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Minimal isometric immersions of spherical space forms in spheres

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Introduction

A number of authors [C], [DW1], [DW2], [L], [T] have studied minimal isometric immersions of Riemannian manifolds into round spheres, and in particular of round spheres into round spheres. As was observed by T. Takahashi [T], if $\Phi : M \to S^N(r) \subset \mathbb{R}^{N+1}$ is such a minimal immersion, then the components of Φ must all be eigenfunctions of the Laplace operator on M corresponding to the same eigenvalue. Conversely, if Φ is an isometric immersion such that all the components are eigenfunctions of the Laplace operator for the same eigenvalue, then Φ is a minimal isometric immersion into a round sphere. Takahashi also observed that if M is an isotropy-irreducible Riemannian homogeneous space, i.e., if the isotropy group of a point acts irreducibly on the tangent space, then an orthonormal basis of each eigenspace automatically gives rise to a minimal isometric immersion into a round sphere. These are called the standard minimal immersions.

In particular, if $M = S^n(1)$ one obtains a sequence of such standard minimal isometric immersions, one for each nonzero eigenvalue. For the first such eigenvalue one obtains the standard embedding into $\mathbb{R}^{n+1}$, and for the second eigenvalue an immersion into $S^{n(n+3)/2-1}(\sqrt{n/(2(n+1))})$, which gives rise to the Veronese embedding of $\mathbb{R}P^n$. For odd-numbered eigenvalues the images of the standard minimal immersions are all embedded spheres and for even-numbered eigenvalues the images are all embedded real projective spaces. E. Calabi [C] showed that *every* minimal isometric immersion of the two-dimensional sphere into $S^N(r)$ is congruent to one of these standard eigenspace immersions. On the other hand, M. Do Carmo and N. Wallach [DW2] showed that in higher dimensions there are in general many minimal isometric immersions of $S^n(1)$ into $S^N(r)$, and that they are parametrized by a compact convex body in a finite-dimensional vector space.

P. Li [L] generalized this result to arbitrary isotropy-irreducible homogeneous spaces and also claimed that the image of a minimal isometric immersion of an isotropy-irreducible homogeneous space is still an isotropy-irreducible homogeneous space. He went on to apply this theorem to the case where M is also a sphere, and ultimately concluded that the image of a minimal isometric immersion of a sphere into a sphere must be either a sphere or a real projective space. This would of course imply that there exists no minimal isometric immersion of a lens space or any other more complicated spherical space form into a sphere.

That this is indeed not correct was first observed by K. Mashimo [Ma1], who gave an example of a minimal isometric immersion of $S^3(1)$ into $S^6(\frac{1}{4})$, whose image is at least a 6-fold subcover of S^3 (but he did not identify the image completely). Later, in [WZ], M. Wang and the second author showed that certain quotients of S^3 (by the so-called polyhedral groups) are in fact isotropy irreducible, and so by the above-mentioned theorem of Takahashi, the polyhedral manifolds $S^3/\mathbf{T}^*$, $S^3/\mathbf{O}^*$ and $S^3/\mathbf{I}^*$ admit minimal isometric immersions into spheres. Also, the first author obtained some explicit minimal isometric embeddings of certain three-dimensional lens spaces. This then raises the question of just which spherical space forms *do* admit minimal isometric imersions or embeddings into spheres. The purpose of the present paper is to give a partial answer to this question. We will show

THEOREM A. *Every homogeneous spherical space form admits a minimal isometric embedding into a standard sphere* (*of sufficiently high dimension and appropriate radius*).

Spherical space forms, i.e. compact manifolds of constant curvature $+1$, have been completely classified [W]. Only few of them are homogeneous, see [W], Theorem 2.7.1, for a description. It seems likely that most if not all spherical space forms admit a minimal isometric immersion into a sphere.

The interior points of the compact convex body parametrizing minimal isometric immersions of spheres into spheres correspond to immersions which use a full basis of the eigenspace corresponding to a given eigenvalue as the coordinates of the immersion. In [WZ] it was observed that these immersions are $SO(n+1)$-equivariant immersions into $\mathbb{R}^{N+1}$ (although they are not equivariant into $S^N(r)$), and hence their images must be embedded spheres or real projective spaces. The minimal immersions in the above Theorem must therefore correspond to boundary points in the convex body. They are still equivariant immersions, but only with respect to a proper subgroup $G \subset SO(n+1)$ that acts transitively on S^n. Their images are therefore G-homogeneous embedded submanifolds. We doubt that there are any minimal isometric immersions whose image is not embedded.

Such equivariant immersions, in the case of $G = SU(2)$ acting transitively on $S^3(1)$, are examined in some detail by K. Mashimo [Ma1], [Ma2], but he does not attempt to identify their images. In [P] F. J. Pedit constructs $U(n)$-equivariant isometric embeddings of $(2n-1)$-dimensional lens spaces into spheres, but they are not minimal.

One should also mention a theorem by Hsiang and Lawson [HL] which states that every homogeneous space G/H admits a minimal isometric immersion (not necessarily an embedding!) into a sphere of sufficiently high dimension, with respect to some G-invariant metric. But in this result, the metric cannot be chosen *apriori.* In particular, for a homogeneous space form, there are in general many G-invariant metrics.

Another question that is interesting in this context was asked by DoCarmo and Wallach [DW2], Remark 1.6: For a given n, what is the smallest dimension N for which there exist minimal isometric immersions of $S^n(1)$ into $S^N(r)$ which are not totally geodesic? In this question one can also specify r, i.e. fix the eigenvalue one wants to consider. A lower bound was given by J. D. Moore [Mr] who showed that no such immersions exist if $N \leq 2n-1$. In [DW2] they guessed the probable answer to be $N = n(n+3)/2 - 1$, which is achieved by the Veronese embedding. That this is false, at least for $n = 3$, was first observed by N. Ejiri [E] who showed that there exists a minimal isometric immersion of $S^3(1)$ into $S^6(\frac{1}{4})$ which is not totally geodesic. He also showed that the immersion is totally real with respect to the natural almost-complex structure on S^6. Notice though that his construction is not explicit, since it uses the fundamental theorem for isometric immersions to prove existence. In [Ma1] Mashimo constructed this immersion more explicity as an $SU(2)$-equivariant immersion. In [Ma2] he shows that it is also an orbit of a subgroup of G_2 acting on S^6 and proves that every totally real immersion of $S^3(1)$ into $S^6(\frac{1}{4})$ is congruent to this example. In [DVV] it was observed that the immersion is a 24-fold cover onto its image. In our paper we will be able to identify the image as the tetrahedral manifold $S^3/\mathbf{T}^*$. We can also easily describe it explicitly as follows. We start with an isometric immersion of $S^3(1)$ obtained by sending $(a, b) \in S^3$, $|a|^2 + |b|^2 = 1$, into:

$$\Big(\frac{1}{4}\sqrt{6}\,(\bar{a}\bar{b}^5 - \bar{a}^5\bar{b}),\ \frac{1}{4}\bar{b}^4(5|a|^2 - |b|^2) + \frac{1}{4}\bar{a}^4(5|b|^2 - |a|^2),$$
$$\frac{1}{4}\sqrt{10}\,[a\bar{b}^3(2|a|^2 - |b|^2) + \bar{a}^3 b(|a|^2 - 2|b|^2)],\ \frac{1}{2}\sqrt{15}\,(|a|^2 - |b|^2)\,\mathrm{Im}\,(a^2\bar{b}^2)\Big).$$

One easily shows that this isometrically immerses $S^3(1)$ into $S^6(\frac{1}{4}) \subset \mathbb{C}^3 \oplus \mathbb{R} = \mathbb{R}^7$ and hence is a minmal isometric immersion. This map is clearly invariant under $\alpha(a, b) = (ia, -ib)$, $\beta(a, b) = (-b, a)$, and $\gamma(a, b) = (\frac{1}{2}(1+i)(a-b), \frac{1}{2}(1-i)(a+b))$. α, β, and γ generate a group of order 24 isomorphic to the binary tetrahedral group $\mathbf{T}^*$ and we will see that the immersion defines an embedding of $S^3/\mathbf{T}^*$ into $S^6(\frac{1}{4})$. We will also prove the following uniqueness property of this immersion:

THEOREM B. *Every $SU(2)$-equivariant minimal isometric immersion of $S^3(1)$ into $S^6(r)$ which is not totally geodesic, is congruent to the above immersion of $S^3(1)$ into $S^6(\frac{1}{4})$, whose image is an embedded $S^3/\mathbf{T}^*$.*

We suspect that this result may be true without the assumption of equivariance. Notice also by Moore's theorem, six is the smallest ambient dimension for which $S^3(1)$ admits a non-totally-geodesic minimal isometric immersion.

In §1 we give some geometric preliminaries, in §2 we prove Theorem A in the three-dimensional case and in §4, §5 in the higher-dimensional cases. In §3 we discuss the moduli space of $SU(2)$-equivariant minimal isometric immersions of $S^3(1)$ and prove Theorem B.

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1. Geometric preliminaries

Let M be an n-dimensional compact Riemannian manifold, and Δ be the Laplacian on $L^2(M)$. If $\phi : M \to \mathbb{R}^N$ is an *isometric* immersion of M into Euclidean space, then the mean curvature vector H of the immersion satisfies

$$\Delta\phi = nH.$$

If furthermore, the coordinate functions of the immersion are all eigenfunctions of Δ corresponding to the same eigenvalue λ, then we have $H = n\phi/\lambda$. Since $\langle H, d\phi\rangle = 0$, this implies that $\langle \phi, d\phi\rangle = 0$, and hence $|\phi|^2$ is constant. Thus ϕ is actually an immersion of M into the sphere S^{N-1} whose radius must be $\sqrt{n/\lambda}$ because of the value of H. Furthermore the immersion is a *minimal* immersion into the sphere, since the mean curvature vector is orthogonal to the sphere. Reversing the reasoning shows the converse: if $\phi : M^n \to S^{N-1}(r)$ is a minimal isometric immersion, then $\Delta\phi = (n/r^2)\phi$. These results were obtained by Takahashi [T] (see also [DW1]).

In order to minimally isometrically immerse a manifold M into a sphere, we must therefore find eigenvalues of the Laplacian of M of sufficiently high multiplicity to provide the coordinate functions of the immersions.

Another result of Takahashi [T] is that certain homogeneous Riemannian manifolds $M = G/H$ do admit such immersions, namely those for which the isotropy group H acts irreducibly on the tangent space. To see this, we consider the eigenspace E_λ to a fixed eigenvalue $\lambda \neq 0$. On E_λ we have the inner product induced by the one on $L^2(M)$, and the group G acts on E_λ by isometries. If we let $\{\phi_1, \ldots, \phi_N\}$ be an orthonormal basis of E_λ, then $\Sigma\, d\phi_i^2$ must be a multiple of the metric on M since both are invariant under G and hence at every point they are invariant under the irreducible action of H. Therefore, after multiplying the metric on M by a constant, $\phi = (\phi_1, \ldots, \phi_N) : M \to \mathbb{R}^N$ is an isometric immersion, which by the above comments give rise to a minimal isometric immersion into a sphere. This immersion is called the *standard minimal immersion of degree d* if λ is the dth nonzero eigenvalue. Notice that a different choice of orthonormal basis for E_λ gives rise to a congruent immersion.

