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Autor(en): **Flucher, Martin**

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Extremal functions for the Trudinger–Moser inequality in 2 dimensions

MARTIN FLUCHER

Abstract. We prove that the *Trudinger–Moser constant*

$$\sup\left\{\int_\Omega \exp(4\pi u^2)\,dx : u \in H_0^{1,2}(\Omega), \int_\Omega |\nabla u|^2\,dx \le 1\right\}$$

is attained on every 2-dimensional domain. For disks this result is due to Carleson–Chang. For other domains we derive an isoperimetric inequality which relates the ratio of the supremum of the functional and its maximal limit on concentrating sequences to the corresponding quantity for disks. A conformal rearrangement is introduced to prove this inequality.
I would like to thank Jürgen Moser and Michael Struwe for helpful advice and criticism.

1. Introduction

Consider functionals of the form

$$F_\Omega(u) = \int_\Omega f(x, u(x))\,dx$$

on a bounded domain $\Omega \subset \mathbb{R}^n$. The function u is supposed to lie in the unit ball

$$B_\Omega := \left\{u \in H_0^{1,2}(\Omega) : \int_\Omega |\nabla u|^2\,dx \le 1\right\}.$$

We ask for conditions under which the supremum

$$\sup F_\Omega := \sup_{u \in B_\Omega} F_\Omega(u)$$

is attained. The particular functional we have in mind is

$$F_\Omega(u) = \int_\Omega \exp(\alpha u^2)\,dx.$$

Trudinger [15] proved that the latter is bounded on B_Ω for sufficiently small α. Moser [9] found that it is bounded for $\alpha \le 4\pi$ and unbounded for $\alpha > 4\pi$, i.e. $\alpha = 4\pi$ is the *critical exponent*. Later, Carleson–Chang [4] found that the supremum is attained even for the critical exponent, if the domain is the unit disk D. Unfortunately their method is limited to disks. However, our main result (Corollary 7) says that the supremum is attained on arbitrary domains. This is in striking contrast to the fact that for bounded domains of dimension $n \ge 3$ the supremum of

$$F_\Omega(u) = \int_\Omega |u|^p \, dx$$

on B_Ω is not attained for the critical Sobolev exponent $p = 2n/(n-2)$.

Moreover Pohozaev's non-existence result [10] and the results of Bahri–Coron [3] show that the solvability of the corresponding Euler equation depends on the topology of the domain. In contrast to this, Adimurthi [1] shows that the Euler equation

$$\Delta u + \lambda u \exp(\alpha u^2) = 0 \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

has a positive solution for any $\alpha > 0$ and $0 < \lambda < \lambda_1$ on any domain (see [2] for the proof). All the same we cannot deduce anything about the existence of maximizers for $F_\Omega(u) = \int_\Omega \exp(4\pi u^2)$ from Adimurthi's result.

2. Preliminaries

The difficulty in finding a maximizer for the Trudinger–Moser functional stems from its lack of compactness, i.e. its discontinuity with respect to weak convergence in $H_0^{1,2}(\Omega)$. To see this consider the sequence

$$u_k(x) = \begin{cases} k & \text{if } 0 \le |x| < \exp(-2\pi k^2), \\ \dfrac{-\log|x|}{2\pi k} & \text{if } \exp(-2\pi k^2) \le |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

We assume $D \subset \Omega$. Then (u_k) is a sequence on ∂B_Ω tending weakly to 0, but

$$\lim_{k \to \infty} F_\Omega(u_k) > F_\Omega(0).$$

Thus the functional is not compact up to the boundary of B_Ω. However it is compact in its interior.

DEFINITION (Compactness in the interior of B_Ω). We say that a general functional $F_\Omega(u) = \int_\Omega f(\cdot, u)\,dx$ is *compact in the interior of* B_Ω if $\limsup \|v_i\| < 1$ and $v_i \rightharpoonup v$ weakly implies $f(\cdot, v_i) \to f(\cdot, v)$ in $L^1(\Omega)$ for a subsequence.

For the Trudinger–Moser functional this property follows from its boundedness on B_Ω via Vitali's theorem. In contrast, for $n \geq 3$ the functional $F_\Omega(u) = \int_\Omega |u|^{2n/(n-2)}\,dx$ is not compact in the interior of B_Ω.

In order to find a maximizer for a general functional $F_\Omega(u) = \int_\Omega f(\cdot, u)\,dx$ consider a maximizing sequence (u_i) and extract a weakly converging subsequence $u_i \rightharpoonup u$ such that the measures $|\nabla u_i|^2\,dx$ tend weakly to some Borel measure $d\mu$.

DEFINITION (Concentration). We say that a sequence (u_i) *concentrates* at x if $u_i \in B_\Omega$ and $|\nabla u_i|^2\,dx \rightharpoonup \theta\delta_x$. Clearly $x \in \bar{\Omega}$ and $0 \leq \theta \leq 1$.

By the following theorem it suffices to exclude this phenomenon.

THEOREM 1 (Concentration-compactness alternative). *Assume Ω is a bounded domain in $\mathbb{R}^n$ of dimension $n \geq 2$. If F_Ω is compact in the interior of B_Ω, then for every sequence (u_i) in B_Ω with $u_i \rightharpoonup u$ and $|\nabla u_i|^2\,dx \rightharpoonup d\mu$ there is a subsequence such that either (u_i) concentrates at a point $x \in \bar{\Omega}$ and $u = 0$ or compactness holds in the sense $f(\cdot, u_i) \to f(\cdot, u)$ in $L^1(\Omega)$. If (u_i) concentrates at x, then $f(\cdot, u_i)\,dx \rightharpoonup f(\cdot, 0)\,dx + \gamma\delta_x$ for some $\gamma \in \bar{\mathbb{R}}$.*

For $F_\Omega(u) = \int_\Omega \exp(4\pi u^2)\,dx$ this result is due to P. L. Lions [8]. Unlike Lions' proof our proof is based on capacity methods. Thus we do not need any information about the structure of F_Ω except the compactness in the interior of B_Ω. Compactness does not imply convergence in $H_0^{1,2}(\Omega)$. In particular concentration and compactness can hold simultaneously. This is the case for compact functionals. Compactness in the interior of B_Ω implies continuity up to the boundary by application of the alternative to converging sequences. Another simple application is to domains with symmetry.

COROLLARY 2. *If Ω is invariant under a group G of diffeomorphisms without fixed points in $\bar{\Omega}$ and F_Ω is compact in the interior of B_Ω, then for G-invariant sequences only compactness occurs.*

Proof. If a G-invariant sequence would concentrate at some point $x \in \bar{\Omega}$, then it would concentrate on the whole orbit Gx. This is a contradiction. □

The concentration-compactness alternative provides the subsequent criterion for the existence of maximizers.

DEFINITION (Concentration-function). For $x \in \bar{\Omega}$ we denote by

$$F_\Omega^\delta(x) := \sup \left\{ \limsup_{i \to \infty} F_\Omega(u_i) \colon (u_i) \text{ concentrates at } x \right\}$$

the *concentration-function* of F_Ω at x and call $\sup F_\Omega^\delta := \sup_{x \in \bar{\Omega}} F_\Omega^\delta(x)$ the *critical level* of F_Ω.

THEOREM 3 (General existence theorem). *Assume the compactness of F_Ω in the interior of B_Ω and suppose the* compactness-criterion

$$\sup F_\Omega^\delta < \sup F_\Omega$$

holds. Then $\sup F_\Omega$ *is attained.*

Proof. From a maximizing sequence (u_i) for F_Ω choose a subsequence such that $u_i \rightharpoonup u \in B_\Omega$ and $|\nabla u_i|^2 \, dx \rightharpoonup d\mu$. The case of concentration is excluded by hypothesis. Therefore $F_\Omega(u_i) \to F_\Omega(u)$ and u realizes $\sup F_\Omega$. □

For 2-dimensional domains and space homogeneous f we will prove that F_Ω^δ is a continuous function on $\bar{\Omega}$ with $F_\Omega^\delta = F_\Omega(0)$ on $\partial\Omega$ and we will see that the critical level depends sensitively on the geometry of the domain.

