r is odd, and $\Omega_1 = 4\omega_1$ if $r = 4k$.

For x_0 in V_A^0:

$$\langle x_0, \Omega_1 \rangle_1 = \langle t(x_0 \Omega_1) \rangle = \langle G(x_0) \rangle = \langle x_0 \rangle \langle t\Omega_1 \rangle.$$

We can deduce that ω_1 satisfies the condition (a) of II.3.

We can show that in V_A: $z\omega_0 = \delta\omega_1$, so ω_0 satisfies condition II.3.(b). We will now prove that $\langle t\omega_1 \rangle$ is invertible.

$$\langle t\Omega_1 \rangle = \sum_{j=0}^{r-1} \mu_{2j+1} \langle e_{2j+1} \rangle^2,$$

$$\begin{aligned}
(A^2 - A^{-2})^2 \langle t\Omega_1 \rangle &= -\sum_{j=0}^{r-1} A^{4j^2+4j+1+4j+2} (A^{4j+4} - A^{-4j-4})^2 \\
&= -A^3 \sum_{j=0}^{r-1} q^{j^2+2j} (q^{2j+2} - 2 + q^{-2j-2}) \\
&= -A^3 \sum_{j=0}^{r-1} q^{(j+2)^2-2} - 2q^{(j+1)^2-1} + q^{j^2-2} \\
&= (q^{-2} - q^{-1}) 2A^3 g,
\end{aligned}$$

$$\langle t\Omega_1 \rangle (A^2 - A^{-2}) = 2A^{-3} g.$$

Here g is the Gauss sum:

$$g = \sum_{j=0}^{r-1} q^{j^2}.$$

g is well known for $q = e^{2i\pi/r} = \zeta_r$:

$$g = \frac{1}{2}\sqrt{r}\,(1+i)(1+e^{-i\pi r/2}) \quad \text{(cf [BE])}.$$

Now we consider the morphism: $\varphi\mathbb{Z}[\zeta_r] \to \Lambda$, which sends ζ_r to q. We claim:

$$-\langle t\Omega_1\rangle\langle t^{-1}\Omega_1\rangle(A^2-A^{-2})^2 = 4\varphi(g\bar{g}) = \begin{cases} 4r & \text{if } r \text{ is odd} \\ 8r & \text{if } r = 4k \end{cases}.$$

For r odd, $\langle t\omega_1\rangle\langle t^{-1}\omega_1\rangle(A^2-A^{-2})^2 = -r$ is invertible.

For $r = 4k$, $\langle t\omega_1\rangle\langle t^{-1}\omega_1\rangle(A^2-A^{-2})^2 = -2k$ is also invertible.

Thus $\langle t\omega_1\rangle$ is invertible in every case.

(4) *Some special cases*

If $\Lambda = \mathbb{C}$, we can write: $\langle t\omega_1\rangle = \beta\gamma$, with: $\beta > 0$ and $|\gamma| = 1$. The invariant can then be expressed by:

$$\theta_A(M_{(L,K)}) = \beta^{-n+\nu}\gamma^{-\sigma}\langle\omega_{c_1}, \ldots, \omega_{c_n}\rangle_L.$$

Here $\sigma = b_+ - b_-$ is the index of W_L, and ν is the nullity of the linking matrice (the first Betti number of M_L).

We are going to study the cases: $A = \zeta_{12}$ and $A = \zeta_{16}$ ($\zeta_n = e^{2i\pi/n}$). We denote by $\mu \in \mathbb{Z}/16$ the Rochlin invariant of (M^3, s).

PROPOSITION (II.9). *For* $A = \zeta_{12}$, $\theta_A(M^3, s) = i^\mu$, *and for* $A = \zeta_{16}$, $\theta_A(M^3, s) = (-\zeta_{16})^{-3\mu}$.