An obvious example of such a homogeneous Riemannian manifold is the n-dimensional sphere, realized as the homogeneous space $SO(n+1)/SO(n)$. The eigenfunctions of S^n are simply the restrictions of harmonic homogeneous polynomials on $\mathbb{R}^{n+1}$ to $S^n(1)$. All the harmonic homogeneous polynomials of degree d restrict to eigenfunctions on S^n with the same eigenvalue $\lambda_d = d(d+n-1)$ and the dimension of this eigenspace is equal to $N_d = (2d+n-1)(d+n-2)!/(d!(n-1)!)$. For odd d, the standard minimal isometric immersion is a minimal isometric *embedding* of S^n into $S^{N_d-1}(\sqrt{n/\lambda_d})$. For even d, all the components of the immersion are invariant under the antipodal map, and we get a minimal isometric embedding of $\mathbb{R}P^n$ into $S^{N_d-1}(\sqrt{n/\lambda_d})$.

In [DW2] the space of all minimal isometric immersions of $S^n(1)$ into $S^N(r)$ was examined in some detail, and it was shown that for $n > 2$ there are many minimal isometric immersions other than the ones described above. If we fix $r = \sqrt{n/\lambda_d}$, or equivalently if we only consider harmonic homogeneous polynomials of degree d, then these minimal isometric immersions (up a rotation of the ambient space) are parametrized by a convex body in a finite-dimensional vector space, which we will now describe.

Let $\phi_0 : S^n(1) \to S^{N_d-1}(\sqrt{n/\lambda_d})$ be the standard minimal isometric imersion of degree d. Then any other isometric immersion ϕ of degree d is given by $A \circ \phi_0$ where A is an $N_d \times N_d$ matrix. Since we can write $A = R \circ P$ where R is orthogonal and P symmetric and positive semidefinite, $A \circ \phi_0$ is congruent to $P \circ \phi_0$. Moreover, one easily checks that $P \circ \phi_0$ is an isometric immersion if and only if $P^2 - Id$ is orthogonal to $\mathrm{Sym}^2\,((\phi_0)_*(TS^n)) \subset \mathrm{Sym}^2\,\mathbb{R}^{N_d}$. If we let W_d be the vector space of all symmetric matrices with this property and $B_d = \{P \in W_d \mid P + Id \geq 0\}$, then $P \circ \phi_0$ is an isometric immersion precisely when $P^2 - Id \in B_d$. One easily shows that $P \in W_d$ implies $\operatorname{tr} P = 0$ and hence B_d is a compact convex body which parametrizes all congruence classes of minimal isometric immersions of degree d. An explicit parametrization is given by $P \in B_d \mapsto \sqrt{P + Id} \circ \phi_0$. In [DW2] it is shown that for $n = 2$ and any d and for $d = 2, 3$ and any n, the space B_d is a point, i.e. any such minimal isometric immersion is congruent to the standard one ϕ_0. For any other value of n and d it is shown that dim $B_d \gtrsim 18$ and that dim B_d grows very quickly with n or d. It seems to be a very difficult problem to determine the dimension of B_d exactly. In [Mu] Y. Muto showed that dim $B_d = 18$ if $n = 3$ and $d = 4$.

From this description it follows immediately that the interior points of the convex body B_d correspond to isometric immersions which use a full basis of E_λ as their components. For these immersions it was observed in [WZ] that they are $SO(n+1)$-equivariant immersions into $\mathbb{R}^{N_d}$ and hence are embeddings of S^n for d odd and of $\mathbb{R}P^n$ for d even. On the other hand, it seems that immersions using only a subspace of E_λ, which correspond to boundary points of the convex body, have not been systematically studied before in the literature. These boundary-type immersions produce the minimal isometric embeddings in Theorem A.

There is also a "gauge group" acting on B_d. If $g \in O(n+1)$ and if $P \circ \phi_0$ is an isometric immersion, then $P \circ \phi_0 \circ g$ is another one. The equivariance properties of ϕ_0 imply that $P \circ \phi_0 \circ g = P \circ \rho(g) \circ \phi_0$ where $\rho(g)$ is the orthogonal matrix of the isometry g acting on the eigenspace E_{λ_d} with respect to the orthonormal basis defining ϕ_0. Since $P \circ \rho(g) \circ \phi_0$ is congruent to $\rho(g)^{-1} \circ P \circ \rho(g) \circ \phi_0$, we have that $O(n+1)$ induces an action on B_d given by $T \in B_d \mapsto \rho(g)^{-1} \circ T \circ \rho(g)$. It follows that $g \in O(n+1)$ lies in the isotropy group of this action at $T \in B_d$ if and only if the corresponding immersion $\sqrt{T + Id} \circ \phi_0$ is equivariant with respect to g. Since ρ induces an absolutely irreducible representation of $SO(n+1)$ on E_{λ_d}, the only matrix T which commutes with every $\rho(g)$ are the multiples of the identity, but $a\,\mathrm{Id} \in B_d$ if and only if $a = 0$. Hence the origin is the only fixed point of the $O(n+1)$ action, corresponding to the fact that ϕ_0 is the only $O(n+1)$-equivariant immersion.

If we fix a subgroup $G \subset SO(n+1)$, then the set of all G-equivariant minimal isometric immersions corresponds to the set of all $T \in B_d$ which commute with every $g \in G$. This set is a convex sub-body of B_d. Of course, G is contained in the isotropy group of every point of this sub-body. But notice that if $P \circ \phi_0$ is G-equivariant and if $g \in SO(n+1) \backslash G$, then $P \circ \phi_0 \circ g$ is in general no longer G-equivariant unless g is in the centralizer of G. On the other hand, $P \circ \phi_0 \circ g$ is equivariant with respect to $gGg^{-1} \subset SO(n+1)$.

Our construction of minimal isometric embeddings for space forms will use G-equivariant immersions, where G is a subgroup of $SO(n+1)$ that still acts transitively on S^n. Given such a group G, we have that $S^n = G/H$ and we let $V^H \subset E_{\lambda_d}$ be the subspace on which $H \subset G$ acts trivially. For every $v \in V^H$ we obtain a map $\Phi_v : G/H \to E_{\lambda_d}$ given by $\Phi(gH) = gv$. The image of this map is obviously contained in the sphere of radius $\|v\|$, and if we pull back the metric on E_{λ_d}, we get a left-invariant symmetric two-tensor on S^n which may or may not agree with the constant-curvature metric. Our goal is therefore to find a vector v such that this pull-back metric has curvature 1. Then Φ_v will be an isometric immersion, which by the previous remarks, must be a minimal isometric immersion of $S^n(1)$ into $S^{N-1}(\sqrt{n/\lambda_d})$. Here N is the smallest integer such that $\Phi_v(M)$ lies in an N-dimensional subspace $E \subset E_{\lambda_d}$.

The image $\Phi_v(M) \subset E$ must of course be an embedded submanifold, namely the orbit of v under the action of G on E. Hence $\Phi_v(M) = G/H^*$, where H^* is the isotropy group of v. Of course $H \subset H^*$ and H^*/H is finite. Therefore, G/H^* is a subcover of S^n and Φ_v gives rise to a minimal isometric embedding of G/H^* into $S^{N-1}(\sqrt{n/\lambda_d})$. Thus, to find an isometric embedding of a given space form G/H^* we need to find a $v \in V^H$ such that H^* is the full isotropy group of v. We call this process of manufacturing an isometric minimal embedding the "equivariant construction" since the embedding is indeed G-equivariant.

We can usually guarantee that $N < N_d$ by the following remark. Although $SO(n+1)$ acts irreducibly on E_{λ_d}, the subgroup $G \subset SO(n+1)$ usually does not. Indeed, if $v \in V^H$ is a vector which lies in a subspace invariant under G, then the whole orbit lies in this subspace. Hence, to produce equivariant immersions of smallest codimension, we choose v in a G-invariant subspace of smallest dimension. Equivalently we could also consider a class-one representation of G with respect to H, i.e. a representation of G which has a fixed vector when restricted to H, and then take the orbit of G through such a fixed vector.

Before we proceed, we will need an explicit expression of the metric on E_{λ_d}, the space of homogeneous harmonic polynomials on $\mathbb{R}^{n+1}$ of degree d. We first remark that the action of $A \in SO(n+1)$ on $p \in \mathbb{R}[x_1, \ldots, x_{n+1}]$ is given by $A \cdot p(x) = p(A^{-1}x) = p(A^t x)$ where $x \in \mathbb{R}^{n+1}$. Since this action is irreducible, the metric is uniquely determined up to a multiple. Now we define

$$\langle p, q \rangle = p\left(\frac{\partial}{\partial x_i}\right) q$$

which must be a real number since both p and q have the same degree. One easily verifies that this inner product is invariant under the action of $SO(n+1)$ (see [V] for details) and hence is our desired inner product.

When $n+1$ is even, we can also express polynomials in E_{λ_d} using complex notation as $p(z_i, \bar{z}_i)$ and, to within a factor 2^d, the above inner product is the same as the one given by

$$\langle p(z_i, \bar{z}_i), q(z_i, \bar{z}_i) \rangle = \mathrm{Re}\left\{ p\left(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial z_i}\right) \bar{q} \right\}.$$

This last inner product is the one we will use. Note that this inner product is easy to work with: for monomials, we have

$$\left\langle \prod_{i=1}^{n} z_i^{k_i}\bar{z}_i^{l_i}, \prod_{i=1}^{n} z_i^{m_i}\bar{z}_i^{n_i} \right\rangle = 0$$

unless $k_i = m_i$ and $l_i = n_i$ for all i, in which case we have

$$\left\langle \prod_{i=1}^{n} z_i^{k_i}\bar{z}_i^{l_i}, \prod_{i=1}^{n} z_i^{k_i}\bar{z}_i^{l_i} \right\rangle = \prod_{i=1}^{n} k_i!\,l_i!\,.$$

We will refer to this as the "unitary metric" on the space of homogeneous polynomials.

Finally, we list the homogeneous space forms. Each homogeneous spherical space form must belong to one of the following classes:

(i) $M = S^3/\Gamma$, where Γ is a finite subgroup of $S^3 = SU(2) = Sp(1)$;
(ii) $M = S^{2n-1}/\mathbf{C}_d$, where $\mathbf{C}_d$ is generated by $e^{2\pi i/d}$ and acts on $\mathbb{R}^{2n} = \mathbb{C}^n$ by multiplication on each complex coordinate;
(iii) $M = S^{4n-1}/\Gamma$, where Γ is any finite subgroup of $Sp(1)$ acting on $\mathbb{R}^{4n} = \mathbb{H}^n$ by multiplication on each quaternionic coordinate from the left.

Minimal isometric embeddings for space forms in the first class are produced in §2, the second one in §4 and the third one in §5.

2. The three-dimensional case

The case of quotients of the three-sphere S^3 is separated from the rest because S^3 is itself a group, rather than simply a homogeneous space. We may consider S^3 either as the group of unit imaginary quaternions $Sp(1)$, or as the special unitary group $SU(2)$. The homogeneous three-dimensional spherical space forms can all be written as S^3/Γ where Γ is an arbitrary finite subgroup of S^3. The homogeneous lens spaces can also be written as quotients of $U(2)$, but the minimal isometric embeddings one obtains in this fashion (see §4) have higher codimension.