Remark. The above theorem does not apply to $F_\Omega(u) = \int_\Omega |u|^{2n/(n-2)} \, dx$ with $n \geq 3$ for two reasons: the lack of compactness in the interior of B_Ω and the failure of the compactness-criterion. The first objection is not serious because every maximizing sequence of this functional automatically concentrates at a single point as follows from a concentration-compactness lemma due to P. L. Lions [8] (Lemma I.1). To see that the compactness-criterion fails choose $u \in B_\Omega$ such that $F_\Omega(u)$ is close to $\sup F_\Omega = S_n^{n/(2-n)}$. ($S_n$ denotes the best Sobolev constant in $\mathbb{R}^n$). For fixed $x \in \Omega$ set $u_t(x+y) := t^{(n-2)/2} u(x+ty)$. Then for t large enough $u_t \in B_\Omega$,

$F_\Omega(u_t) = F_\Omega(u)$ and (u_t) concentrates at x as $t \to \infty$. This proves $F_\Omega^\delta(x) = \sup F_\Omega$ which is fatal.

Several authors tried to describe the asymptotic behavior of maximizing sequences for this functional consisting of solutions of a subcritical problem. The most precise description was recently given by Han [6]. He considers maximizing sequences consisting of solutions of

$$\Delta u + n(n-2)u^{(n+2)/(n-2)-\varepsilon} = 0$$

and proves concentration of a subsequence of (u_ε) at a critical point of $\operatorname{Tr} H_\Omega$ (defined in Section 4.2 of this paper). A similar problem has been studied by O. Rey [12]. The maximizing sequences in the results of Han and Rey concentrate at specific points because they are chosen in a particular way. But of course there are maximizing sequences which concentrate at any given point $x \in \bar{\Omega}$.

3. Main results

Throughout the remaining sections except Section 4.1 a *domain* will be an open, bounded and connected subset of $\mathbb{R}^2$ with smooth boundary. To every domain Ω associate its *symmetrized domain* $\Omega^* := \{x \in \mathbb{R}^2 : |x| < R_\Omega\}$ having the same area as Ω, i.e. $R_\Omega = \sqrt{|\Omega|/\pi}$. As a reference domain we take $D := \{x \in \mathbb{R}^2 : |x| < 1\}$ on which we consider the space of radially symmetric functions.

DEFINITION (F_{rad}). Denote by $H^{1,2}_{0,\text{rad}}(D)$ the space of radially symmetric functions in $H^{1,2}_0(D)$ which are non-increasing in radial direction and by B_{rad}, $F_{\text{rad}} : B_{\text{rad}} \to \mathbb{R}$ and $F^\delta_{\text{rad}} : \bar{D} \to \mathbb{R}$ the corresponding unit ball, functional and concentration-function.

As to f we make the following general assumptions.

(A) f is space homogenous, i.e. independent of x, continuous and $f(0) = 0$.
(B) $f(|t|) \geq f(t)$.
(C) f is non-decreasing on $\mathbb{R}^+$.
(D) $\sup F_{\text{rad}} < \infty$.

The function $f(t) = \exp(4\pi t^2) - 1$ satisfies (A) . . . (D). For the radially symmetric case on the unit disk Carleson–Chang have computed the critical level of this functional and – in the case of a disk – found a function u with $F_{\text{rad}}(u)$ above this level. In our terms their result reads as follows.

LEMMA 4 (Carleson–Chang [4]). *For* $f(t) = \exp(4\pi t^2) - 1$ *one has*

$$\sup F^{\delta}_{\text{rad}} = \pi e,$$

$$\sup F_{\text{rad}} > \pi e.$$

The general existence theorem implies that the Trudinger–Moser constant is achieved for disks. By stability of the compactness-criterion under small perturbations of the domain this result carries over to domains which are close to a disk in measure (see Struwe [14]). However, in general replacing D by another domain with the same area decreases both sides of the compactness-criterion by a factor which is not necessarily close to 1. Thus the compactness-criterion might fail. Fortunately – and this is our main point – the ratio $\sup F_{\Omega}/\sup F^{\delta}_{\Omega}$ can only increase.

THEOREM 5 (Functional isoperimetric inequality). *Assume* (A) . . . (D). *Then*

$$\frac{\sup F_{\Omega}}{\sup F^{\delta}_{\Omega}} \geq \frac{\sup F_{\text{rad}}}{\sup F^{\delta}_{\text{rad}}}$$

for every domain Ω.

Thus $\sup F_{\Omega}/\sup F^{\delta}_{\Omega}$ is minimal for disks and this case is worst with respect to the compactness-criterion. Equality holds if and only if Ω is a disk. The functional isoperimetric inequality makes the general existence theorem much more applicable, because verifying the compactness-criterion in the radially symmetric case is a 1-dimensional problem.

COROLLARY 6 (Special existence theorem). *Assume* (A) . . . (D), *the compactness of* F_{Ω} *in the interior of* B_{Ω} *and the* radial compactness-criterion

$$\sup F^{\delta}_{\text{rad}} < \sup F_{\text{rad}}.$$

Then $\sup F_{\Omega}$ *is attained.*

Together with the result of Carleson–Chang this answers our main question.

COROLLARY 7. *The Trudinger–Moser constant*

$$\sup_{u \in B_\Omega} \int_\Omega \exp(4\pi u^2)\, dx$$

is attained on every bounded domain $\Omega \subset \mathbb{R}^2$.

4. Tools and proofs

4.1. *Concentration-compactness alternative*

DEFINITION (Capacity). For every set $A \subset \Omega$ define its *capacity* with respect to Ω

$$C_\Omega(A) := \inf_{\substack{u \in H_0^{1,2}(\Omega)\\ u = 1 \text{ on } A}} \int_\Omega |\nabla u|^2\, dx.$$

The key to the proof of Theorem 1 is the following observation. (As norm on $H_0^{1,2}(\Omega)$ we use $\|u\|^2 := \int_\Omega |\nabla u|^2\, dx$.)

LEMMA 8. *Assume* $n \geq 2$ *and* $u \in H_0^{1,2}(\Omega)$. *Then*

$$t^2 C_\Omega(\{u > t\}) \to 0,$$
$$\inf_{\substack{v \in H_0^{1,2}(\Omega)\\ v = u \text{ on } \{u > t\}}} \|v\| \to 0$$

as $t \to \infty$.

Proof. Assume the contrary of the first claim, i.e. $t_i^2 C_\Omega(\{u > t_i\}) \geq \varepsilon$ for some $\varepsilon > 0$ and a sequence of levels $0 = t_0 < t_1 < t_2 < \cdots \to \infty$. For a subsequence $(t_i - t_{i-1})^2 C_\Omega(\{u > t_i\}) \geq \varepsilon/2$ which leads to the contradiction

$$\int_\Omega |\nabla u|^2\, dx = \sum_{i=1}^{\infty} \int_{\{t_{i-1} \leq u < t_i\}} |\nabla u|^2\, dx \geq \sum_{i=1}^{\infty} (t_i - t_{i-1})^2 C_\Omega(\{u > t_i\}) = \infty.$$

As to the second claim fix $\varepsilon > 0$ and choose t so large that $\int_{\{u > t\}} |\nabla u|^2 < \varepsilon$ and $t^2 C_\Omega(\{u > t\}) < \varepsilon$. By definition of capacity there is a function $w \in H_0^{1,2}(\Omega)$ such that $w = 1$ on $\{u > t\}$ and $t^2 \int_\Omega |\nabla w|^2\, dx < \varepsilon$. Thus

$$v := \begin{cases} tw & \text{on } \{u \le t\}, \\ u & \text{on } \{u > t\} \end{cases}$$

is in $H_0^{1,2}(\Omega)$ and $\int_\Omega |\nabla v|^2\, dx < 2\varepsilon$. This completes the proof of the Lemma. □

Now we come to the proof of Theorem 1. Let f, (u_i), u and μ be as in the theorem. We show that if μ is not a Dirac measure of mass 1, then compactness holds.

STEP 1. *If (u_i) concentrates at a point x, then it tends weakly to* 0.

Proof. For every test function ϕ and $r > 0$ use Cauchy's inequality to estimate

$$\begin{aligned}\int_\Omega \nabla\phi\, \nabla u_i\, dx &= \int_{\Omega\setminus B(x,r)} \nabla\phi\, \nabla u_i\, dx + \int_{\Omega\cap B(x,r)} \nabla\phi\, \nabla u_i\, dx \\ &\le \|\phi\| \sqrt{\int_{\Omega\setminus B(x,r)} |\nabla u_i|^2\, dx} + \|\phi\|_{C^1(\Omega)} \sqrt{\pi r^2}\, \|u_i\|\end{aligned}$$

which is arbitrary small if we choose r small and i large enough. Thus $u_i \rightharpoonup 0$ in $H_0^{1,2}(\Omega)$. □

One can show that $|\nabla u|^2\, dx \le d\mu$ (see P. L. Lions [8]) which also yields the claim.