Proof. For $A = \zeta_{12}, r = 3, \omega_0 = 1, \omega_1 = -z, \langle t\omega_1\rangle = -i$. For $A = \zeta_{16}$, $r = 4$, $\tilde{\omega}_0 = 1, \tilde{\omega}_1 = -(1/\sqrt{2})z, \langle t\tilde{\omega}_1\rangle = -(\zeta_{16})^3$. One can evaluate the invariant using a link with empty characteristic sublink; in this case, W_L is spin and its index gives μ.

Remark. For $A = -\zeta_{16}$, $\theta_A(M^3, s) = (\zeta_{16})^{-3\mu}$ is exactly the invariant $\tau_4(M^3, s)$ of Kirby–Melvin (see [KM2], theorem 7.1).

Case $r = 1$. For all j, $\lambda_{2j} = \delta$, $\mu_{2j} = 1$ and $\mu_{2j+1} = -A^3$; one can easily deduce that (ω_0, ω_1) satisfies conditions II.3.

Using a presentation $(L, \varnothing)$ as above, we show that the invariant is: $(-A)^\mu$.

§III. Some symmetry formulas and their consequences

(1) *The odd case*

A good question is: What is the influence of the spin structure s when one calculates the invariant $\theta_A(M^3, s)$ defined in II.5? Here is the answer for odd r:

THEOREM (III.1). *For odd r, the following reduction holds for the spin invariant defined in* II.5:

$$\theta_A(M^3, s) = \alpha^{\mu}\tilde{\theta}_A(M^3),$$

with $\alpha = -A^{r^2}$ *and*

$$\tilde{\theta}_A(M^3) = \frac{\langle \omega_0, \ldots, \omega_0 \rangle_L}{\langle t\omega_0 \rangle^{b_+} \langle t^{-1}\omega_0 \rangle^{b_-}}.$$

We recall that μ is the Rochlin invariant. We observe that $\alpha^4 = 1$; more precisely, the order of α is 2, 1 or 4 according to the order of A, which may be r, $2r$ or $4r$.

If one constructs invariants for manifolds M^3 equipped with a cohomology class in $H^1(M^3, \mathbb{Z}/2)$ as indicated in the Remark 3 of §II.4.(3), the invariant $\tilde{\theta}_A(M^3)$ is the one obtained for the zero cohomology class. The formula above can be written:

$$\theta_A(M^3, s) = \theta_{A^{r^2}}(M^3, s)\tilde{\theta}_A(M^3).$$

We can show that the same formula holds for the 'unspun' invariant constructed with ω when the order of A is $4r$, r odd:

$$\theta_A(M^3) = \theta_{A^{r^2}}(M^3)\tilde{\theta}_A(M^3).$$

When the order of A is $2r$, r odd, the 'unspun' invariant $\theta_A(M^3)$ is exactly $\tilde{\theta}_A(M^3)$.

The following lemma gives some needed symmetry formulas, and is easily established.

LEMMA (III.2). *For every integer i*:
(a) $\langle e_{r-2-i} \rangle = A^{2r}\langle e_i \rangle$,
(b) $\lambda_{r-2-i} = A^{2r}\lambda_i$,
(c) $\mu_{r-2-i} = -A^{r^2+2r(1+i)}\mu_i$.

Proof of theorem III.1

Using Lemma III.2, we can show that: $\langle t\omega_1\rangle = -A^{-r^2}\langle t\omega_0\rangle$. We claim that: $\forall x_0 \in K_A^0(S^1 \times D^2)$ $\langle tx_0, t\omega_0\rangle = \langle x_0\rangle\langle t\omega_0\rangle$, so that $\tilde{\theta}_A$ is well defined.

Now we can evaluate the two invariants using a presentation $(L, \emptyset)$...

(2) *The case* $r = 4k$

Let us begin by giving an example which shows that no reduction analogous to III.1 holds in this case.

The lens space $L(n, 1)$ is obtained with an unknotted circle weighted by n. For even n, this space has two spin structures, and the respective Rochlin invariants are: $\operatorname{sgn}(n)$ and $\operatorname{sgn}(n) - n$ (here $\operatorname{sgn}(n)$ is the sign of n: 0 or ± 1). This can be seen using $\theta_{\zeta_{16}}$.