We start by listing the possible groups Γ. As is well-known [W], [Mo], the following is an exhaustive list of the finite proper subgroups of $Sp(1)$:

(i) the cyclic groups $\mathbf{C}_d = \{e^{2\pi k i/d}\colon k = 0, 1, \ldots, d-1\}$ for $d \geq 2$;
(ii) the binary dihedral groups $\mathbf{D}_d^* = \mathbf{C}_{2d} \cup \mathbf{C}_{2d}j$ where j is the usual generator of the quaternions over $\mathbb{C}$, for $d \geq 1$ (note that $d = 1$ gives a cyclic group

isomorphic to $\mathbf{C}_4$, and $d = 2$ gives what is usually called the "quaternionic group", $\{\pm 1, \pm i, \pm j, \pm k\}$);

(iii) the binary tetrahedral group $\mathbf{T}^* = \mathbf{D}_2^* \cup \{\frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$ of order 24 (this is the double cover of the group of symmetries of the tetrahedron);

(iv) the binary octahedral group $\mathbf{O}^* = \mathbf{T}^* \cup e^{\pi i/4}\mathbf{T}^*$ of order 48 (this is the double cover of the group of symmetries of the octahedron);

(v) the binary icosahedral group $\mathbf{I}^* = \mathbf{T}^* \cup x\mathbf{T}^* \cup x^2\mathbf{T}^* \cup x^3\mathbf{T}^* \cup x^4\mathbf{T}^*$, where $x = a + i + (1/a)j$ and a is the golden ratio $(1+\sqrt{5})/2$. This group has order 120 and is the double cover of the group of symmetries of the icosahedron.

Furthermore, any pair of finite subgroups of $Sp(1)$ which are isomorphic are in fact conjugate to each other in $Sp(1)$.

Corresponding to each of these finite subgroups of $Sp(1)$, we get a homogeneous three-dimensional spherical space-form:

(i) the lens spaces $L(d; 1) = Sp(1)/\mathbf{C}_d$ for $d \geq 2$ (note that $L(2; 1)$ is the real projective space $\mathbb{R}P^3$);

(ii) the "prism manifolds" $Sp(1)/\mathbf{D}_d^*$ for $d \geq 2$;

(iii) the "tetrahedral manifold" $Sp(1)/\mathbf{T}^*$;

(iv) the "octahedral manifold" $Sp(1)/\mathbf{O}^*$;

(v) the "icosahedral manifold" $Sp(1)/\mathbf{I}^*$.

For later purposes, we list here all possible inclusions among these groups:

(i) $\mathbf{C}_d \subset \mathbf{C}_{nd}$;

(ii) $\mathbf{C}_d \subset \mathbf{C}_{2d} \subset \mathbf{C}_{2nd} \subset \mathbf{D}_{nd}^*$; $\mathbf{D}_d^* \subset \mathbf{D}_{nd}^*$;

(iii) $\mathbf{C}_2 \subset \mathbf{C}_4 \subset \mathbf{T}^*$; $\mathbf{C}_3 \subset \mathbf{C}_6 \subset \mathbf{T}^*$; $\mathbf{D}_2^* \subset \mathbf{T}^*$;

(iv) $\mathbf{C}_2 \subset \mathbf{C}_4 \subset \mathbf{C}_8 \subset \mathbf{O}^*$; $\mathbf{C}_3 \subset \mathbf{C}_6 \subset \mathbf{O}^*$; $\mathbf{D}_2^* \subset \mathbf{D}_4^* \subset \mathbf{O}^*$; $\mathbf{D}_3^* \subset \mathbf{O}^*$; $\mathbf{T}^* \subset \mathbf{O}^*$;

(v) $\mathbf{C}_2 \subset \mathbf{C}_4 \subset \mathbf{I}^*$; $\mathbf{C}_3 \subset \mathbf{C}_6 \subset \mathbf{I}^*$; $\mathbf{C}_5 \subset \mathbf{C}_{10} \subset \mathbf{I}^*$; $\mathbf{D}_2^* \subset \mathbf{I}^*$; $\mathbf{D}_3^* \subset \mathbf{I}^*$; $\mathbf{D}_5^* \subset \mathbf{I}^*$.

To verify these inclusions for the subgroups of the binary polyhedral groups, one first determines the subgroups of the polyhedral groups $\mathbf{T}$, $\mathbf{O}$, $\mathbf{I}$ in $SO(3)$ by observing that $\mathbf{T}$ and $\mathbf{I}$ are isomorphic to the alternating groups A_4 and A_5 and that $\mathbf{O}$ is isomorphic to the symmetric group S_4. Under the projection from $Sp(1)$ to $SO(3)$ the inverse image of a polyhedral group is the corresponding binary polyhedral group, the inverse image of a dihedral group D_d is a binary dihedral group $\mathbf{D}_d^*$, and the inverse image of a cyclic group $\mathbf{C}_d$ is the cyclic group $\mathbf{C}_{2d}$. In addition, for a cyclic group of odd order in $SO(3)$, there exists a cyclic group of the same order in $Sp(1)$, for which the projection gives rise to an isomorphism. All this follows from the fact that all subgroups of $Sp(1)$ contain the center $\{\pm 1\}$ of $Sp(1)$, except for the cyclic subgroups of odd order.

To see the subgroups of $Sp(1)$ as subgroups of $SU(2)$, we simply identify the quaternion $a + bj$ with the matrix

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}.$$

If $a + bj \in Sp(1)$, i.e. $|a|^2 + |b|^2 = 1$, then the corresponding matrix is in $SU(2)$. Thus the action of the quaternion $a + bj$ on the polynomial $p(z, w, \bar{z}, \bar{w})$ is given by

$$((a + bj) \cdot p)(z, w, \bar{z}, \bar{w}) = p(\bar{a}z - bw, \bar{b}z + aw, a\bar{z} - \bar{b}\bar{w}, b\bar{z} + \overline{aw}).$$

Instead of looking at the action of $Sp(1)$ on the full space of homogeneous harmonic polynomials in four real variables, we only consider the following subspace. Let W_d be the space of homogeneous complex polynomials of degree d in two complex variables z, w. If we regard W_d as a real vector space by taking real and imaginary parts, we obtain a $2(d + 1)$-dimensional subspace of the $(d + 1)^2$-dimensional space of homogeneous harmonic polynomials in four real variables. The natural action of $Sp(1)$ on z and w induces an action of $Sp(1)$ on W_d which is the same as the action of $Sp(1)$ on E_{λ_d} restricted to W_d. Hence we only need to find polynomials $p(z, w)$ in W_d such that Γ is the stabilizer group of p and such that the orbit $Sp(1) \cdot p$ has constant curvature 1.

We can reduce the codimension of the embedding in some cases, by observing that, if the degree is even, say $2d$, then the irreducible representation of $Sp(1)$ on W_{2d} is the complexification of a real representation of dimension $2d + 1$. The conjugation which gives rise to this real subspace is given by the complex antilinear map which sends $z^k w^{d-k}$ to $(-1)^k z^{d-k} w^k$. Hence the real subspace $R_{2d} \subset W_{2d}$ has as a basis

$$z^{2d} + w^{2d},\ i(z^{2d} - w^{2d}),\ z^{2d-1}w - zw^{2d-1},\ i(z^{2d-1}w + zw^{2d-1}), \ldots, i^d z^d w^d$$

and $Sp(1)$ leaves this subspace R_{2d} invariant. Hence if p is a polynomial in R_{2d}, then the orbit $Sp(1) \cdot p$ also lies in R_{2d}.

It is a fact (see [Mi] for a résumé and [K] for a beautiful classical exposition) that the subalgebra of $\mathbb{C}[z, w]$ left invariant by the action of any finite subgroup of $Sp(1)$ is generated by three homogeneous polynomials which satisfy one algebraic relation. We list these polynomials and relations for each of the above groups:

(i) For the cyclic group $\mathbf{C}_d$ the algebra of invariant polynomials is generated by $p = z^d$, $q = w^d$, and $r = zw$, with the obvious relation $pq = r^d$.

(ii) For the binary dihedral group $\mathbf{D}_d^*$, the algebra of invariant polynomials is generated by $P = z^{2d} + w^{2d}$, $Q = z^{2d+1}w - w^{2d+1}z$, and $R = z^2w^2$. The relation is given by $P^2R - Q^2 - 4R^{d+1} = 0$.

(iii) For the binary tetrahedral group $\mathbf{T}^*$, the algebra of invariant polynomials is generated by $\alpha = zw^5 - wz^5$, $\beta = z^8 + 14z^4w^4 + w^8$ and $\gamma = z^{12} - 33z^8w^4 - 33z^4w^8 + w^{12}$. The relation is $108\alpha^4 - \beta^3 + \gamma^2 = 0$.

(iv) For the binary octahedral group $\mathbf{O}^*$, we can express the generators in terms of those of $\mathbf{T}^*$, since $\mathbf{T}^* \subset \mathbf{O}^*$. The generators are β, α^2 and $\alpha\gamma$, and the relation is $(\alpha^2)(\beta)^3 - 108(\alpha^2)^3 - (\alpha\gamma)^2 = 0$.

(v) Finally, for the binary icosahedral group we discover that the realization of $\mathbf{I}^*$ as a subgroup of $Sp(1)$ given above, while easy to describe, is not so convenient for computing the invariant polynomials. For example, the generator of lowest degree has degree 12, and is $22(5 + 8a)\alpha^2 - (11 + 18a)\gamma$, in terms of the generators of $\mathbf{T}^*$ given above. For convenience later, we perform a conjugation in $Sp(1)$ (which places a vertex of the icosahedron on the z-axis in $\mathbb{R}^3$, as opposed to a vertex of the dual dodecahedron), to realize the binary icosahedral group as the following set of quaternions:

$$\{\pm\epsilon^\mu, \pm\epsilon^\mu j : \mu = 0, \ldots, 4\}$$
$$\cup \left\{\frac{\pm\epsilon^\nu}{\sqrt{5}}((\epsilon^4 - \epsilon)\epsilon^\mu + (\epsilon^2 - \epsilon^3)\epsilon^{-\mu}j) : \mu, \nu = 0, \ldots, 4\right\}$$
$$\cup \left\{\frac{\pm\epsilon^\nu j}{\sqrt{5}}((\epsilon^4 - \epsilon)\epsilon^\mu + (\epsilon^2 - \epsilon^3)\epsilon^{-\mu}j) : \mu, \nu = 0, \ldots, 4\right\}$$

where $\epsilon = e^{2\pi i/5}$. For this presentation of $\mathbf{I}^*$, the algebra of invariant polynomials is generated by $A = zw(z^{10} + 11z^5w^5 - w^{10})$, $B = (z^{20} + w^{20}) - 228(z^{15}w^5 - z^5w^{15}) + 494z^{10}w^{10}$ and $C = (z^{30} + w^{30} + 522(z^{25}w^5 - z^5w^{25}) - 10{,}005(z^{20}w^{10} + z^{10}w^{20})$. These are algebraically related by the equation $C^2 - B^3 + 1728A^5 = 0$.

Armed with the generators of the algebras of invariant polynomials for each of the finite subgroups of $Sp(1)$, we are now in position to carry out the "equivariant construction" of minimal isometric embeddings. One should be careful in applying the above description of invariant polynomials since it depends completely on the embedding of the subgroup Γ chosen. If we change the embedding by a conjugacy in $Sp(1)$, then the description of the set of invariant polynomials changes correspondingly. This applies in particular when we claim that a given group Γ is the full isotropy group of a polynomial p: It is not enough that p is simply not on the list of invariant polynomials for a bigger group; rather, we must check that p is not *conjugate* to anything on the bigger group's list.