STEP 2. *If μ is not a Dirac measure of mass* 1, *then compactness holds.*

Proof. We distinguish the cases $u \in L^\infty(\Omega)$ and $u \notin L^\infty(\Omega)$. First assume $u \in L^\infty(\Omega)$. Since μ is not a Dirac measure of mass 1, there is a radius $R > 0$ such that $\mu(B(x, R)) < 1$ for every $x \in \bar{\Omega}$. Fix $\gamma > 0$ and choose $r \in (0, R)$ such that a function η exists which is harmonic on $B(0, R)\setminus B(0, r)$ with $\eta = 0$ on $\mathbb{R}^n\setminus B(0, R)$, $\eta = 1$ on $B(0, r)$ and $\int_{\mathbb{R}^n} |\nabla\eta|^2\, dx < \gamma$. With $\eta^x(y) := \eta(y - x)$ we get

$$\begin{aligned}&\limsup_{i\to\infty} \int_\Omega |\nabla(\eta^x u_i)|^2\, dx \\ &\quad\le (1+\varepsilon) \limsup_{i\to\infty} \int_\Omega |\eta^x|^2 |\nabla u_i|^2\, dx + c(\varepsilon) \limsup_{i\to\infty} \int_\Omega |\nabla\eta^x|^2\, |u_i|^2\, dx \\ &\quad\le (1+\varepsilon)\mu(B(x, R)) + c(\varepsilon) \int_\Omega |\nabla\eta^x|^2 |u|^2\, dx\end{aligned}$$

because $u_i \to u$ in $L^2(\Omega)$ and $\eta \in H^{1,\infty}$. Since u is bounded we can make the above <1 by choosing ε and γ small enough. Compactness in the interior of B_Ω provides a subsequence for which $f(\cdot, \eta^x u_i) \to f(\cdot, \eta^x u)$ in $L^1(\Omega)$ hence $f(\cdot, u_i) \to f(\cdot, u)$ in $L^1(B(x,r) \cap \Omega)$. Covering $\bar{\Omega}$ by finitely many balls $B(x,r)$ yields $f(\cdot, u_i) \to f(\cdot, u)$ in $L^1(\Omega)$.

If $u \notin L^\infty(\Omega)$ we can assume that $\operatorname{ess\,sup} u = +\infty$. Otherwise consider $\tilde{u} = -u$, $\tilde{u}_i = -u_i$ and $\tilde{f}(x,t) = f(x,-t)$. By Lemma 8 there is a function $v \in H_0^{1,2}(\Omega)$ such that $v = u$ on $\{u > t\}$ for some t and $\|v\| < \|u\|$. For the convergence on $\{u > t\}$ set $v_i := u_i - u + v$. Then $v_i \rightharpoonup v$ and

$$\limsup_{i\to\infty} \|v_i\|^2 = \limsup_{i\to\infty} \|(v_i - v)\|^2 + \|v\|^2 = \limsup_{i\to\infty} \|u_i\|^2 - \|u\|^2 + \|v\|^2 < 1.$$

Compactness in the interior of B_Ω yields $f(\cdot, v_i) \to f(\cdot, v)$ in $L^1(\Omega)$ for a subsequence, hence $f(\cdot, u_i) \to f(\cdot, u)$ in $L^1(\{u > t\})$. For the convergence on $\{u \le t\}$ set $v(x) := \min\{u(x), t\}$. Then $\|v\| < \|u\|$ since $\operatorname{ess\,sup} u = +\infty$. The same argument as above shows $f(\cdot, u_i) \to f(\cdot, u)$ in $L^1(\{u \le t\})$. Together compactness is proved.

STEP 3. *If (u_i) concentrates at x, then for a subsequence*

$$f(\cdot, u_i)\, dx \rightharpoonup f(\cdot, 0)\, dx + \gamma \delta_x$$

with some $\gamma \in \bar{\mathbb{R}}$.

Proof. For a subsequence the limit $\gamma := \lim \int_\Omega (f(\cdot, u_i) - f(\cdot, 0))\, dx$ exists in $\bar{\mathbb{R}}$. For $r > 0$ choose a cut-off function $\eta \in C^\infty(\mathbb{R}^n)$ with $\eta(x) = 0, \eta = 1$ on $\mathbb{R}^n \backslash B(x,r)$. Then

$$\int_\Omega |\nabla(\eta u_i)|^2\, dx \le 2 \int_\Omega |\eta|^2 |\nabla u_i|^2\, dx + 2 \int_\Omega |\nabla \eta|^2 |u_i|^2\, dx$$

which tends to 0 because $\eta(x) = 0$ and $u_i \to 0$ in $L^2(\Omega)$. Compactness in the interior of B_Ω provides a subsequence for which $f(\cdot, u_i) \to f(\cdot, 0)$ in $L^1(\Omega \backslash B(x,r))$. Furthermore $\int_{B(x,r)} (f(\cdot, u_i) - f(\cdot, 0))\, dx \to \gamma$ by definition of γ. Thus $(f(\cdot, u_i) - f(\cdot, u))\, dx \rightharpoonup \gamma \delta_x$ since r was arbitrary. □

This completes the proof of Theorem 1. We add a stronger version of Step 3.

PROPOSITION 9. *Assume f is space homogenous and (u_i) concentrates at x. Then $\int_{\{|u_i| < t\}} f(u_i)\, dx \to \int_\Omega f(0)\, dx$ for every $t > 0$.*

Proof. Observe that $|\{|u_i| \geq t\}| \to 0$ because $u_i \to 0$ in $L^2(\Omega)$. Thus

$$\left|\int_{\{|u_i|<t\}} f(u_i)\,dx - \int_\Omega f(0)\,dx\right|$$

$$= \left|\int_{\{|u_i|<t\}} (f(u_i)\,dx - f(0))\,dx\right|$$

$$\leq \left|\int_{\{|u_i|<t\}\cap B(x,r)} (f(u_i)\,dx - f(0))\,dx\right| + \left|\int_{\Omega\setminus B(x,r)} (f(u_i)\,dx - f(0))\,dx\right|.$$

The first term is $\leq c(t)r^2$. By the previous step the second term tends to 0 as $i \to \infty$. $\square$

4.2. *Green's function, conformal radius*

In 2 dimensions the Green's function has the form

$$G_{\Omega,x}(y) = -\frac{1}{2\pi}\log|x-y| - H_{\Omega,x}(y).$$

The regular part $H_{\Omega,x}$ is a harmonic function with the same boundary data as the singular part. By

$$\operatorname{Tr} H_\Omega : x \to H_{\Omega,x}(x)$$

we denote its trace on the diagonal. On the unit disk $H_{D,0} \equiv 0$.

DEFINITION (Approximately small disks). We say that the sets (B_i) form a sequence of approximately small disks of radii ρ_i at x if $B(x, \rho_i - \delta_i) \subset B_i \subset B(x, \rho_i + \delta_i)$ with $\delta_i/\rho_i \to 0$.

LEMMA 10 (Asymptotic analysis of the Green's function). *For every* $t > 0$

$$\int_{\{G_{\Omega,x}<t\}} |\nabla G_{\Omega,x}|^2\,dx = t,$$

$$\int_{\{G_{\Omega,x}=t\}} |\nabla G_{\Omega,x}|\,ds = 1.$$

As $t \to \infty$ *the sets* $\{G_{\Omega,x} \ge t\}$ *form a sequence of approximately small disks of radii* $\rho_t = r_\Omega(x) \exp(-2\pi t)$ *and*

$$|\nabla G_{\Omega,x}(y)| = \frac{1}{2\pi\rho_t} + O(1)$$

uniformly for $y \in \{G_{\Omega,x} = t\}$. *In particular*

$$\lim_{t\to\infty} \frac{|\{G_{\Omega,x} \ge t\}|}{\exp(-4\pi t)} = \pi r_\Omega^2(x),$$

$$\lim_{t\to\infty} \frac{\exp(2\pi t)}{|\nabla G_{\Omega,x}(y)|} = 2\pi r_\Omega(x).$$