For $n = 0$, the two spin structures are equivalent (the space is $S^1 \times S^2$). The Rochlin invariant distinguishes the two spin structures if n is not in $16\mathbb{Z}$.

For $n = 16k$, $k \neq 0$, the invariants for $A = \zeta_{32k}$ are opposite and not zero, and so the two spin structures are distinguished.

Proof. These two invariants are:

$$I_0 = \langle t\omega_1\rangle^{-\operatorname{sgn}(k)}\langle t^{16k}\omega_0\rangle = \langle t\omega_1\rangle^{-\operatorname{sgn}(k)}\langle \omega_0\rangle,$$

and

$$I_1 = \langle t\omega_1\rangle^{-\operatorname{sgn}(k)}\langle t^{16k}\omega_1\rangle = \langle t\omega_1\rangle^{-\operatorname{sgn}(k)}\langle -\omega_1\rangle.$$

We have: $\langle \omega_0\rangle = \langle \omega_1\rangle = (-r/2)\langle A^2 - A^{-2}\rangle^{-2}$, so $I_0 = -I_1 \neq 0$.

We now give the decomposition theorem. A similar proposition was announced by Turaev in [T2] and appears in [KM2]. The hypotheses are those of II.3.

THEOREM (III.3). *For* $r = 4k$, *the global invariant* (*see* II.4.3) *decomposes as following*:

$$\theta_A(M_L) = \sum_{s \in \operatorname{Spin}(M_L)} \theta_A(M_L, s).$$

We need some symmetry formulas again.

LEMMA (III.4). *If $r = 4k$, for all integers i, j:*
(a) $\langle e_{r-2-i} \rangle = \langle e_i \rangle$.
(b) $\lambda_{r-2-i} = -\lambda_i$.
(c) $\mu_{r-2-i} = (-1)^{i+1}\mu_i$.
(d) $\langle z^j, e_{r-2-i} \rangle = (-1)^j \langle z^j, e_i \rangle$.

In V_A we set

$$N_+ = \text{Span}\,\{e_{2k-1-i} + e_{2k-1+i}, 0 \le i \le 2k-1\} = N^0_+ \oplus N^1_+,$$

$$N_- = \text{Span}\,\{e_{2k-1-i} + e_{2k-1+i}, 1 \le i \le 2k-1\} = N^0_- \oplus N^1_-.$$

COROLLARY (III.5). (a) *For the scalar product $\langle\ ,\ \rangle_0$:*

$$V^0_A = (N_-)^\perp \quad \textit{and} \quad V^1_A = (N_+)^\perp.$$

(b) $t(N^1_+) = N^1_+$, $t(N^1_-) = N^1_-$, $t(N^0_+) = N^0_-$, $t(N^0_-) = N^0_+$.

Proof of Theorem III.3

We can deduce from III.5 that for every k:

$t^k\omega_1$ is in N^1_+, $t^{2k}\omega_0$ is in N^0_+ and $t^{2k+1}\omega_0$ is in N^0_-.

Now we claim that we can use a link L, with only unknotted components: any surgery presentation can be reduced to such a link, using K-moves.

Let us write:

$$\langle \omega, \ldots, \omega \rangle_L = \sum_{K \subset L} \langle \omega_{a_1}, \ldots, \omega_{a_n} \rangle_L.$$

Here a_i is one if and only if the component L_i is in the sublink K. We are going to show that, in this sum, the contribution of a term corresponding to a sublink which is not characteristic is zero. Let K be a sublink of L, which contains a component L_i, such that:

$$[L_i] \cdot [K] \ne [L_i] \cdot [L_i] \quad \text{(modulo 2)}.$$

(a) If L_i is not in K, then: $lk(L_i, K) = p_i + 1 \pmod 2$, p_i being the coefficient of the component L_i. If p_i is even, the term corresponding to K can be written:

$\langle t^{2k}\omega_0, x_1\rangle_0$, with x_1 odd, and is zero, according to III.5.