CASE I: THE CYCLIC GROUPS $\mathbf{C}_d$. The quotients $S^3/\mathbf{C}_d$ are the lens spaces $L(d; 1)$. By the results of the previous section, we need to find a homogeneous polynomial $p(z, w)$, invariant under the action of $\mathbf{C}_d$, so that the metric induced by the "unitary metric" on the orbit of p agrees with the constant-curvature 1 metric at p. The tangent space to $SU(2)$ at the identity is the Lie algebra $\mathfrak{su}(2)$, and an orthonormal basis for $\mathfrak{su}(2)$ in the constant-curvature 1 metric is given by the matrices:

$$Z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

If $\Phi_p : SU(2) \to \mathbb{C}[z, w]$ is the map $\Phi_p(a + bj) = p(\bar{a}z - bw, \bar{b}z + aw)$, then one easily computes that for $p(z, w) = z^a w^b$,

$$\begin{aligned}
\Phi_{p*}(Z) &= \Phi_{p*}(i) = i(b - a)z^a w^b, \\
\Phi_{p*}(U) &= \Phi_{p*}(j) = -az^{a-1}w^{b+1} + bz^{a+1}w^{b-1}, \\
\Phi_{p*}(V) &= \Phi_{p*}(ij) = -i(az^{a-1}w^{b+1} + bz^{a+1}w^{b-1}).
\end{aligned}$$

Any invariant polynomial for $\mathbf{C}_d$ consists of sums and products of z^d, w^d, and zw. One easily checks that none of the polynomials cz^{kd}, cw^{kd}, $c(zw)^k$, $c_1 z^d + c_2 w^d$ give rise to an isometric immersion. If we set $f_{2d}(z, w) = c_1 z^{2d} + c_2 z^d w^d$, then

$$\begin{aligned}
\Phi_{f_{2d}*}(Z) &= -2c_1 diz^{2d}, \\
\Phi_{f_{2d}*}(U) &= -2c_1 dz^{2d-1}w + c_2 d(z^{d+1}w^{d-1} - z^{d-1}w^{d+1}), \\
\Phi_{f_{2d}*}(V) &= -i(2c_1 dz^{2d-1}w + c_2 d(z^{d-1}w^{d+1} + z^{d+1}w^{d-1})).
\end{aligned}$$

Provided $d \geq 3$, these three polynomials are orthogonal with respect to the "unitary metric". If $d \geq 3$ we have

$$\begin{aligned}
\|\Phi_{f_{2d}*}(Z)\|^2 &= 4|c_1|^2 d^2 (2d)!, \\
\|\Phi_{f_{2d}*}(U)\|^2 &= \|\Phi_{f_{2d}*}(V)\|^2 = 4|c_1|^2 d^2 (2d-1)! + 2|c_2|^2 d^2 (d-1)!(d+1)!\,.
\end{aligned}$$

If we set

$$|c_1|^2 = \frac{1}{4d^2(2d)!} \quad \text{and} \quad |c_2|^2 = \frac{2d-1}{4d^2(d!)(d+1)!},$$

then the push-forwards of Z, U and V will be orthonormal, and the $Sp(1)$ orbit of f_{2d} will provide an isometric minimal immersion of the lens space $L(d; 1)$ into the $4d+1$-dimensional sphere of radius $\sqrt{3/(4d(d+1))}$. The polynomial f_{2d} is of course also invariant under $\mathbf{C}_{2d}$ and by equating the coefficients of w^{2d} and zw^{2d-1} in $gf_{2d} = f_{2d}$ for $g \in SU(2)$, a calculation shows that the isotropy group of f_{2d} is in fact equal to $\mathbf{C}_{2d}$. Hence $\Phi_{f_{2d}}$ gives rise to a minimal isometric embedding of $L(2d; 1)$, $d \geq 3$, into $S^{4d+1}(\sqrt{3/(4d(d+1))})$. As we will see shortly, the codimension can actually be improved if $d \geq 4$.

Since we only need the absolute value of c_1 and c_2, it seems that we have a two-parameter family of solutions. But one parameter is due to the ambient congruence of W_{2d} which takes c_i to $e^{i\theta}c_i$. The other parameter is due to the fact that if f_{2d} is a solution, then so is $\begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} f_{2d} = (\bar{a}^{2d}c_1)z^{2d} + c_2 z^d w^d$. Hence the solutions give rise to a one parameter family of orbits of constant curvature one, all of which are congruent to each other. Each of these orbits corresponds to the same three-parameter family of solutions in the moduli space B_{2d}, where all the members of this family are equivalent to each other with respect to the gauge group.

For $d = 2$ not all of the terms in the polynomials $\Phi_{f_{2d}*}(U)$ and $\Phi_{f_{2d}*}(V)$ are orthogonal to each other. In fact, for $L(4; 1)$ we will see in §3 that we cannot define an isometric embedding using degree 4 or (real) degree 6 polynomials. But one can easily find one using degree 8 polynomials. In fact, the polynomial $p_8 = c_1 z^8 + c_2 z^2 w^6$ gives rise to an isometric embedding if and only if $|c_1| = 1/(480\sqrt{21})$ and $|c_2| = \sqrt{7}/(240\sqrt{6})$. A calculation again shows that the isotropy group of p_8 is equal to $\mathbf{C}_4$ and hence we obtain a minimal isometric embedding of $L(4; 1)$ into $S^{17}(\sqrt{\tfrac{3}{80}})$. One can also improve the codimension by using real degree 10 polynomials.

To obtain a minimal isometric embedding of $L(d; 1)$ for d odd, we must use a polynomial of degree $3d$. In particular, if we set

$$k_{3d} = c_1 z^{3d} + c_2 z^{2d} w^d$$

one shows as above that the orbit through k_{3d} has constant curvature 1 if and only if

$$|c_1|^2 = \frac{d+1}{4d^2(3d+2)(3d)!} \quad \text{and} \quad |c_2|^2 = \frac{3d-1}{4d^2(3d+2)(d!)(2d)!}.$$

Furthermore, the stabilizer group of k_{3d} is equal to $\mathbf{C}_d$ and hence we get a minimal isometric embedding of $L(d; 1)$, for odd $d \geq 3$, into $S^{6d+1}(\sqrt{1/(d(3d+2))})$.

CASE II: THE BINARY DIHEDRAL GROUPS $\mathbf{D}_d^*$. The quotients $S^3/\mathbf{D}_d^*$ are usually called "prism manifolds" (see [Mo]). One easily checks that powers of the invariant polynomials P, Q and R do not give rise to isometric immersions, hence we need to take linear combinations. If d is even, then the polynomial $g_{2d}(z, w) = c_1(z^{2d} + w^{2d}) + c_2 z^d w^d$ is invariant under the dihedral group action, and we calculate:

$$\begin{aligned}
\Phi_{g_{2d}*}(Z) &= -2c_1 di(z^{2d} - w^{2d}),\\
\Phi_{g_{2d}*}(U) &= -2c_1 d(z^{2d-1}w - w^{2d-1}z) - c_2 d(z^{d-1}w^{d+1} - z^{d+1}w^{d-1}),\\
\Phi_{g_{2d}*}(V) &= -i(2c_1 d(z^{2d-1}w + w^{2d-1}z) + c_2 d(z^{d-1}w^{d+1} + z^{d+1}w^{d-1})).
\end{aligned}$$

Provided $d \geq 3$, these three polynomials are clearly orthogonal. This is the case even for $d = 2$, but notice that for $d = 2$ not all of the polynomials in the image of U (or V) are orthogonal to each other. Hence if $d \geq 3$ we compute

$$\begin{aligned}
&\|\Phi_{g_{2d}*}(Z)\|^2 = 8|c_1|^2 d^2 (2d)!,\\
&\|\Phi_{g_{2d}*}(U)\|^2 = \|\Phi_{g_{2d}*}(V)\|^2 = 8|c_1|^2 d^2 (2d-1)! + 2|c_2|^2 d^2 (d-1)!(d+1)!.
\end{aligned}$$

If we set

$$|c_1|^2 = \frac{1}{8d^2(2d)!} \quad \text{and} \quad |c_2|^2 = \frac{2d-1}{4d^2(d!)(d+1)!},$$

then the push-forwards of Z, U and V will be orthonormal, and the $Sp(1)$-orbit of g_{2d} will provide an isometric minimal immersion of the prism manifold $S^3/\mathbf{D}_d^*$, d even ≥ 4. If c_1 and c_2 are real, the polynomial g_{2d} also lies in the real subspace $R_{2d} \subset W_{2d}$ (since d is even) and hence $\Phi_{g_{2d}}$ provides a minimal isometric immersion into the $2d$-dimensional sphere. We now need to see whether $\mathbf{D}_d^*$ is the full isotropy group of g_{2d}. If $d = 4$, this is actually not the case, since the polynomial is identical to one of the invariant polynomials of $\mathbf{O}^*$. To see whether there exists an invariance group K for g_{2d} with $\mathbf{D}_d^* \subset K$ in any of the other cases, we use, besides the list of possible inclusions among the finite subgroups of $Sp(1)$, the fact that the orders of $\mathbf{D}_d^*$ must divide the order of K, that K must have an invariant polynomial of degree $2d$, and that the invariant polynomials for K must occur in a degree for which $\mathbf{D}_d^*$ also has an invariant polynomial. These conditions already exclude all but the possibility that $K = \mathbf{D}_{kd}^*$, but this can easily be excluded since the only invariant polynomial for $\mathbf{D}_{kd}^*$ in degree $2d$ is $z^d w^d$ and we already saw that this polynomial does not give rise to an isometric immersion. Hence $\Phi_{g_{2d}}$ provides a minimal isometric embedding of $S^3/\mathbf{D}_d^*$ into $S^{2d}(\sqrt{3/(4d(d+1))})$ for even $d \geq 6$.

To obtain an isometric embedding for $S^3/\mathbf{D}_4^*$, instead of choosing c_1 and c_2 real in the polynomial g_8, we can choose e.g. c_1 real and c_2 imaginary. Indeed, one shows, by equating the coefficients of z^8 and w^8 in $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} g_8 = g_8$, that c_1/c_2 must be real if $ab \neq 0$ for some element of Γ, and that $ab = 0$ for all elements of Γ implies that the invariance group is $\mathbf{D}_4^*$. Hence, if c_1/c_2 is not real, the invariance group of g_8 is equal to $\mathbf{D}_4^*$. Of course, in this case the polynomial no longer lies in a real subspace and hence we obtain an minimal isometric embedding of $S^3/\mathbf{D}_4^*$ into $S^{17}(\sqrt{\frac{3}{80}})$.

If $d = 2$, then the correct formula for the lengths of the images of Z, U and V is given by

$$\|\Phi_{g_4 *}(Z)\|^2 = 32 \cdot 4!|c_1|^2,$$

$$\|\Phi_{g_4 *}(U)\|^2 = 2 \cdot 4!|2c_1 - c_2|^2, \qquad \|\Phi_{g_4 *}(V)\|^2 = 2 \cdot 4!|2c_1 + c_2|^2.$$

They will be orthonormal if $c_1 = 1/(16\sqrt{3})$ and $c_2 = i/8$. By equating coefficients again, one shows that the invariance group of g_4 is equal to $\mathbf{D}_2^*$ and so we get a minimal isometric embedding of $S^3/\mathbf{D}_2^*$ into $S^9(\sqrt{\frac{1}{8}})$.

One can actually improve the codimension of the latter embedding somewhat. The orbit of the polynomial $\psi_8 = c_1(z^8 + w^8) + c_2(z^6w^2 + z^2w^6) + tz^4w^4$ has constant curvature one if we set $c_1 = -1/(512\sqrt{35})$ and $c_2 = \sqrt{7}/(384\sqrt{5})$ and $t = \sqrt{7}/(768\sqrt{5})$. The only possible invariance groups for this polynomial are $\mathbf{D}_2^*$ and $\mathbf{O}^*$ (since $\mathbf{T}^*$ and $\mathbf{D}_4^*$ have the same invariant polynomials in degree 8), but then the orbit of ψ_8 would have to go through the "standard" invariant polynomial for $\mathbf{O}^*$ and one easily shows that this is not the case. Hence one obtains a minimal isometric embedding of $S^3/\mathbf{D}_2^*$ into $S^8(\sqrt{\frac{3}{80}})$.