Proof. By definition of the Green's function $\int_\Omega \nabla G_{\Omega,x} \nabla f\, dx = f(x)$ for all test functions f. Choosing a smooth approximation of $f(y) := \inf\{G_{\Omega,x}(y), t\}$ the first claim follows. Integration by parts yields the second identity. Solving $G_{\Omega,x}(y) = t$ for $|y - x|$ yields $|y - x| = \exp(-2\pi H_{\Omega,x}(y)) \exp(-2\pi t)$. By smoothness of $H_{\Omega,x}$ the corresponding level set is close to a circle. As to the gradient on this level

$$|\nabla G_{\Omega,x}(y)| = \left| -\frac{1}{2\pi} \frac{(y-x)}{|y-x|^2} - \nabla H_{\Omega,x}(y) \right| = \frac{1}{2\pi\rho_t} + O(1)$$

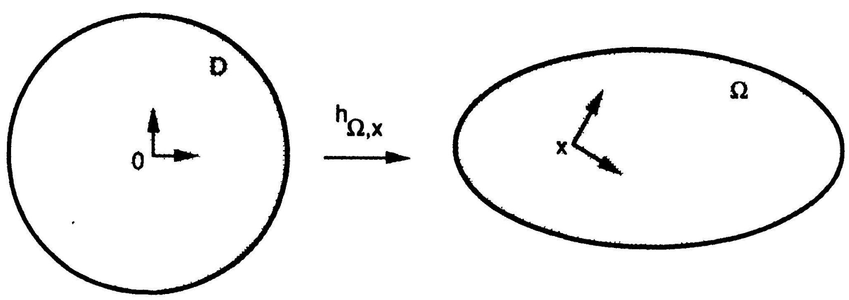
by the previous claim. □

DEFINITION (Conformal radius and conformal incenters). For $x \in \Omega$ define

$$r_\Omega(x) := \exp(-2\pi \operatorname{Tr} H_\Omega(x)).$$

The points where the conformal radius is maximal – i.e. where $\operatorname{Tr} H_\Omega$ is minimal – are called *conformal incenters* of Ω.

On simply connected domains the conformal radius has a simple geometric interpretation. In this case the Riemann mapping theorem provides for given $x \in \Omega$ a conformal diffeomorphism $h_{\Omega,x} : D \to \Omega$ with $h_{\Omega,x}(0) = x$.



This map is unique up to rotations of D. Thus $|h'_{\Omega,x}(0)|$ is a well defined number ($'$ denotes the complex derivative). We claim

$$r_\Omega(x) = |h'_{\Omega,x}(0)|.$$

This is the standard definition of the conformal radius on simply connected domains. It is consistent with the above definition by conformal invariance of the Green's function: $G_{D,0}(z) = G_{\Omega,x}(h_{\Omega,x}(z))$, i.e.

$$-\frac{1}{2\pi}\log|z| = -\frac{1}{2\pi}\log|h_{\Omega,x}(0) - h_{\Omega,x}(z)| - H_{\Omega,x}(h_{\Omega,x}(z))$$

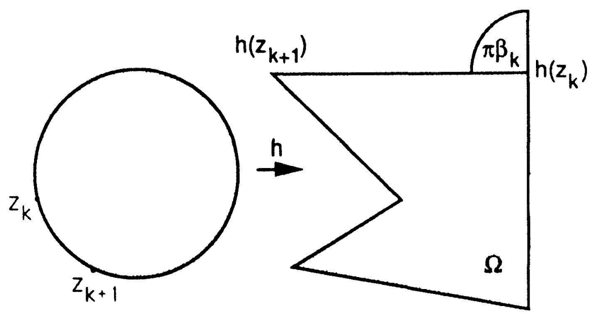
which is equivalent with

$$H_{\Omega,x}(h_{\Omega,x}(z)) = -\frac{1}{2\pi}\log\left|\frac{h_{\Omega,x}(0) - h_{\Omega,x}(z)}{z}\right|.$$

In the limit as $z \to 0$ this equality tends to what we claimed. The conformal radius of the unit disk is $r_D(x) = 1 - |x|^2$ as can be seen from appropriate Möbius transformations. More generally $r_{\Omega^*}(x) = R_\Omega(1 - |x|^2/R_\Omega^2)$. The conformal radius of any simply connected domain can be computed from a single conformal diffeomorphism $h : D \to \Omega$ via $r_\Omega(h(z)) = |h'(z)|(1 - |z|^2)$. For polygons the conformal radius can be computed from the Schwarz–Christoffel map

$$h(z) = c\int_0^z \prod_{k=1}^K (\xi - z_k)^{-\beta_k}\,d\xi + d$$

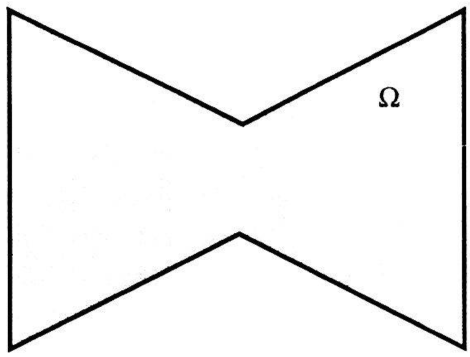
which provides a conformal transformation of the unit disk to a polygon.



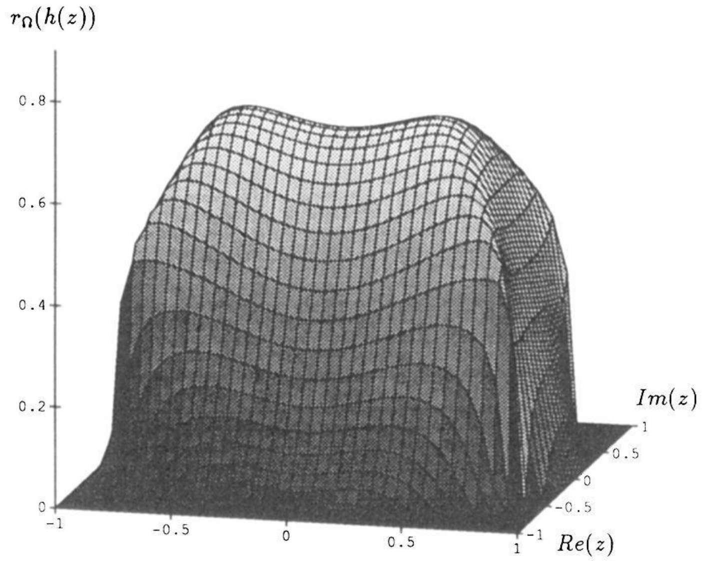
Combining this with a Möbius transformation of the unit disk yields

$$r_\Omega(h(z)) = r_\Omega(h(0)) \frac{1 - |z|^2}{\prod_{k=1}^{K} |z - z_k|^{\beta_k}} .$$

The conformal radius is small near the boundary and large at points which are far from the boundary. All the same it can attain several maxima, i.e. multiple conformal incenters, as it does on the domain below.



We have plotted the conformal radius for this domain parameterized over D, i.e. the function $r_\Omega \circ h$.



It shows 2 maxima on the same level. They correspond to 2 different conformal incenters of Ω. However, there is a single conformal incenter if the domain is strictly convex.

PROPOSITION 11. *Assume Ω is strictly convex. Then*

$$\Delta \operatorname{Tr} H_\Omega - \frac{2}{\pi} \exp(4\pi \operatorname{Tr} H_\Omega) = 0 \quad \text{in } \Omega$$

$$\operatorname{Tr} H_\Omega(x) \to \infty \quad \text{uniformly as } x \to \partial\Omega.$$

In particular Ω has a single conformal incenter.

Proof. Since Ω is simply connected there is a conformal bijection $f: \Omega \to D$ from which the conformal radius can be computed as $r_\Omega(x) = (1 - |f(x)|^2)/|f'(x)|$. Equivalently $\operatorname{Tr} H_\Omega = -(2\pi)^{-1}(\log(1 - f\bar{f}) - \frac{1}{2}\log(f'\bar{f}'))$. With $\Delta \operatorname{Tr} H_\Omega = 4\partial_{\bar{z}}\partial_z \operatorname{Tr} H_\Omega$ the claim follows after a simple computation. I thank G. Philippin for this remark. As to the boundary condition see Proposition 12 below. A theorem due to A. Kennington (see Kawohl [7], Theorem 3.13) implies that on strictly convex domains the solutions of such boundary value problems are strictly convex. In particular they attain their minimum at a single point. □

Some properties of the conformal radius follow immediately by application of the maximum principle to the regular part of the Green's function.

PROPOSITION 12. *The conformal radius of any domain satisfies*
1. $r_\Omega \in C(\Omega, \mathbb{R}^+)$.
2. $r_\Omega(x) \to 0$ *as* $x \to \partial\Omega$.
3. $\sup r_\Omega \le \sup r_{\Omega^*} = r_{\Omega^*}(0) = R_\Omega$.