If p_i is odd the term is written:

$\langle t^{2k+1}\omega_0, x_0\rangle_0$, with x_0 even ...

(b) If L_i is in K, then $lk(L_i, K - L_i) = 1 \pmod 2$.
The corresponding term is:

$\langle t^{k}\omega_1, x_1\rangle_0$, with x_1 odd ...

A particular case is that, for $\varepsilon = \pm 1$, $\langle t^{\varepsilon}\omega\rangle = \langle t^{\varepsilon}\omega_1\rangle$.
The theorem is proven.

§IV. Uniqueness

We are going to study the conditions of Proposition II.3. This will justify the choice of roots of unity, and will be used to prove the following uniqueness theorem.

THEOREM (IV.1). *Suppose Λ is an integral domain containing a root of unity A, of order ρ, and θ_A is the invariant associated to a solution (w_0, w_1) of conditions II.3.*
There exists a unit λ in Λ, a homomorphism $f: \Lambda_\rho \to \Lambda$ induced by $\mathbb{Z}[A, A^{-1}] \to \Lambda$ such that:

$$\theta_A(M, s) = \lambda^{\nu} f(\theta_\rho(M, s)).$$

Here ν is the first Betti number of M.

See II.6 for the definition of Λ_ρ and θ_ρ.

(1) *Diagonalisation of the bilinear form*

$$\langle\, ,\rangle_0 \quad K^0(S^1 \times D^2) \otimes K^1(S^1 \times D^2) \to \mathbb{Z}[A, A^{-1}].$$

Recall that the eigenvalues of the operator c are

$$\lambda_n = -A^{2n+2} - A^{-2n-2} \quad (n \geq 0).$$

We define the elements Q_{2k} and Q_{2k+1} ($k \geq 0$) respectively in $K^0(S^1 \times D^2)$ and $K^1(S^1 \times D^2)$ by:

$$Q_0 = 1, \qquad Q_1 = z$$

and for $k \geq 1$:

$$Q_{2k} = \prod_{i=0}^{k-1} (z^2 - \lambda_{2i+1}^2),$$

$$Q_{2k+1} = z \prod_{i=0}^{k-1} (z^2 - \lambda_{2i}^2).$$

PROPOSITION (IV.2)
(a) If $l \neq k$, $\langle Q_{2k}, Q_{2l+1} \rangle_0 = 0$.

(b) $d_k = \langle Q_{2k}, Q_{2k+1} \rangle_0 = \dfrac{-1}{A^2 - A^{-2}} \displaystyle\prod_{i=1}^{2k+1} (A^{4i} - A^{-4i})$.

Proof. (a) For $l < k$:

$$\begin{aligned}\langle Q_{2k}, e_{2l+1} \rangle_0 &= \langle Q_{2k}(c) \cdot e_{2l+1} \rangle \\ &= Q_{2k}(\lambda_{2l+1}) \langle e_{2l+1} \rangle = 0.\end{aligned}$$

Thus Q_{2k} is orthogonal to Span $\{z^{2l+1}, l < k\}$. In the same way, Q_{2k+1} is orthogonal to Span $\{z^{2l}, l < k - 1\}$.

(b) $d_k = \langle Q_{2k}, e_{2k+1} \rangle_0$,

$$d_k = \langle Q_{2k}(c) \cdot e_{2k+1} \rangle = Q_{2k}(\lambda_{2k+1}) \langle e_{2k+1} \rangle,$$

$$d_k = (-1)^{2k+1} \frac{A^{4k+4} - A^{-4k-4}}{A^2 - A^{-2}} \prod_{l<k} (\lambda_{2k+1}^2 - \lambda_{2l+1}^2).$$

One can deduce (b) from the identity:

$$\lambda_{2k+1}^2 - \lambda_{2l+1}^2 = (A^{4(k+l)+8} - A^{-4(k+l)-8})(A^{4(k-l)} - A^{-4(k-l)}).$$