In the case of d even ≥ 4, we can consider the orbit through g_{2d} for all allowable values of c_1 and c_2 to obtain a two-parameter family of solutions. One parameter is again due to the ambient congruence which takes c_i to $e^{i\theta}c_i$. But changes in the other parameter, namely c_1/c_2, cannot be accounted for by congruences, or the fact that the polynomials lie on the same orbit. In fact, we obtain a one-parameter family of distinct orbits parametrized by c_1/c_2 (note that $|c_1/c_2|$ is fixed). The orbits of the polynomials with c_1/c_2 real lie in a $2d$-dimensional sphere, and those for c_1/c_2 not real lie in a $(4d+1)$-dimensional sphere. Furthermore for $d = 4$, the orbit for c_1/c_2 real is actually an embedded $S_3/\mathbf{O}^*$.

More generally, we can consider the orbit through $c_1z^{2d} + c_2w^{2d} + c_3z^dw^d$. It has constant curvature one if and only if $|c_1|^2 + |c_2|^2 = 1/(4d^2(2d)!)$ and $|c_3|^2 = (2d-1)/(4d^2(d!)(d+1)!)$, and one shows that this gives rise to a two parameter family of non-congruent orbits. For $|c_1| = |c_2|$, we can assume that $c_1 = c_2$ and recover the previous solutions. For $|c_1| \neq |c_2|$ one can show that the orbits are embedded lens spaces in a $(4d+1)$-dimensional sphere.

We can consider the polynomial g_{2d} for d odd ≥ 3. With the above values for c_1 and c_2, g_{2d} still gives rise to an isometric immersion, but the invariance group is no longer $\mathbf{D}_d^*$ (at least not for our chosen embedding of $\mathbf{D}_d^*$ in $Sp(1)$). g_{2d} is clearly invariant under $\mathbf{C}_{2d}$. If $d = 3$, we will see in §3 that the invariance group is equal to $\mathbf{T}^*$ (but with respect to a different embedding of $\mathbf{T}^*$ than the one chosen earlier) since there is only one orbit of constant curvature one in R_6. On the other hand, if d odd ≥ 5, we can exclude a bigger invariance group than $\mathbf{C}_{2d}$ as we did in the case of g_{2d}, for even $d \geq 6$. If we choose c_1 real and c_2 purely imaginary, g_{2d} lies in R_{2d} and so we get a minimal isometric embedding of $L(2d; 1)$, for odd $d \geq 5$, into $S^{2d}(\sqrt{3/(4d(d+1))})$.

If d is odd, then the polynomial $h_{2d+2}(z, w) = c_1(z^{2d+1}w - w^{2d+1}z) + c_2 z^{d+1} w^{d+1}$ is invariant under the dihedral group $\mathbf{D}_d^*$, and one easily shows that the pushforwards of Z, U and V are orthonormal if and only if

$$|c_1|^2 = \frac{1}{8d^2(2d+1)!} \quad \text{and} \quad |c_2|^2 = \frac{(2d+1)(d-2)}{4d^2(d+1)(d+1)!(d+2)!}.$$

Hence the $Sp(1)$ orbit of h_{2d+2} will provide an isometric minimal immersion of the prism manifold $S^3/\mathbf{D}_d^*$, for d odd and ≥ 3. If $d = 5$ and c_1/c_2 is real, then the polynomial h_{12} is actually the same as the one for $\mathbf{I}^*$ and if $d = 3$ there exists an invariant polynomial for $\mathbf{O}^*$ of the same degree as h_8. But if d odd ≥ 7, and if we choose c_1 and c_2 real, then we obtain a minimal isometric embedding of $S^3/\mathbf{D}_d^*$ into $S^{2d+2}(\sqrt{3/(4(d+1)(d+2))})$. For $d = 3$ and $d = 5$ we can again choose c_1 real and c_2 imaginary to obtain a minimal isometric embedding of $S^3/\mathbf{D}_d^*$ into a $(4d+1)$-dimensional sphere.

For §3 it will actually be of interest to look at the case $d = 3$ in more detail. We will show that $\mathbf{D}_3^*$ is the full invariance group of h_8. Indeed, we only need to exclude the possibility that the invariance group of h_8 is $\mathbf{O}^*$. But the only invariant polynomial for $\mathbf{O}^*$ (with respect to the standard embedding) is equal to $q = c(z^8 + w^8 + 14z^4w^4)$ which for an appropriate choice of c gives rise to a constant curvature one orbit. If the invariance group for h_8 were $g\mathbf{O}^*g^{-1}$ for some $g \in SU(2)$, then gh_8 would have invariance group $\mathbf{O}^*$ and hence $gh_8 = q$. But, by equating coefficients of z^7w, z^6w^2, and z^5w^3 in $gh_8 = q$, one can show that there exists no such g. Hence we obtain three distinct curvature one orbits among the degree 8 polynomials in R_8, an embedded $S^3/\mathbf{D}_3^*$, an embedded $S^3/\mathbf{O}^*$ and an embedded $S^3/\mathbf{D}_2^*$. Hence they cannot be congruent to each other and we obtain three distinct orbits in the moduli space.

Finally, we can consider the polynomial h_{2d+2} for d even. For $d = 2$ this polynomial is the same as the invariant polynomial for $\mathbf{T}^*$, but for even $d \geq 4$ one can show, by choosing c_1 real and c_2 imaginary, that this gives rise to a minimal isometric embedding of $L(2d; 1)$ into $S^{2d+2}(\sqrt{3/(4(d+1)(d+2))})$.

CASE III: THE POLYHEDRAL GROUPS $\mathbf{T}^*$, $\mathbf{O}^*$ and $\mathbf{I}^*$. Because the three-dimensional polyhedral manifolds are isotropy irreducible (i.e., the adjoint actions of $\mathbf{T}^*$, $\mathbf{O}^*$ and $\mathbf{I}^*$ are irreducible on the Lie algebra $\mathfrak{sp}(1)$, see [WZ]), Takahashi's result tells us that the orbit of *any* nonconstant homogeneous harmonic polynomial invariant under a polyhedral group will yield a minimal isometric immersion of the corresponding polyhedral manifold. However, for later purposes we need the exact polynomial that induces a constant curvature 1 metric. With a calculation similar to the lens spaces one easily verifies the following assertions.

For the binary tetrahedral group $\mathbf{T}^*$, the $SU(2)$ orbit through the polynomial $\tilde{\alpha} = (1/(16\sqrt{15}))(zw^5 - wz^5)$ gives rise to a minimal isometric embedding of $S^3/\mathbf{T}^*$ into $S^6(\frac{1}{4})$. This example realizes the smallest codimension of all our examples.

For the binary octahedral group $\mathbf{O}^*$, the orbit through the polynomial $\tilde{\beta} = (1/(384\sqrt{35}))(z^8 + 14z^4w4 + w^8)$ gives rise to a minimal isometric embedding of $S^3/\mathbf{O}^*$ into $S^8(\sqrt{\frac{3}{80}})$.

Finally, for the binary icosahedral group $\mathbf{I}^*$, the orbit through

$$\tilde{A} = \frac{1}{7200\sqrt{154}}(z^{11}w + 11z^6w^6 - zw^{11})$$

gives rise to a minimal isometric embedding of $S^3/\mathbf{I}^*$ into $S^{12}(\sqrt{\frac{1}{56}})$.

To see that these immersions are actually embeddings, we observe that $\mathbf{O}^*$ and $\mathbf{I}^*$ are maximal subgroups in $SU(2)$. Furthermore, for $\mathbf{T}^*$ we are using a degree 6 polynomial and, although $\mathbf{T}^*$ is contained in $\mathbf{O}^*$ and $\mathbf{I}^*$, they have no invariant polynomial of degree 6.

To obtain the explicit form of the isometric embedding of $S^3/\mathbf{T}^*$ mentioned in the introduction, we take the map which sends (a, b) to $\Phi_{\tilde{\alpha}}(a + bj) = \tilde{\alpha}(\bar{a}z - bw, \bar{b}z + aw)$ for $\tilde{\alpha} = (1/(16\sqrt{15}))(zw^5 - wz^5)$ and express the result as a linear combination of the orthonormal basis

$$\{(z^6 + w^6)/12\sqrt{10}, i(z^6 - w^6)/12\sqrt{10}, (z^5w - zw^5)/4\sqrt{15}, i(z^5w + zw^5)/4\sqrt{15},$$
$$(z^4w^2 + z^2w^4_0)/4\sqrt{6}, i(z^4w^2 - z^2w^4)/4\sqrt{6}, iz^3w^3/6\}.$$

The coefficients are then the components of the embedding.

To summarize the results of this section, we present the following table. For each homogeneous three-dimensional spherical space form, we list the polynomial whose orbit provides the minimal isometric embedding of smallest codimension. The constants c_1 and c_2 are real.

Space		Polynomial	Coefficients	Target
$L(d;1)$,	$d=4s$ $d \geq 8$	$h_{d+2}=c_1(z^{d+1}w+w^{d+1}z)$ $+ic_2z^{2s+1}w^{2s+1}$	$c_1^2=\dfrac{1}{2d^2(d+1)!}$ $c_2^2=\dfrac{8s^2-6s-2}{d^2(2s+1)(2s+1)!(2s+2)!}$	$S^{d+2}\left(\sqrt{\dfrac{3}{(d+2)(d+4)}}\right)$
$L(d;1)$,	$d=4s+2$ $d \geq 10$	$g_d=c_1(z^d+w^d)$ $+ic_2z^{2s+1}w^{2s+1}$	$c_1^2=\dfrac{1}{2d^2d!}$ $c_2^2=\dfrac{4s+1}{d^2(2s+1)!(2s+2)!}$	$S^d\left(\sqrt{\dfrac{3}{d(d+2)}}\right)$
$L(d;1)$,	d odd	$k_{3d}=c_1z^{3d}+c_2z^{2d}w^d$	$c_1^2=\dfrac{(d+1)}{4d^2(3d+2)(3d)!}$ $c_2^2=\dfrac{3d-1}{4d^2(3d+2)d!(2d)!}$	$S^{6d+1}\left(\sqrt{\dfrac{1}{d(3d+2)}}\right)$
$L(4;1)$		$p_8=\dfrac{1}{480\sqrt{21}}z^8+\dfrac{\sqrt{7}}{240\sqrt{6}}z^2w^6$		$S^{17}\left(\sqrt{\dfrac{3}{80}}\right)$
$L(6;1)$		$f_6=\dfrac{1}{72\sqrt{5}}z^6+\dfrac{\sqrt{5}}{72}z^3w^3$		$S^{13}\left(\dfrac{1}{4}\right)$
$S^3/\mathbf{D}_d^*$,	d even $d \geq 6$	$g_{2d}=c_1(z^{2d}+w^{2d})$ $+c_2z^dw^d$	$c_1^2=\dfrac{1}{8d^2(2d)!}$ $c_2^2=\dfrac{2d-1}{4d^2(d+1)!d!}$	$S^{2d}\left(\sqrt{\dfrac{3}{4d(d+1)}}\right)$
$S^3/\mathbf{D}_d^*$,	d odd $d \geq 7$	$h_{2d+2}=c_1(z^{2d+1}w-w^{2d+1}z)$ $+c_2z^{d+1}w^{d+1}$	$c_1^2=\dfrac{1}{8d^2(2d+1)!}$ $c_2^2=\dfrac{(2d+1)(d-2)}{4d^2(d+1)(d+1)!(d+2)!}$	$S^{2d+2}\left(\sqrt{\dfrac{3}{4(d+1)(d+2)}}\right)$
$S^3/\mathbf{D}_2^*$		$\psi_8=c\{14z^4w^4+28(z^6w^2+z^2w^6)$ $-3(z^8+w^8)\}$	$c=\dfrac{1}{1536\sqrt{35}}$	$S^8\left(\sqrt{\dfrac{3}{80}}\right)$
$S^3/\mathbf{D}_3^*$		$h_8=c_1(z^7w-w^7z)+c_2z^4w^4$	$c_1=\dfrac{1}{72\sqrt{70}},\quad c_2=\dfrac{i\sqrt{7}}{288\sqrt{5}}$	$S^8\left(\sqrt{\dfrac{3}{80}}\right)$