Proof. 1. We show that $\operatorname{Tr} H_\Omega$ is continuous. Using the symmetry of the regular part of the Green's function in its arguments we can estimate

$$|\operatorname{Tr} H_\Omega(x) - \operatorname{Tr} H_\Omega(y)| \le 2 \max_{z \in \partial\Omega} \left| \frac{1}{2\pi} \log|x - z| - \frac{1}{2\pi} \log|y - z| \right|$$

by application of the maximum principle to the harmonic function

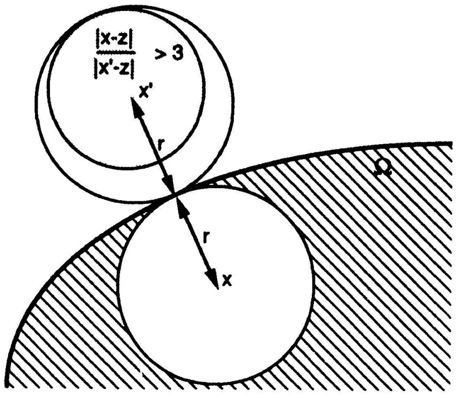
$$z \mapsto H_{\Omega,x}(z) - H_{\Omega,y}(z).$$

Thus

$$|\operatorname{Tr} H_\Omega(x) - \operatorname{Tr} H_\Omega(y)| \le \frac{|x - y|}{\pi \min\{|x - \partial\Omega|, |y - \partial\Omega|\}}.$$

2. The following argument is similar to that used by O. Rey [12] (2.8) showing that in the higher dimensional case $\operatorname{Tr} H_\Omega$ grows like $|x - \partial\Omega|^{2-n}$ as $x \to \partial\Omega$. Denote by R the minimal curvature radius of the arc $\partial\Omega$. Then to every point $x \in \Omega$

within distance $r < R$ of $\partial\Omega$ there is a reflected point x' at distance r from $\partial\Omega$ such that $B(x', r) \cap \partial\Omega = \emptyset$.



For fixed x the function $z \mapsto -(2\pi)^{-1} \log|x' - z| - H_{\Omega,x}(z)$ is harmonic in Ω. Thus

$$-\frac{1}{2\pi}\log|x' - x| - \operatorname{Tr} H_\Omega(x) \le \max_{z\in\partial\Omega}\left(-\frac{1}{2\pi}\log|x' - z| + \frac{1}{2\pi}\log|x - z|\right)$$

by the maximum principle. This means

$$r_\Omega(x) \le |x - x'| \max_{z\in\partial\Omega}\frac{|x - z|}{|x' - z|}$$

and implies $r_\Omega(x) \le 6|x - \partial\Omega|$ because

$$|x - x'| = 2|x - \partial\Omega| \quad \text{and} \quad \{z : |x - z|/|x' - z| > 3\} \subset B(x', r)$$

which is not hit by $\partial\Omega$.

3. We show that $r_\Omega(x) \le r_{\Omega^*}(0)$ for any $x \in \Omega$. This yields the claim because r_{Ω^*} is maximal at the origin. If Ω is simply connected then the mean value theorem for holomorphic functions implies

$$|h'_{\Omega,x}(0)|^2 \le \frac{1}{\pi}\int_D |h'_{\Omega,x}(z)|^2\, dx$$

via Jensen's inequality. (This inequality is strict if $h'_{\Omega,x}$ is not a constant, i.e. if Ω is not a disk.) The integral on the right is just the area of Ω and we get

$$r^2_\Omega(x) \le \frac{|\Omega|}{\pi} = R^2_\Omega = r^2_{\Omega^*}(0)$$

from the definition of the conformal radius for simply connected domains. In the general case we make use of Lemma 10, in particular $\int_{\{G_{\Omega,x} < t\}} |\nabla G_{\Omega,x}|^2\, dx = t$.

Symmetrization does not increase the Dirichlet integral. Replace $G^*_{\Omega,x}$ on $\{G^*_{\Omega,x} < t\}$ by the harmonic function v with the same boundary data. Then $\int_{\{v<t\}} |\nabla v|^2\,dx \le t$. The function v has to be a multiple of the Green's function at the origin: $v = \lambda G_{\Omega^*,0}$. Therefore

$$t \ge \int_{\{v<t\}} |\nabla v|^2\,dx = \lambda^2 \int_{\{G_{\Omega^*,0} < t/\lambda\}} |\nabla G_{\Omega^*,0}|^2\,dx = t\lambda$$

by Lemma 10. Hence $\lambda \le 1$. Using $|\{G_{\Omega,x} \ge t\}| = |\{v \ge t\}| = |\{G_{\Omega^*,0} \ge t/\lambda\}|$ and Lemma 10 we find

$$r_\Omega^2(x) = \lim_{t\to\infty} \frac{|\{G_{\Omega,x} \ge t\}|}{\pi \exp(-4\pi t)} \le \lim_{t\to\infty} \frac{|\{G_{\Omega^*,0} \le t/\lambda\}|}{\pi \exp(-4\pi t/\lambda)} = r_{\Omega^*}^2(0). \quad \square$$

The last inequality is equivalent with $\operatorname{Tr} H_\Omega(x) \ge -(2\pi)^{-1} \log(R_\Omega)$. It is strict if Ω is not a disk. If in addition $|\Omega| = |D|$ it implies $\inf \operatorname{Tr} H_\Omega > 0$ which can be considered as a *positive mass theorem* for 2-dimensional domains.

4.3. *Concentration-formula*

Surprisingly the concentration-function is related with the conformal radius r_Ω via a simple formula.

THEOREM 13 (Concentration-formula). *Assume* (A) . . . (D). *Then*

$$F_\Omega^\delta(x) = r_\Omega^2(x) F_{\mathrm{rad}}^\delta(0)$$

for every $x \in \Omega$.

In particular the concentration-formula says that

$$\limsup_{i\to\infty} F_\Omega(u_i) \le (\sup r_\Omega^2) F_{\mathrm{rad}}^\delta(0)$$

whenever the sequence (u_i) concentrates somewhere and that this inequality is optimal. Furthermore a maximizing sequence which concentrates has to concentrate at a point where the conformal radius is maximal. Clearly these points are independent of f. Observe that $F^{\delta}_{\mathrm{rad}}(0) = \sup F^{\delta}_{\mathrm{rad}} = \sup F^{\delta}_{D}$ by Schwarz symmetrization. Now we can give a precise generalization of the result of Carleson–Chang to arbitrary domains. (Use Lemma 4, Theorem 5 and Theorem 13).

COROLLARY 14. *For* $f(t) = \exp(4\pi t^2) - 1$ *one has*

$$\sup F^{\delta}_{\Omega} = (\sup r^2_{\Omega})\pi e,$$

$$\sup F_{\Omega} > (\sup r^2_{\Omega})\pi e.$$

The concentration-formula allows to deduce non-trivial properties of the concentration-function from those of the conformal radius (Proposition 12).

COROLLARY 15. *Under the general assumptions* (A) ... (D) *the concentration-function satisfies*

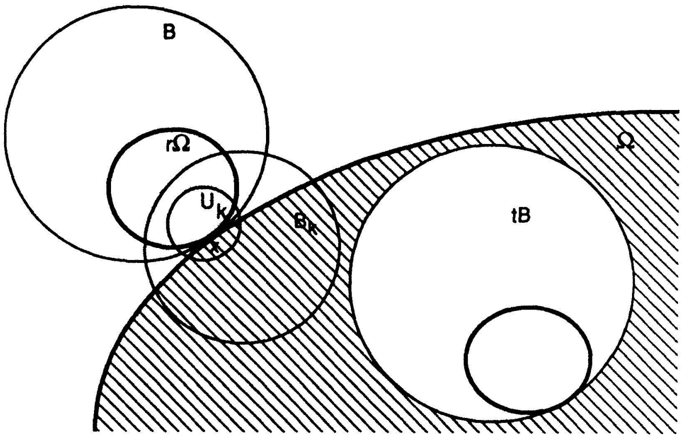
1. $F^{\delta}_{\Omega} \in C(\bar{\Omega}, \mathbb{R}^+)$.
2. $F^{\delta}_{\Omega} \mid \partial\Omega = 0$.
3. *Either* $F^{\delta}_{\Omega} \equiv 0$ *or* $F^{\delta}_{\Omega} > 0$ *on* Ω.
4. *The concentration-functions on a fixed domain but for different functions f are scalar multiples of each other.*
5. $\sup F^{\delta}_{\Omega} \leq \sup F^{\delta}_{\Omega^*}$.