In order to discuss the conditions of II.3, we need the following lemma:

LEMMA (IV.3)

$$\langle t^{-1}Q_{2k}\rangle = q^{-k^2-2k}\prod_{l=1}^{k}(1+q^{2l})(1-q^{2l+1}).$$

We first prove the following technical lemma:

LEMMA (IV.4)
(a) $\tau Q_{2k} = A^{4k}Q_{2k+1}$.
(b) $\tau^2 Q_{2k} = A^{8k+2}Q_{2k+2} + A^{-6}(2 + A^{16k+16} + A^{16k+8})Q_{2k}$
$\qquad + A^{8k-2}(A^{8k+4} - A^{-8k-4})(A^{8k} - A^{-8k})Q_{2k-2}$.

Proof of Lemma IV.4

(a) $\tau Q_{2k} = A^{4k}Q_{2k+1} + \sum_{l=0}^{k-1}\zeta_l Q_{2l+1}$ (see Lemma II.1).

For $l < k$:

$$\langle Q_{2l}, \tau Q_{2k}\rangle_0 = \zeta_l d_l = \langle \tau Q_{2l}, Q_{2k}\rangle_0 = 0.$$

(b) In a similar way:

$$\tau Q_{2k+1} = A^{4k+2}Q_{2k+2} + \xi_k Q_{2k} + \nu_{k-1}Q_{2k-2} \quad (\nu_{-1} = 0).$$

ξ_k and ν_{k-1} satisfy $\xi_k d_k = \langle \tau Q_{2k+1}, Q_{2k+1}\rangle$, and

$$\nu_{k-1}d_{k-1} = \langle \tau Q_{2k+1}, Q_{2k-1}\rangle = A^{4k-2}\, d_k.$$

One easily obtains:

$$\nu_{k-1} = A^{4k-2}(A^{8k+4} - A^{-8k-4})(A^{8k} - A^{-8k}).$$

To calculate ξ_k, we shall use e_{2k} and e_{2k-1}:

$$e_{2k} = Q_{2k} + \eta_{2k-2}Q_{2k-2} + \cdots,$$
$$e_{2k+1} = Q_{2k+1} + \eta_{2k-1}Q_{2k-1} + \cdots.$$

The dots indicate a polynomial which has degree smaller than the preceeding term.

$$\begin{aligned}\langle Q_{2k-1}, e_{2k}\rangle &= \eta_{2k-2}\, d_{k-1} \\ &= Q_{2k-1}(\lambda_{2k})\langle e_{2k}\rangle \\ &= (\lambda_{2k}^2 - \lambda_{2k-2}^2)^{-1}\, d_k.\end{aligned}$$

One deduces:

$$\eta_{2k-2} = \frac{A^{8k+4} - A^{-8k-4}}{A^4 - A^{-4}}.$$

In a similar way:

$$\eta_{2k-1} = \frac{A^{8k} - A^{-8k}}{A^4 - A^{-4}}.$$

Now, we can calculate ξ_k:

$$\begin{aligned}\tau Q_{2k+1} &= \tau e_{2k+1} - \eta_{2k-1}\tau Q_{2k-1} \\ &= A^{4k+2}e_{2k+2} + A^{-4k-6}e_{2k} - \eta_{2k-1}A^{4k-2}Q_{2k} + \cdots \\ &= A^{4k+2}Q_{2k+2} + A^{4k+2}\eta_{2k}Q_{2k} + A^{-4k-6}Q_{2k} - \eta_{2k-1}A^{4k-2}Q_{k2} + \cdots.\end{aligned}$$

One deduces:

$$\xi_k = 2A^{-4k-6} + A^{12k+10} + A^{12k+2}$$

and

$$\begin{aligned}\tau^2 Q_{2k} &= A^{4k}\tau Q_{2k+1} \\ &= A^{4k}(A^{4k+2}Q_{2k+2} + \xi_k Q_{2k} + \nu_{k-1}Q_{2k-2}), \\ \tau^2 Q_{2k} &= A^{8k+2}Q_{2k+2} + A^{-6}(2 + A^{16k+16} + A^{16k+8})Q_{2k} \\ &\quad + A^{8k-2}(A^{8k+4} - A^{-8k-4})(A^{8k} - A^{-8k})Q_{2k-2}.\end{aligned}$$