Table (continued)

Space	Polynomial	Coefficients	Target
$S^3/\mathbf{D}_4^*$	$\tilde{g}_8 = c(z^8 + 14iz^4w^4 + w^8)$	$c = \dfrac{1}{384\sqrt{35}}$	$S^{17}\left(\sqrt{\dfrac{3}{80}}\right)$
$S^3/\mathbf{D}_5^*$	$h_{12} = c(z^{11}w - w^{11}z + 11iz^6w^6)$	$c = \dfrac{1}{7200\sqrt{154}}$	$S^{21}\left(\sqrt{\dfrac{1}{56}}\right)$
$S^3/\mathbf{T}^*$	$\tilde{\alpha} = \dfrac{1}{16\sqrt{15}}(zw^5 - wz^5)$		$S^6\left(\dfrac{1}{4}\right)$
$S^3/\mathbf{O}^*$	$\tilde{\beta} = c(z^8 + 14z^4w^4 + w^8)$	$c = \dfrac{1}{384\sqrt{35}}$	$S^8\left(\sqrt{\dfrac{3}{80}}\right)$
$S^3/\mathbf{I}^*$	$\tilde{A} = c(z^{11}w + 11z^6w^6 - zw^{11})$	$c = \dfrac{1}{7200\sqrt{154}}$	$S^{12}\left(\sqrt{\dfrac{1}{56}}\right)$

3. The equivariant moduli space for S^3

The moduli space of equivariant minimal isometric immersions has some special features in the case of the 3-sphere. The isometry group of S^3 can be described by the action of $Sp(1) \times Sp(1)$ via left and right multiplication of unit quaternions. Any minimal isometric immersion which is equivariant with respect to some transitive group action is also equivariant with respect to either the left or right action of $Sp(1)$ on S^3. The two actions are equivalent to each other under the orientation reversing isometry given by quaternionic conjugation, and hence it is sufficient to look at all minimal isometric immersions equivariant with respect to the left-$Sp(1)$ action. In this section we will examine this set in some detail. As explained above, if p is some polynomial, then the immersion corresponding to p (i.e., the orbit of p) is given by $g \in SU(2) \mapsto gp$. The gauge group $SO(4)$ acts on these immersions in two ways. The left multiplication by a unit quaternion h gives rise to the immersion $g \mapsto hg \mapsto hgp$ which is clearly congruent to the original one. The right multiplication by h gives rise to the immersion $g \mapsto gh \mapsto ghp$ which is the same as the immersion given by the polynomial hp.

As was mentioned in §1, the $SU(2)$-equivariant minimal isometric immersions of S^3 and of degree d forms a convex sub-body of the set B_d of all isometric minimal immersions of degree d. Let us first examine what the possible codimensions of such equivariant minimal immersions are. To see this, we need to determine the respresentations of $SU(2)$ on the full eigenspaces E_{λ_d}. The full isometry group, $SO(4)$, is locally isomorphic to $SU(2) \times SU(2)$ and the representation of $SO(4)$ on E_{λ_d} is isomorphic to $[W_d \otimes W_d]_{\mathbb{R}}$. Here W_d is the irreducible representation of $SU(2)$ of (complex) dimension $d+1$ and the tensor product, being a representation of real type, is the complexification of a real representation (denoted by $[\,]_{\mathbb{R}}$) of real dimension $(d+1)^2$. Hence the restriction from $SO(4)$ to $SU(2)$ is isomorphic to $(d+1)[W_d]_{\mathbb{R}} = (d+1)R_d$ if d is even and to $(k+1)W_d$ if $d = 2k+1$ is odd. Recall that W_d is a representation of real type if d is even and a representation of quaternionic type if d is odd. If we consider a polynomial $p \in E_{\lambda_d}$ whose orbit $SU(2) \cdot p$ lies in a subspace $E \subset E_{\lambda_d}$ of smallest possible dimension, then E must also be invariant under $SU(2)$. Hence the possible ambient dimensions of full minimal isometric $SU(2)$-equivariant immersions are $s(d+1)$, $1 \le s \le d+1$ if d is even and equal to $2s(d+1)$, $1 \le s \le k+1$ if $d = 2k+1$ is odd.

We first discuss, for each d, the smallest possible ambient dimensions. If $d = 2$ or 3, then there exists a unique minimal isometric immersion, which is equivariant with respect to the action of $SO(4)$ (and hence $SU(2)$-equivariant), but only goes into the full eigenspace (ambient dimensions 9 and 16, respectively). If $d = 4$, then there exists no minimal immersion with ambient dimension 5, as follows from Moore's theorem, but we saw in §2 that there exists one (for $p = g_4$) with ambient dimension 10 (and image S^3/D_2^*). If d even ≥ 6, we saw in §2 that there exist minimal immersions (with $p = g_d$) with ambient dimension $d+1$ (whose images are $S^3/\mathbf{T}^*$, $S^3/\mathbf{O}^*$, $S^3/\mathbf{D}_{d/2}^*$, or $L(d; 1)$ depending on the value of d). If d is odd and divisible by 3, we gave examples of minimal immersions with ambient dimension $2(d+1)$ for certain values of d in §2 (embeddings of the lens spaces $L(d/3; 1)$ via k_d). To give examples for all odd $d = 2k+1 \ge 5$, let $\phi_d = c_1 z^{2k+1} + c_2 z^k w^{k+1}$. One easily shows that the orbit through ϕ_d has constant curvature one if and only if

$$|c_1|^2 = \frac{2k^2+5k+2}{(k+1)(2k+3)(2k+1)^2(2k+1)!}, \qquad |c_2|^2 = \frac{1}{(2k+3)[(k+1)!]^2}.$$

The image in this case turns out to be always an embedded sphere. Hence the smallest ambient dimension N of a degree d minimal $SU(2)$-equivariant immersion satisfies: $N = 9$ if $d = 2$, $N = 16$ if $d = 3$, $N = 10$ if $d = 4$, $N = 2k+1$ if $d = 2k \ge 6$, and $N = 4k+4$ if $d = 2k+1 \ge 5$. That these are the smallest ambient dimensions for $SU(2)$-equivariant minimal isometric immersions was already observed in [Ma2], but he did not discuss the nature of the image.

We will now prove some uniqueness theorems for equivariant minimal immersions. For this purpose we first derive the general equations that such immersions satisfy. Let $p = \Sigma_{k=0}^{d} c_k z^{d-k} w^k$ be a general polynomial in the representation W_d.

One easily shows, with the methods developed in §2, that the orbit through p has constant curvature iff the following equations are satisfied

$$\sum_{k=0}^{d} (2k-d)^2(d-k)!k!|c_k|^2 = 1,$$

$$\sum_{k=0}^{d} (d-k)!k!|c_k|^2 = \frac{3}{d(d+2)},$$

$$\sum_{k=0}^{d-2} (k+2)!(d-k)!c_k\bar{c}_{k+2} = 0,$$

$$\sum_{k=0}^{d-1} (d-2k-1)(k+1)!(d-k)!c_k\bar{c}_{k+1} = 0.$$

These are six real equations in the $2(d+1)$ real unknowns $\operatorname{Re}(c_k)$, $\operatorname{Im}(c_k)$.

For d even, if we want the orbit $SU(2)\cdot p$ to lie in the real subspace $R_{2d} \subset W_{2d}$, then we also need to assume that

$$c_{2d} = \bar{c}_0, \quad c_{2d-1} = -\bar{c}_1, \dots, c_{d+1} = (-1)^{d+1}\bar{c}_{d-1}, \quad c_d = i^d t$$

where t is real. Hence the orbit through

$$p = c_0 z^{2d} + \bar{c}_0 w^{2d} + c_1 z^{2d-1} w - \bar{c}_1 z w^{2d-1} + \cdots + i^d t z^d w^d$$

has constant curvature 1 if and only if

$$\sum_{k=0}^{d-1} 2(2k-2d)^2(2d-k)!k!|c_k|^2 = 1,$$

$$\sum_{k=0}^{d-1} 2(2d-k)!k!|c_k|^2 + d!d!t^2 = \frac{3}{4d(d+1)},$$

$$\sum_{k=0}^{d-3} 2(k+2)!(2d-k)!c_k\bar{c}_{k+2} + (-1)^{d+1}(d+1)!(d+1)!c_{d-1}^2$$
$$+ (-i)^d 2d!(d+2)!c_{d-2}t = 0,$$

$$\sum_{k=0}^{d-2} (2d-2k-1)(k+1)!(2d-k)!c_k\bar{c}_{k+1} + (-i)^d d!(d+1)!c_{d-1}t = 0.$$

These are six equations in the $2d-1$ unknowns $c_1, \dots, c_{d-1}, t$.

We first examine these equations for $d=4$ and show:

PROPOSITION 1. *Up to congruences of the ambient space, among the $SU(2)$-orbits of polynomials in $\mathbb{R}^{10}=W_4$ there exists a unique one of constant curvature one, which is isometric to $S^3/\mathbf{D}_2^*$.*

Proof. The equations for $p=c_0z^4+\cdots+c_4w^4$ become:

$$16|c_0|^2+|c_1|^2+|c_3|^2+16|c_4|^2=\frac{1}{24},$$

$$24|c_0|^2+6|c_1|^2+4|c_2|^2+6|c_3|^2+24|c_4|^2=\frac{1}{8},$$

$$4c_0\bar{c}_2+3c_1\bar{c}_3+4c_2\bar{c}_4=0,$$

$$6c_0\bar{c}_1+c_1\bar{c}_2-c_2\bar{c}_3-6c_3\bar{c}_4=0.$$

To simplify the equations we use the following observation from [Ma2]. If $O_1=SU(2)\cdot p_1$ is one orbit in $\mathbb{R}^{10}$, let N be the linear subspace of $\mathbb{R}^{10}$ normal to the tangent space of O_1 at p_1. Then any other orbit O_2 must pass through N. Indeed, all orbits are constant distance apart, and hence there exists a minimal geodesic of $\mathbb{R}^{10}$ from O_1 to O_2 perpendicular to O_1 at p_1. Hence $O_2\cap N\neq\emptyset$. In our case let $p_1=z^4$. Then the tangent space to $SU(2)\cdot p_1$ at p_1 is spanned by iz^4, z^3w, and iz^3w, and so the condition that $p_2\in N$ in particular implies $c_1=0$. By multiplying p with $\begin{bmatrix} a & 0\\ 0 & \bar{a}\end{bmatrix}$ we can also change the variables c_0, c_1, c_2, c_3, c_4 to $\bar{a}^4c_0$, $\bar{a}^2c_1$, c_2, a^2c_3, a^4c_4 and we can also apply the ambient congruence which takes c_i to $e^{i\theta}c_i$. Both operations preserve the condition $c_1=0$ and hence we can assume, in addition to $c_1=0$, that two of the remaining variables are real.