Proof. The conformal radius is continuous on Ω, hence so is F^{δ}_{Ω}. Since $r_{\Omega}(x) \to 0$ as $x \to \partial\Omega$ the same holds for F^{δ}_{Ω}. By definition F^{δ}_{Ω} is lower semi continuous. Thus $F^{\delta}_{\Omega} = 0$ on $\partial\Omega$. Since $r_{\Omega} > 0$ one has $F^{\delta}_{\Omega} > 0$ in the interior of Ω if $F^{\delta}_{\mathrm{rad}}(0) > 0$ and $F^{\delta}_{\Omega} \equiv 0$ if $F^{\delta}_{\mathrm{rad}}(0) = 0$. All concentration-functions on Ω are scalar multiples of r^2_{Ω}. From the last item of Proposition 12 we get $\sup F^{\delta}_{\Omega} = (\sup r_{\Omega})^2 F^{\delta}_{\mathrm{rad}}(0) \leq (\sup r_{\Omega^*})^2 F^{\delta}_{\mathrm{rad}}(0) = \sup F^{\delta}_{\Omega^*}$. □

We give an alternate, more geometric proof of the fact that the concentration-function vanishes at the boundary. It can be generalized easily to $H^{1,n}_0(\Omega)$ on n-dimensional domains and it requires only that Ω satisfies the exterior ball condition.

Choosing a subsequence (u_i) of a sequence which concentrates at a boundary point x we can assume that $F_{\Omega}(u_i) \to \limsup F_{\Omega}(u_i)$. Since Ω satisfies the exterior ball condition there is a ball B such that $\bar{B} \cap \bar{\Omega} = \{x\}$. Choosing B small enough some translate tB is entirely contained in Ω. Special conformal diffeomorphisms are circle reflections. The circle itself is a fixed point set. If two circles intersect orthogonally, then each of them is invariant under the reflection with respect to the other. The reflection r with respect to the circle ∂B maps Ω into B. The sequence

$u_i \circ r$ concentrates at $r(x) = x$ and $\lim F_B(u_i \circ r) = \lim F_\Omega(u_i)$ since $|r'(x)|^2 = 1$ (see Lemma 16 below).



Next choose circle reflections r_k with respect to circles ∂B_k mapping B onto itself having their center so close to x that $|(r_k^{-1})'(x)|^2 \geq 2k$. On some neighborhood U_k of x we still have $|(r_k^{-1})'|^2 \geq k$. By Step 3

$$\lim_{i\to\infty} \int_B f(u_i \circ r)\, dx = \lim_{i\to\infty} \int_{U_k} f(u_i \circ r)\, dx$$

for every fixed k. Thus for a subsequence of (u_i)

$$\int_{U_i} f(u_i \circ r)\, dx \geq \frac{1}{2} \lim_{i\to\infty} \int_B f(u_i \circ r)\, dx.$$

Set $w_i := u_i \circ r \circ r_i \circ t^{-1}$. It's support is contained in $tB \subset \Omega$. Therefore $w_i \in B_\Omega$ and

$$\sup F_\Omega \geq \lim_{i\to\infty} F_\Omega(w_i) = \lim_{i\to\infty} F_B(u_i \circ r \circ r_i) = \lim_{i\to\infty} \int_B f(u_i \circ r)|(r_i^{-1})'|^2\, dx$$
$$\geq \lim_{i\to\infty} \int_{U_i} f(u_i \circ r)|(r_i^{-1})'|^2\, dx \geq \lim_{i\to\infty} \left(i \int_{U_i} f(u_i \circ r)\, dx \right).$$

But since $\sup F_\Omega < \infty$ by (D) this is only possible if $\int_{U_i} f(u_i \circ r)\, dx \to 0$. Together we conclude $F_\Omega(u_i) \to 0$.

4.3.1. *Proof of Theorem* 13 *on simply connected domains.* In the simply connected case tools from complex analysis provide a particularly simple proof. In order to exploit the conformal equivalence of the domain with the unit disk we need a transformation rule for concentrating sequences.

LEMMA 16. *Assume $h : \Omega' \to \Omega$ is a conformal diffeomorphism and $h(x') = x$. If a sequence (u_i) in B_Ω concentrates at x, then $(u_i \circ h)$ is a sequence in $B_{\Omega'}$ concentrating at x' and*

$$\lim_{i\to\infty} F_\Omega(u_i) = |h'(x')|^2 \lim_{i\to\infty} F_{\Omega'}(u_i \circ h)$$

if the limits exist.

Proof. In two dimensions the Dirichlet integral is invariant under conformal transformations. Therefore $u_i \circ h \in B_{\Omega'}$. For every $r > 0$

$$\int_{\Omega'\setminus B(x',r)} |\nabla(u_i \circ h)|^2\, dx = \int_{\Omega\setminus hB(x',r)} |\nabla u_i|^2\, dx$$

which tends to 0 as $i \to \infty$ because $hB(x', r)$ is a neighborhood of x. This means that $(u_i \circ h)$ concentrates at x'. Applying Step 3 to the sequence $(u_i \circ h)$ yields

$$\lim_{i\to\infty} F_{\Omega'}(u_i \circ h) = \lim_{i\to\infty} \int_{B(x',r)} f(u_i \circ h)\, dx = \lim_{i\to\infty} \int_{hB(x',r)} f(u_i)|(h^{-1})'|^2\, dx$$
$$= (|(h^{-1})'(x)|^2 + O(r)) \lim_{i\to\infty} \int_{hB(x',r)} f(u_i)\, dx$$

as $r \to 0$. Step 3 yields the claim. □

Now we can prove Theorem 13 for simply connected domains. First we construct a sequence showing that $r_\Omega^2(x)F_{\mathrm{rad}}^\delta(0)$ is a lower bound for $F_\Omega^\delta(x)$. Then we have to show that this is indeed the worst what happens.

Choose a sequence (v_i) realizing $F_{\mathrm{rad}}^\delta(0)$, i.e. a sequence in B_{rad} concentrating at 0 such that $F_{\mathrm{rad}}(v_i) \to F_{\mathrm{rad}}^\delta(0)$. Then $u_i := v_i \circ h_{\Omega,x}^{-1}$ concentrates at x and $\lim F_\Omega(u_i) = r_\Omega^2(x) \lim F_{\mathrm{rad}}(v_i)$ by the previous lemma. Thus $F_\Omega^\delta(x) \ge r_\Omega^2(x)F_{\mathrm{rad}}^\delta(0)$. For the opposite inequality choose a sequence (u_i) realizing $F_\Omega^\delta(x)$. By (B) we can assume $u_i \ge 0$. Set $v_i := u_i \circ h_{\Omega,x}$, then by the previous lemma

$$F_\Omega^\delta(x) = \lim_{i\to\infty} F_\Omega(u_i) = r_\Omega^2(x) \lim_{i\to\infty} F_D(v_i) \le r_\Omega^2(x)F_{\mathrm{rad}}^\delta(0)$$

because (v_i) concentrates at 0. This proves Theorem 13 for simply connected domains.

4.3.2. *Proof of Theorem* 13 *on general domains.* To estimate F_Ω^δ from below choose (v_i) realizing $F_{\mathrm{rad}}^\delta(0)$. The conformally rearranged sequence $(v_{i_{\Omega,x}})$ (Section

4.4) concentrates at x and $\lim F_\Omega(v_{i_{\Omega,x}}) = r_\Omega^2(x) \lim F_{\text{rad}}(v_i)$ by Theorem 18. Thus $F_\Omega^\delta(x) \geq r_\Omega^2(x) F_{\text{rad}}^\delta(0)$. For the opposite inequality choose a sequence (u_i) realizing $F_\Omega^\delta(x)$. We will show that there is a sequence $(\bar{u}_i)$ also realizing $F_\Omega^\delta(x)$, $r_i \to 0$ and $\lambda_i \to \infty$ such that:

1. $\bar{u}_i$ is harmonic for values <1 and $\{\bar{u}_i \geq 1\} \subset B(x, r_i)$.
2. $\bar{u}_i \to 0$ in $C_{\text{loc}}^k(\Omega \backslash \{x\})$ for all $k \geq 0$.
3. $\lambda_i \bar{u}_i \to G_{\Omega,x}$ in $C_{\text{loc}}^k(\Omega \backslash \{x\})$ for all $k \geq 0$.
4. The sets $\{\bar{u}_i \geq 1\}$ form a sequence of approximately small disks of radii $\rho_i = r_\Omega(x) \exp(-2\pi\lambda_i)$ at x.