Proof of Lemma IV.3

Let: $P_{2k} = t^{-1}Q_{2k}$. Using Lemma II.1, IV.4(b) can be written:

$$\begin{aligned}A^{-6}z^2P_{2k+2} &= A^{8k+2}P_{2k+2} + A^{-6}(2 + A^{16k+16} + A^{16k+8})P_{2k} \\ &\quad + A^{8k-2}(A^{8k+4} - A^{-8k-4})(A^{8k} - A^{-8k})P_{2k-2}.\end{aligned}$$

Let $u_k = P_{2k}(\delta)$.

$$u_{k+1} = A^{-8k-8}(\delta^2 - 2 - A^{16k+16} - A^{16k+8})u_k$$
$$- A^{-4}(A^{8k+4} - A^{-8k-4})(A^{8k} - A^{-8k})u_{k-1}.$$

Lemma IV.3 can now be proven by induction.

(2) *Condition* II.3 *studied*

Suppose A is in a field Λ (for example, the quotient field of an integral domain). We want to study the following equation:

$$\forall x_0 \in K^0_\Lambda(S^1 \times D^2) \quad \langle x_0, v_1\rangle_0 = \langle t^{-1}x_0\rangle\langle v_1\rangle. \tag{E_1}$$

We are looking for a solution $v_1 \in K^1_\Lambda(S^1 \times D^2)$ such that: $\langle v_1\rangle \neq 0$. Let: $w = \langle v_1\rangle^{-1}v_1$. (E_1) becomes:

$$\forall x_0 \in K^0_\Lambda(S^1 \times D^2) \quad \langle x_0, w\rangle_0 = \langle t^{-1}x_0\rangle. \tag{E_2}$$

It is clear that w determines v_1, up to a multiplicative coefficient.

Write: $w = \Sigma\, w_k Q_{2k+1}$. (E_2) becomes:

$$\forall k \quad w_k d_k = u_k.$$

A polynomial solution can exist only if u_k is zero for k big. Thus q must be a root of unity, the order of which is again denoted by r. The smallest integer k for which d_k is zero is the smallest for which:

$$q^{4k} = 1 \quad \text{or} \quad q^{4k+2} = 1.$$

The order of q^2 is $2k$ or $2k+1$.

Case 1: r is odd, so $r = 2k+1$.

(E_2) has a unique solution v of minimal degree: $\deg(v) = 2k-1$. The other solutions are: $v + x$ with $x \in N_k = \operatorname{Span}\{Q_{2l+1}, l \geq k\}$.

Case 2: r is congruent to 2 modulo 4.

The order of q^2 is: $r/2 = 2k+1$, but u_k is not zero: $q^{2k+1} = -1$. (E_2) has no solution.

Case 3: r is congruent to 0 modulo 4.

The order of q^2 is: $r/2 = 2k$, $q^{2k} = -1$, and for $l \geq k$, $u_l = 0$. The conclusion is the same as in case 1.

Conclusion

The discussion above shows that an invariant can be defined, using Proposition II.3, only if A is a root of unity whose order is not congruent to 8, modulo 16. Furthermore, the solution ω_1, given in theorem II.5, has minimal degree for odd r; for $r = 4k$, a solution of minimal degree is $\tilde{\omega}_1$.

(3) *Proof of Theorem* IV.1

Recall the hypothesis: A is a root of unity in an integral domain Λ, and (w_0, w_1) is a solution of condition II.3. The discussion above shows that the order ρ of A is not congruent to eight modulo 16, or equivalently, the order r of $q = A^4$ is not congruent to 2 modulo 4. In the following we suppose $r > 1$; the case $r = 1$ is left to the reader.