If $c_1=0$ the last two equations become $c_0\bar{c}_2=-c_2\bar{c}_4$, $c_2\bar{c}_3=-6c_3\bar{c}_4$. If $c_2\neq 0$, $c_3\neq 0$ we can assume that c_2 and c_3 are real and obtain $c_0=-\bar{c}_4$, $c_2=-6\bar{c}_4$. The first two equations then become $|c_3|^2+32|c_4|^2=\frac{1}{24}$ and $2|c_3|^2+64|c_4|^2=\frac{1}{24}$ which clearly has no solutions.

If $c_3=0$, we can assume that c_4 is real and c_2 is imaginary. Then $c_0=c_4$ and one obtains the solution $g_4=(1/(16\sqrt{3}))(z^4+w^4)+(i/8)\,z^2w^2$ the orbit of which, is, according to §2, the dihedral manifold $S^3/\mathbf{D}_2^*$.

If $c_2=0$ and $c_3\neq 0$ we need $c_4=0$ and the first two equations become $16|c_0|^2+|c_3|^2=\frac{1}{24}$ and $8|c_0|^2+2|c_3|^2=\frac{1}{24}$. Since we can assume that c_0 and c_3 are real, we obtain the solution $q=\frac{1}{24}z^4+(1/(6\sqrt{2}))zw^3$. We claim that the $SU(2)$-orbit through q is congruent to the $SU(2)$-orbit through g_4. To see this consider

$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \cdot q$. By looking at the coefficients of z^3w and zw^3 one easily shows that there exists a polynomial in the orbit of q whose coefficients of z^3w and zw^3 are 0 and is hence of the form $c_0z^4 + c_4w^4 + c_2z^2w^2$. But the argument in the case of $c_1 = c_3 = 0$ now implies that, up to congruence, the orbit through this polynomial is the same as the orbit through g_4. □

PROPOSITION 2. *Among the $SU(2)$-orbits of polynomials in $\mathbb{R}^7 = [W_6]_{\mathbb{R}}$ there exists a unique one of curvature one, which is isometric to $S^3/\mathbf{T}^*$.*

Proof. The equations for $p = c_0z^6 + \bar{c}_0w^6 + c_1z^5w - \bar{c}_1zw^5 + \cdots - itz^3w^3$ become

$$135|c_0|^2 + 10|c_1|^2 + |c_2|^2 = \frac{1}{384},$$

$$60|c_0|^2 + 10|c_1|^2 + 4|c_2|^2 + \frac{3}{2}t^2 = \frac{1}{384},$$

$$10c_0\bar{c}_2 + 2c_2^2 + 5ic_1t = 0,$$

$$25c_0\bar{c}_1 + 5c_1\bar{c}_2 + ic_2t = 0.$$

As in the proof of Proposition 1, we first simplify the equations. This time we consider the orbit through z^3w^3. The tangent space of this orbit is spanned by $z^4w^2 + z^2w^4$ and $i(z^4w^2 - z^2w^4)$, and so the normal space coincides with $c_2 = 0$. Since every orbit intersects the normal space, we can assume $c_2 = 0$. With this assumption, the last two of the above equations become $c_0\bar{c}_1 = 0$ and $c_1t = 0$. If $c_1 = 0$, they are automatically satisfied. By modifying the polynomial, we can assume that c_0 is real and hence we obtain the solution $q = (1/(72\sqrt{10}))(z^6 + w^6) - i(\sqrt{5}/72)z^3w^3$.

If $c_1 \neq 0$ we need $c_0 = t = 0$ and we get the solution $\tilde{\alpha} = (1/(16\sqrt{15}))(z^5w - zw^5)$ whose orbit is the tetrahedral manifold $S^3/\mathbf{T}^*$. We now claim that q lies in the $SU(2)$-orbit of $\tilde{\alpha}$. Indeed, considering $r = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \cdot \tilde{\alpha}$ one easily shows that one can choose a and b so that the coefficients of z^5w and z^4w^2 in r are 0 and hence r is of the form $c_0z^6 + \bar{c}_0w^6 - itz^3\omega^3$. We can furthermore assume that c_0 is real, but since the orbit must have constant curvature one, it must agree with q. □

Finally, we give some (partially) heuristic arguments as to the dimension of the set of $SU(2)$-equivariant minimal isometric immersions. We start with degree 4. As explained earlier, the first time we can expect solutions is if the ambient space is

$2R_4 = W_4$. One obtains 6 equations in 10 unknowns and hence a 4-dimensional soution set. (In fact, at a specific solution one easily checks that the equations have maximal rank). But one has a one-dimensional family of ambient congruences coming from $c_i \to e^{i\theta}c_i$, and so there is at most a three-dimensional family of solutions in the moduli space B_4, which agrees with the result in Proposition 1 that there is only one orbit up to congruence. If we consider orbits in kR_4, $3 \le k \le 5$ of constant curvature 1, we obtain 6 equations in $5k$ unknowns giving rise to a $(5k - 6)$-dimensional solution set. However, we obtain a large group of ambient congruences from the group of orthogonal transformations on kR_4 which commute with the representation of $SU(2)$ on kR_4. This group is isomorphic to $SO(k)$ since R_4 is absolutely irreducible. Hence in $3R_4$ we obtain a 6-dimensional solution set, in $4R_4$ an 8-dimensional solution set, and in $5R_4 = E_{\lambda_4}$ a 9-dimensional solution set of equivariant solutions in B_4. Recall that $\dim B_4 = 18$.

For $d = 5$, we consider orbits in kW_5, $1 \le k \le 3$. In this case the group of orthogonal transformations commuting with the action of $SU(2)$ is isomorphic to $Sp(k)$ since W_5 is a quaternionic representation. Hence a calculation as above shows that among the orbits in W_5 we obtain a 3-dimensional solution set, in $2W_5$ an 8-dimensional solution set, and in $3W_5 = E_{\lambda_5}$ a 9-dimensional solution set. This argument at least shows that the orbit of ϕ_5 in W_5 discussed at the beginning of this section is isolated among all equivariant solutions.

Similar calculations can be carried out for larger values of d. The only other cases where one obtains a 3-dimensional solution set and hence an isolated (if not unique) orbit is for $d = 6$ and orbits in R_6 (corresponding to the unique solution in Proposition 2) and for $d = 8$ and orbits in R_8. In the latter case we have three solutions from §2, the orbit $S^3/\mathbf{O}^*$ of $\tilde{\beta}$, the orbit $S^3/\mathbf{D}_3^*$ of h_8, and the orbit $S^3/\mathbf{D}_2^*$ of ψ_8. One easily checks the maximal rank condition at these three solutions and hence it follows that they are isolated among the equivariant immersions. They cannot be congruent since their images are distinct. We doubt that there are any other solutions for $d = 8$.

We suspect that in general the only congruences that one obtains between orbits of the same representation are orthogonal transformations which commute with the representation. It would then follow that the set of $SU(2)$-equivariant minimal isometric immersions of degree d form a convex body of dimension $2k^2 + 3k - 5$ if $d = 2k + 1$ or $d = 2k$.

4. Higher-dimensional lens spaces

To realize the higher dimensional lens spaces as homogeneous spaces we write S^{2n-1} as $U(n)/U(n - 1)$ where $U(n - 1)$ is the subgroup of n by n unitary matrices with a 1 in the upper left hand corner. The subgroup C_d generated by

$$\begin{bmatrix} e^{2\pi i/d} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \end{bmatrix}$$

commutes with $U(n-1)$ and the homogeneous space $U(n)/(\mathbf{C}_d \times U(n-1))$ is the lens space $L(d; 1, \ldots, 1) = S^{2n-1}/\mathbf{C}_d$ where $\mathbf{C}_d$ acts on $\mathbb{R}^{2n} = \mathbb{C}^n$ by multiplication on each coordinate. This homogeneous space is reductive, i.e. there is an ad-$\mathfrak{u}(n-1)$-invariant subspace of $\mathfrak{u}(n)$, namely, the subspace $\mathfrak{m}$ of skew-hermitian matrices whose *only* nonzero entries are in the first row and column. We can identify $\mathfrak{m}$ with the tangent space to S^{2n-1} at $(1, 0, \ldots, 0)$ and a left-invariant metric on the lens space with an ad-$\mathfrak{u}(n-1)$-invariant inner product on $\mathfrak{m}$. One easily verifies that for the inner product on $\mathfrak{m}$ which gives rise to the constant curvature 1 metric on the lens space, the following is an orthonormal basis (each of the vectors in the basis is a skew-hermitian matrix A, and only the nonzero entries of A are given, the rest being assumed to vanish):

$$Z: \quad a_{11} = i,$$

$$X_k: \quad a_{1k} = -a_{k1} = 1, \qquad k = 2, \ldots, n,$$

$$Y_k: \quad a_{1k} = a_{k1} = i, \qquad k = 2, \ldots, n.$$

The homogeneous harmonic polynomials in the $2n$ real variables we write again as polynomials in the complex variables $z_k, \bar{z}_k$ $(k = 1, \ldots, n)$. As before, for any polynomial in the variables z_i, the real and imaginary parts are automatically harmonic, and the action of $U(n) \subset SO(2n)$ on the space of harmonic polynomials restricts to the action of $U(n)$ on $\mathbb{C}[z_1, \ldots, z_n]$ where $A \in U(n)$ acts on $p \in \mathbb{C}[z_1, \ldots, z_n]$ by replacing z_i by A^{-1} acting on z_i.

For any $\mathbf{C}_d \times U(n-1)$-invariant homogeneous polynomial $p \in \mathbb{C}[z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n]$ we define the map

$$\Phi_p : U(n)/(\mathbf{C}_d \times U(n-1)) \to \mathbb{C}[z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n]$$

given by

$$\Phi_p(g(\mathbf{C}_d \times U(n-1))) = gp$$

which we will try to make into an isometric embedding.

One easily shows that for $p_a(z_1, \dots, z_n) = (1/(a\sqrt{a!}))z_1^a$ we have

$$\Phi_{p_a*}(Z) = \frac{-i}{\sqrt{a!}} z_1^a,$$

$$\Phi_{p_a*}(X_k) = \frac{-1}{\sqrt{a!}} z_1^{a-1} z_k, \qquad k = 2, \dots, n,$$

$$\Phi_{p_a*}(Y_k) = \frac{-i}{\sqrt{a!}} z_1^{a-1} z_k, \qquad k = 2, \dots, n.$$

These polynomials are orthogonal, but their norms are not equal, in particular

$$\|\Phi_{p_a*}(Z)\|^2 = 1,$$

$$\|\Phi_{p_a*}(X_j)\|^2 = \|\Phi_{p_a*}(Y_j)\|^2 = \frac{1}{\sqrt{a}}.$$

Note that $\Phi_{p_a*}(X_j)$ and $\Phi_{p_a*}(Y_j)$ are *shorter* than $\Phi_{p_a*}(Z)$.