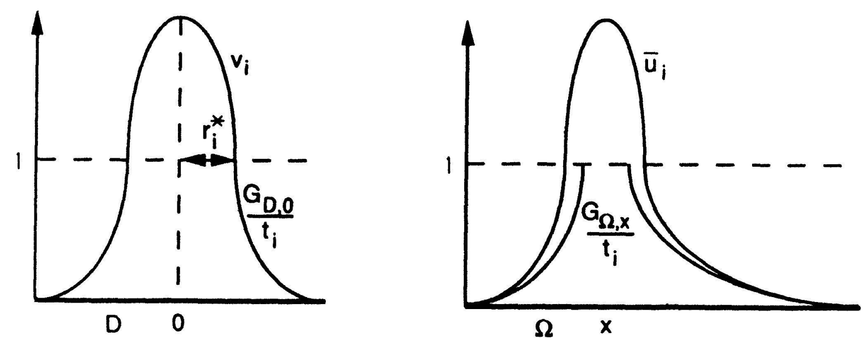
Once the sequence $(\bar{u}_i)$ is constructed proceed as follows. Replace λ_i by the least level $t_i \geq \lambda_i$ for which $\{G_{\Omega,x} \geq t_i\} \subset \{\bar{u}_i \geq 1\}$. By Lemma 10 the sets $\{G_{\Omega,x} \geq t_i\}$ are also approximately small disks of radii ρ_i at x. By Dirichlet's principle

$$\int_{\{G_{\Omega,x}/t_i < 1\}} \left| \nabla \frac{G_{\Omega,x}}{t_i} \right|^2 dx \leq \int_{\{\bar{u}_i < 1\}} |\nabla \bar{u}_i|^2 \, dx.$$

The left side is independent of Ω and x. In particular we can replace (Ω, x) by $(D, 0)$. Set

$$v_i(z) := \begin{cases} \dfrac{G_{D,0}(z)}{t_i} & \text{for values } < 1, \\[2ex] \bar{u}_i^* \left(\dfrac{\rho_i^*}{r_i^*} z \right) & \text{for values } \geq 1, \end{cases}$$

where ρ_i^* denotes the radius of the disk $\{\bar{u}_i \geq 1\}^*$ and $r_i^* := \exp(-2\pi t_i)$ is chosen such that the two pieces of v_i fit together.



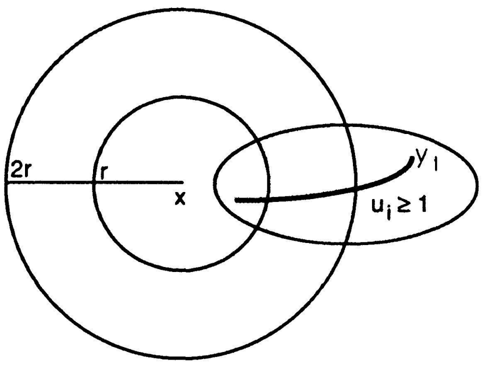
By construction $\|v_i\| \leq \|u_i\| \leq 1$, i.e. $v_i \in B_{\text{rad}}$. As to the functional observe that $\rho_i^* = \rho_i + o(\rho_i)$ and $r_i^* = \exp(-2\pi\lambda_i) + o(\exp(-2\pi\lambda_i))$, hence $\rho_i^*/r_i^* \to r_\Omega(x)$.

By Proposition 9

$$\lim_{i\to\infty} F_\Omega(\bar u_i) = \lim_{i\to\infty}\int_{\{\bar u_i \ge 1\}} f(\bar u_i)\,dx = \lim_{i\to\infty}\left(\frac{\rho_i^*}{r_i^*}\right)^2\int_{\{v_i\ge 1\}} f(v_i)\,dx$$
$$= r_\Omega^2(x)\lim_{i\to\infty} F_{\text{rad}}(v_i) \le r_\Omega^2(x) F_{\text{rad}}^\delta(0)$$

because (v_i) concentrates at 0. This completes the proof of Theorem 13 up to the construction of the sequence $(\bar u_i)$. Now we make up for this.

1. Fix $r > 0$ and assume there is a point $y_i \in \{u_i \ge 1\}\backslash B(x, 2r)$.



If y_i could be connected with $B(x, r)$ by an arc which is entirely contained in $\{u_i \ge 1\}$, then (u_i) could not concentrate at x because in 2 dimensions the capacity of a connected set of diameter r is bounded below by a positive number which only depends on r and Ω. This contradiction implies that for i large enough every component of $\{u_i \ge 1\}$ which intersects $B(x, r)$ is contained in $B(x, 2r)$. It allows to replace u_i by a function $\bar u_i \in H_0^{1,2}(\Omega)$ which coincides with u_i on the connected components of $\{u_i \ge 1\}$ which intersect $B(x, r)$ and is harmonic otherwise. Thus $\|\bar u_i\| \le \|u_i\|$ by Dirichlet's principle. Since r was arbitrary we can choose $r_i \to 0$ and a subsequence of $(\bar u_i)$ such that $\{\bar u_i \ge 1\} \subset B(x, r_i)$. By Step 3 and Proposition 9 there is a subsequence of (u_i) such that $\int_{(\Omega\backslash B(x,r_i))\cup\{u_i<1\}} f(u_i)\,dx \to 0$. The same holds for the sequence $(\bar u_i)$ which also concentrates at x. Therefore the limit of the functional remains unchanged.

2. The following argument is similar to that given by Schoen [13] (Theorem 3.3) for the Yamabe functional. Fix $r > 0$ such that $\Omega\backslash\bar B(x, r)$ is connected and a compact subset $K \subset \Omega\backslash\bar B(x, r)$. For i large enough $\bar u_i$ is a positive harmonic function on $\Omega\backslash\bar B(x, r)$. Since $\inf_K \bar u_i \to 0$ also $\sup_K \bar u_i \to 0$ by Harnack's inequality. Schauder's estimate implies $\sup_K |\nabla \bar u_i| \to 0$. By iterative application of Schauder's estimate the same follows for all derivatives of $\bar u_i$, since they are harmonic on $\Omega\backslash\{x\}$ as well. Since r was arbitrary we find $\bar u_i \to 0$ in $C^k_{\text{loc}}(\Omega\backslash\{x\})$ for all k.

3. Fix $y \neq x$ and set $\lambda_i := G_{\Omega,x}(y)/\bar{u}_i(y)$ and $u_i' := \lambda_i \bar{u}_i$. Then $\lambda_i \to \infty$. For K as before but containing y, Harnack's inequality and Schauder's estimate imply that (u_i') is bounded in all $C^k(K)$. By Ascoli's compactness theorem and since K was arbitrary there is a subsequence of (u_i') which converges in all $C^k_{\mathrm{loc}}(\Omega\backslash\{x\})$ to a function u'. By uniform convergence u' has to satisfy

$$\Delta u' = 0 \quad \text{on } \Omega\backslash\{x\}$$

$$u' = 0 \quad \text{on } \partial\Omega$$

and $u'(y) = G_{\Omega,x}(y)$. The only function with these properties is $u' = G_{\Omega,x}$ itself.

4. By Lemma 10 the sets $\{G_{\Omega,x} \geq \lambda_i\}$ form a sequence of approximately small disks of radii ρ_i. Since $\lambda_i \bar{u}_i \to G_{\Omega,x}$ in $C^1_{\mathrm{loc}}(\Omega\backslash\{x\})$ a subsequence of $\{\lambda_i \bar{u}_i \geq \lambda_i\}$ also consists of approximately small disks of radii ρ_i at x.

4.4. *Conformal rearrangement, mean value inequality*

On simply connected domains the mean value theorem implies what we call the *mean value inequality*

$$\frac{1}{2\pi r} \int_{\{|z| = r\}} |h'_{\Omega,x}(z)|^2 \, ds \geq |h'_{\Omega,x}(0)|^2.$$

We prove a generalization of this inequality to arbitrary domains. It will be essential in the proof of the functional isoperimetric inequality.

THEOREM 17 (Mean value inequality). *For any* $r \in (0, 1]$

$$\frac{1}{(2\pi r)^2} \int_{\{G_{\Omega,x} = -(2\pi)^{-1} \log(r)\}} \frac{ds}{|\nabla G_{\Omega,x}|} \geq r^2_{\Omega}(x).$$

This inequality tends to an equality as $r \to 0$.