Let us work with the quotient field $Q(\Lambda)$. First of all, we reduce the problem, using the quotient space

$$V_A = K(S^1 \times D^2) \otimes Q(\Lambda)/(e_{r-1}).$$

LEMMA (IV.5). *The meta-bracket is well defined on* V_A.

Proof. Suppose x is in the ideal (e_{r-1}), we want to show that:

$$\langle \ldots, x, \ldots \rangle_L = 0.$$

Let L_i be the component where x is satellized. Note that changing any crossing on the diagram, adding around it an unknotted circle, weighted by ± 1, and satellized with ω_1, does not change the nullity of the meta-bracket. Hence we can suppose that L_i is unknotted.

Then we can write: $\langle \ldots, x, \ldots \rangle_L = \langle t^k(x), y \rangle_0 = 0$.

(a) *Case* $r = 2k + 1$. The diagonalization of §IV.1 shows that the bilinear form:

$\langle\, , \rangle_0 \quad V_A^0 \otimes V_A^1 \to Q(\Lambda)$ is non-singular.

Observe that $(Q_{2j})_{0 \leq j \leq k-1}$ and $(Q_{2j+1})_{0 \leq j \leq k-1}$ are bases of V_A^0 and V_A^1, respectively, and that $(e_{r-1}) = (e_{2k}) = \mathrm{Span}\,\{Q_l, l \geq 2k\} = (Q_{2k})$. Using the unique solution of (E_2), one can see that in V_A^1

$$\langle tw_1 \rangle \omega_1 = \langle t\omega_1 \rangle w_1.$$

We can write $w_1 = \lambda\omega_1$, and also $w_0 = \lambda\omega_0$. λ is the constant of w_0 (reduced in V_A^0), so is in Λ.

$\langle tw_1 \rangle = \lambda \langle t\omega_1 \rangle$ is invertible, so λ is invertible.

(b) *Case* $r = 4k$. Now the bilinear form: $\langle\, , \rangle_0\ V_A^0 \otimes V_A^1 \to Q(\Lambda)$, is singular.

Let: $W_A^1 = V_A^1/N_-^1$, and $W_A^0 = V_A^0/N_+^0$.

We can deduce from III.5 that the bilinear form $\langle\, , \rangle_0$ is defined on $W_A^0 \otimes W_A^1$; the diagonalisation of IV.1) shows that it is non singular. In W_A^1 and V_A^0/N_-^0, we have respectively: $w_1 = \lambda\omega_1 = \lambda\tilde{\omega}_1$, and $w_0 = \lambda\omega_0 = \lambda\tilde{\omega}_0$. λ is the constant of w_0, so is in Λ.

$\langle tw_1 \rangle = \lambda \langle t\tilde{\omega}_1 \rangle$ is invertible, so λ is invertible. One can check that $\langle t\tilde{\omega}_1 \rangle$ lives in Λ, more precisely in the homomorphic image of $\mathbb{Z}[A, A^{-1}]$.

The proof is achieved as in the odd case using the following lemma. This lemma justifies the use of the reduced elements $\tilde{\omega}_0$ and $\tilde{\omega}_1$.

LEMMA (IV.6). *Suppose $(L, K) = (L_i, p_i, c_i)_{1 \leq i \leq n}$ is a link with characteristic sublink. Every expression $\langle x_1, \ldots, x_n \rangle_L$, in which each x_i has the parity of c_i, is zero if for some j, x_j is in N_-^0, or in N_-^1.*

Proof. As already seen, we can suppose that L_j is unknotted.

(a) Suppose L_j is in K: $lk(L_j, K - L_j) \equiv 0 \pmod 2$. The expression can be written: $\langle y, t^\alpha x_j \rangle_0$ with y even. $t^\alpha x_j$ is in N_-^1, so the scalar product is zero.

(b) Suppose L_j is not in K: $lk(L_j, K) \equiv p_j \pmod 2$. The expression can be written $\langle t^{p_j} x_j, y \rangle_0$. The parity of y is given by p_j; the scalar product is zero again.

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