We need another $\mathbf{C}_d \times U(n-1)$-invariant polynomial. To be $U(n-1)$-invariant, the only way it can depend upon $z_2, \dots, z_n$ is to be a function of $\sigma = |z_2|^2 + \cdots + |z_n|^2$. We thus search for harmonic homogeneous polynomials which are functions of σ and $\rho = |z_1|^2$. A calculation shows that the unique such polynomial (up to scaling) of degree $2c$ is given by

$$F_{2c}(z_1, \dots, z_n) = \sum_{k=0}^{c} (-1)^k a_k \sigma^{c-k} \rho^k,$$

where $a_k = \binom{c}{k}\binom{n+c-2}{k}$. We then calculate that

$$\Phi_{F_{2c}*}(Z) = 0,$$

$$\Phi_{F_{2c}*}(X_j) = \sum_{k=0}^{c-1} (-1)^k \sigma^{c-k-1} \rho^k (z_k \bar{z}_1 + z_1 \bar{z}_k)((a_k(c-k) + a_{k+1}(k+1)),$$

$$\Phi_{F_{2c}*}(Y_j) = \sum_{k=0}^{c-1} (-1)^k \sigma^{c-k-1} \rho^k i(z_k \bar{z}_1 - z_1 \bar{z}_k)((a_k(c-k) + a_{k+1}(k+1)),$$

and hence

$$\|\Phi_{F_{2c}*}(Z)\| = 0, \qquad \|\Phi_{F_{2c}*}(X_j)\| = \|\Phi_{F_{2c}*}(Y_j)\| \neq 0.$$

Using this, we see that the three polynomials $\Phi_{F_{2d}*}(Z)$, $\Phi_{F_{2d}*}(Y_k)$ and $\Phi_{F_{2d}*}(X_k)$ are orthogonal, and they are orthogonal to the images under $\Phi_{p_{2d}*}$. Since $\Phi_{F_{2d}*}(Z) = 0$, we see that we can make up for the deficiency in the length of $\Phi_{p_{2d}*}(X_k)$ and $\Phi_{p_{2d}*}(Y_k)$ by adding the appropriate multiple of F_{2d} to p_{2d}. The correct choice of scale factors will then provide us with a minimal isometric immersion of $L(2d; 1, \dots, 1) = U(n)/(\mathbf{C}_{2d} \times U(n-1))$ into the $N_{2d} - 1$-dimensional sphere of radius $\sqrt{(2n-1)/(4d(d+n-1))}$. One also easily shows that $\mathbf{C}_{2d}$ is the full isotropy group of this polynomial and hence this immersion is an embedding.

We can improve our measure of the codimension of the embedding if we recall that the representation of $U(n)$ on the space H_n of homogeneous harmonic polynomials in $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$ is reducible. In fact, the irreducible pieces are the spaces $H_{k,l}$ (with $k + l = n$) of harmonic polynomials which are bihomogeneous of degree k in $z_1, \dots, z_n$ and degree l in $\bar{z}_1, \dots, \bar{z}_n$ (see [G]). The *real* dimension of $H_{k,l}$ is $2(\binom{n+k-1}{k}\binom{n+l-1}{l} - \binom{n+k-2}{k-1}\binom{n+l-2}{l-1})$. Since $p_{2d} \in H_{2d,0}$ and $F_{2d} \in H_{d,d}$, the orbit of their weighted sum is contained in $H_{2d,0} \oplus H_{d,d}$.

For d odd one shows that the orbit through the harmonic homogeneous polynomial

$$c_1 z_1^{3d} + c_2 z_1^d \sum_{k=0}^{d} (-1)^k \frac{\binom{d}{k}\binom{2d}{k}}{\binom{n+k-2}{k}} |z_1|^{2(d-k)}(|z_2|^2 + \cdots + |z_n|^2)^k$$

provides, for appropriate choice of c_1 and c_2, a minimal isometric embedding of $L(d; 1, \dots, 1)$ into sphere of radius $\sqrt{(2n-1)/(3d(3d+2n-2))}$ in $H_{3d,0} \oplus H_{2d,d}$.

5. Space forms of dimension $4n - 1$

Finally, we turn to the spherical space forms which are realized as homogeneous spaces of the symplectic group $Sp(n)$. Recall that the sphere S^{4n-1} can be realized as the homogeneous space $Sp(n)/Sp(n-1)$, where $Sp(n-1)$ acts on the last $n-1$ variables. Then, any finite subgroup Γ of $Sp(1)$ (these were listed in §2) can act on the first component of the quaternionic Euclidean space $\mathbb{H}^n$, yielding a homogeneous space $Sp(n)/(\Gamma \times Sp(n-1))$. This manifold is also equal to S^{4n-1}/Γ where Γ acts on $\mathbb{H}^n$ in each variable by multiplication on the left.

The Lie algebra $\mathfrak{sp}(n)$ has the ad-$\mathfrak{sp}(n-1)$-invariant splitting $\mathfrak{sp}(n) = \mathfrak{sp}(n-1) \oplus \mathfrak{m}$, where an orthonormal basis of $\mathfrak{m}$ is given by the following set of $4n - 1$ quaternionic matrices (in each case, the matrix is given in the form $A + Bj$, where A is a skew-hermitian and B is a symmetric complex matrix. Only the nonzero elements of A or B are listed, and all other elements of A and B are taken to be zero):

$$Z_k:\quad a_{1k} = a_{k1} = i, \qquad k = 1, \ldots, n,$$
$$U_k:\quad b_{1k} = b_{k1} = 1, \qquad k = 1, \ldots, n,$$
$$V_k:\quad b_{1k} = b_{k1} = i, \qquad k = 1, \ldots, n,$$
$$W_k:\quad a_{1k} = -a_{k1} = 1, \qquad k = 2, \ldots, n.$$

Identifying $\mathbb{C}^{2n}$ with $\mathbb{H}^n$ via

$$(z_1, \ldots, z_n, w_1, \ldots, w_n) \to (z_1 + w_1 j, \ldots, z_n + w_n j),$$

$Sp(n)$ becomes a subgroup of $U(2n)$ where $A + Bj \in Sp(n)$ becomes

$$\begin{bmatrix} A & B \\ -\bar{B} & \bar{A} \end{bmatrix}.$$

We now need to find $\Gamma \times Sp(n-1)$-invariant polynomials in the $2n$ complex variables z_k and w_k $(k = 1, \ldots, n)$, where of course, an element $A + Bj$ of $Sp(n)$ acts on $p(z_k, w_k)$ by replacing z_k and w_k with

$$\begin{bmatrix} \bar{A}^t & -B^t \\ \bar{B}^t & A^t \end{bmatrix}$$

acting on $(z_1, \ldots, z_n, w_1, \ldots, w_n)$. Given such a polynomial p, we get a map

$$\Phi_p : Sp(n)/(\Gamma \times Sp(n-1)) \to \mathbb{C}[z_k, w_k]$$

and we compute that, for $p_{ab}(z_k, w_k) = z_1^a w_1^b$,

$$\Phi_{p_{ab}*}(Z_1) = i(b-a) z_1^a w_1^b,$$
$$\Phi_{p_{ab}*}(Z_k) = -i(a z_k w_1 - b w_k z_1) z_1^{a-1} w_1^{b-1},$$
$$\Phi_{p_{ab}*}(U_1) = -a z_1^{a-1} w_1^{b+1} + b z_1^{a+1} w_1^{b-1},$$
$$\Phi_{p_{ab}*}(U_k) = -(a w_k w_1 - b z_k z_1) z_1^{a-1} w_1^{b-1},$$
$$\Phi_{p_{ab}*}(V_1) = -i(a z_1^{a-1} w_1^{b+1} + b z_1^{a+1} w_1^{b-1}),$$
$$\Phi_{p_{ab}*}(V_k) = -i(a w_k w_1 + b z_k z_1) z_1^{a-1} w_1^{b-1},$$
$$\Phi_{p_{ab}*}(W_k) = -(a z_k w_1 + b z_1 w_k) z_1^{a-1} w_1^{b-1},$$

where k always runs from 2 to n. It turns out, as for the lens spaces, that we will need a polynomial that depends on $\rho = |z_1|^2 + |w_1|^2$, and $\sigma = |z_2|^2 + |w_2|^2 + \cdots + |z_n|^2 + |w_n|^2$. A calculation shows that a harmonic polynomial of degree $2d$ that depends on ρ and σ is

$$F_{2d} = \sum_{k=0}^{d} (-1)^k a_k \rho^{d-k} \sigma^k,$$

where

$$a_k = \frac{\binom{d}{k}\binom{d+1}{k}}{\binom{2n+k-3}{k}}.$$

As in the case of the lens spaces, one now easily checks that $\Phi_{F_{2d}*}(Z_k)$, $\Phi_{F_{2d}*}(U_k)$, $\Phi_{F_{2d}*}(V_k)$ and $\Phi_{F_{2d}*}(W_k)$ are orthogonal to each other and to the images under any $\Phi_{p_{ab}*}$ as long as $a+b=2d$. Furthermore $\Phi_{F_{2d}*}(Z_1) = \Phi_{F_{2d}*}(U_1) = \Phi_{F_{2d}*}(V_1) = 0$ and

$$\|\Phi_{F_{2d}*}(Z_k)\|^2 = \|\Phi_{F_{2d}*}(U_k)\|^2 = \|\Phi_{F_{2d}*}(V_k)\|^2 = \|\Phi_{F_{2d}*}(W_k)\|^2 \neq 0, \qquad k \geq 2$$

the latter simply following from the fact that F_{2d} is invariant under the action of $Sp(n-1)$ and $Sp(n-1)$ acts transitively on the subspace generated by Z_k, U_k, V_k, W_k, $k \geq 2$.

We now need to add to the polynomial F_{2d} one of the polynomials $q(z_1, w_1)$ invariant under Γ, as described in §2. Since all such q are linear combinations of p_{ab} with $a+b=2d$, it follows from the above that all the images under $\Phi_{F_{2d}*}$ and Φ_{q*} are still orthogonal to each other and by the construction in §2, we have $\|\Phi_{q*}(Z_1)\| = \|\Phi_{q*}(U_1)\| = \|\Phi_{q*}(V_1)\| = 1$. Hence we only need to check that

$$\|\Phi_{q*}(Z_k)\|^2 = \|\Phi_{q*}(U_k)\|^2 = \|\Phi_{q*}(V_k)\|^2 = \|\Phi_{q*}(W_k)\|^2 < 1.$$

(The equality of the length of these vectors is again clear from the fact that q is invariant under $Sp(n-1)$.) It will then follow that, for appropriate choice of c_1 and c_2, $c_1 F_{2d} + c_2 q$ provides a minimal isometric immersion of S^{4n-1}/Γ into the $N_{2d}-1$-dimensional sphere of radius $\sqrt{(4n-1)/(4d(d+2n-1))}$. The fact that this immersion is an embedding then follows as in §2.

We now check the deficiency in length of the images under Φ_{q*}. One easily shows, using the explicit formulas for q in §2, that the length squared L^2 of these images is as shown in the following table.

Space		Polynomial	L^2
$L(d;1)$,	$d=4s\geq 8$	h_{d+2}	$3/(d+4)$
$L(d;1)$,	$d=4s+2\geq 10$	g_d	$3/(d+2)$
$L(d;1)$,	d odd	k_{3d}	$3/(3d+2)$
$L(4;1)$		p_8	$3/10$
$L(6;1)$		f_6	$3/8$
$S^3/\mathbf{D}_d^*$,	even $d\geq 6$	g_{2d}	$3/(2d+2)$
$S^3/\mathbf{D}_d^*$,	odd $d\geq 7$	h_{2d+2}	$3/(2d+4)$
$S^3/\mathbf{D}_2^*$		ψ_8	$3/10$
$S^3/\mathbf{D}_3^*$		h_8	$3/10$
$S^3/\mathbf{D}_4^*$		$\tilde{g}_8$	$3/10$
$S^3/\mathbf{D}_5^*$		h_{12}	$3/14$
$S^3/\mathbf{T}^*$		$\tilde{\alpha}$	$3/8$
$S^3/\mathbf{O}^*$		$\tilde{\beta}$	$3/10$
$S^3/\mathbf{I}^*$		$\tilde{A}$	$3/14$

Thus in all cases, $L^2<1$, which also finishes this case and finishes the proof of our Theorem.

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