Proof. The isoperimetric inequality for planar domains implies

$$4\pi |A| \leq \left(\int_{\partial A} ds\right)^2 \leq \left(\int_{\partial A} |\nabla u| \, ds\right)\left(\int_{\partial A} \frac{1}{|\nabla u|} \, ds\right).$$

In our case

$$\int_{\{G_{\Omega,x}=-(2\pi)^{-1}\log(r)\}}\frac{ds}{|\nabla G_{\Omega,x}|}\geq\frac{4\pi\left|\left\{G_{\Omega,x}>-\frac{1}{2\pi}\log(r)\right\}\right|}{\int_{\{G_{\Omega,x}=-(2\pi)^{-1}\log(r)\}}|\nabla G_{\Omega,x}|\,ds}$$

The denominator is $=1$ by Lemma 10. As to the numerator set

$$\Omega_r:=\{G_{\Omega,x}>-(2\pi)^{-1}\log(r)\}.$$

Then $G_{\Omega_r,x}=G_{\Omega,x}+(2\pi)^{-1}\log(r)$ and $H_{\Omega_r,x}=H_{\Omega,x}-(2\pi)^{-1}\log(r)$ and therefore $r_{\Omega_r}(x)=r_\Omega(x)r$. From Proposition 12 we know that $\pi R_\Omega^2\geq\pi r_\Omega^2(x)$, hence $|\Omega_r|\geq\pi r_{\Omega_r}^2(x)$. Plugging this into the above inequality yields the first claim. By Lemma 10

$$\left|\left\{G_{\Omega,x}=-\frac{1}{2\pi}\log(r)\right\}\right|=2\pi r_\Omega(x)r+O(r^2)$$

and

$$\frac{1}{|\nabla G_{\Omega,x}|}=2\pi r_\Omega(x)r+O(r^2)$$

on this level set. Thus

$$\frac{1}{(2\pi r)^2}\int_{\{G_{\Omega,x}=-(2\pi)^{-1}\log(r)\}}\frac{ds}{|\nabla G_{\Omega,x}|}=\frac{1}{(2\pi r)^2}(2\pi r_\Omega(x)r+O(r^2))^2$$

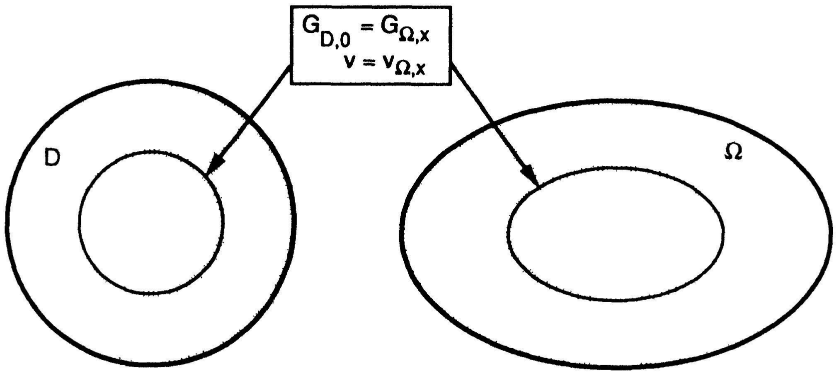
which tends to $r_\Omega^2(x)$ as $r\to 0$. □

This inequality is a generalization of the mean value inequality for simply connected domains as can be derived from the conformal invariance of the Green's function. For simply connected domains the proof of the functional isoperimetric inequality uses the conformal transformation of radially symmetric functions into functions on Ω. For the general case we introduce a rearrangement which generalizes this transformation and also preserves the Dirichlet integral.

DEFINITION (Conformal rearrangement). To every $v\in H_{0,\mathrm{rad}}^{1,2}(D)$ and $x\in\Omega$ associate its *conformal rearrangement* on Ω at x

$$v_{\Omega,x}:=v\circ G_{D,0}^{-1}\circ G_{\Omega,x}.$$

The function $v_{\Omega,x}$ is a constant on each level set of $G_{\Omega,x}$ namely the same constant as v takes on the corresponding level set of $G_{D,0}$. Equivalently, if $v = \phi \circ G_{D,0}$, then $v_{\Omega,x} = \phi \circ G_{\Omega,x}$.



If Ω is simply connected, then $v_{\Omega,x} = v \circ h_{\Omega,x}^{-1}$ by conformal invariance of the Green's function.

THEOREM 18. *Assume* $v \in H_{0,\mathrm{rad}}^{1,2}(D)$. *Then*
1. $v_{\Omega,x} \in H_0^{1,2}(\Omega)$ *and* $\|v_{\Omega,x}\| = \|v\|$.
2. *For every* $f \in C(\mathbb{R}, \mathbb{R}^+)$

$$\int_\Omega f(v_{\Omega,x})\,dx \geq r_\Omega^2(x) \int_D f(v)\,dx.$$

3. *If* (v_i) *concentrates at* 0, *then* $(v_{i_{\Omega,x}})$ *concentrates at* x *and*

$$\lim_{i\to\infty} F_\Omega(v_{i_{\Omega,x}}) = r_\Omega^2(x) \lim_{i\to\infty} F_{\mathrm{rad}}(v_i)$$

if the limits exits.

Proof. For simplicity we assume $\nabla v \neq 0$ except at the origin.

1. For simply connected domains this is just the conformal invariance of the Dirichlet integral. For the general case choose $y \in \Omega$ and $z \in D$ such that $G_{\Omega,x}(y) = G_{D,0}(z)$. Then $\nabla v_{\Omega,x}(y) = (|\nabla v(z)|/|\nabla G_{D,0}(z)|)\nabla G_{\Omega,x}(y)$. By the co-area formula (Federer [5] Theorem 3.2.12)

$$\|v_{\Omega,x}\|^2 = \int_0^\infty \int_{\partial\{v_{\Omega,x} > t\}} |\nabla v_{\Omega,x}|\,ds\,dt$$
$$= \int_0^\infty \frac{|\nabla v(z(t))|}{|\nabla G_{D,0}(z(t))|} \int_{\partial\{G_{\Omega,x} > G_{D,0}(z(t))\}} |\nabla G_{\Omega,x}|\,ds\,dt$$

where $z(t) \in v^{-1}(t)$. By Lemma 10 the inner integral is $=1$ independently of Ω and x. In particular we can replace (Ω, x) by $(D, 0)$ which yields the analogous expression for $\|v_{D,0}\| = \|v\|$.

2. This inequality follows from the mean value inequality using the radial symmetry of v. By the co-area formula

$$\begin{aligned}\int_\Omega f(v_{\Omega,x})\,dx &= \int_0^\infty f(t) \int_{\partial\{v_{\Omega,x}>t\}} \frac{ds}{|\nabla v_{\Omega,x}|}\,dt \\ &= \int_0^\infty f(t) \frac{|\nabla G_{D,0}(z(t))|}{|\nabla v(z(t))|} \int_{\partial\{G_{\Omega,x}>G_{D,0}(z(t))\}} \frac{ds}{|\nabla G_{\Omega,x}|}\,dt.\end{aligned}$$

Using $|\nabla G_{D,0}(z(t))| = (2\pi|z(t)|)^{-1}$ and $(|\nabla v(z(t))|)^{-1} = (2\pi|z(t)|)^{-1}\int_{\partial\{v>t\}}|\nabla v|^{-1}\,ds$ we can write this as

$$\int_0^\infty f(t)\int_{\partial\{v>t\}}\frac{ds}{|\nabla v|}\left[\frac{1}{(2\pi|z(t)|)^2}\int_{\partial\{G_{\Omega,x}>G_{D,0}(z(t))\}}\frac{ds}{|\nabla G_{\Omega,x}|}\right]dt.$$

By the mean value inequality the expression in square brackets is $\geq r_\Omega^2(x)$ and the claim follows.

3. From the first item we already know that $v_{i_{\Omega,x}} \in B_\Omega$. Furthermore

$$\int_{\{G_{\Omega,x}<t\}} |\nabla v_{i_{\Omega,x}}|^2\,dx = \int_{\{G_{D,0}<t\}} |\nabla v_i|^2\,dx$$

which tends to 0 for every $t > 0$. This shows that $(v_{i_{\Omega,x}})$ concentrates at x. As in the previous item the limit of $F_\Omega(v_{i_{\Omega,x}})$ can be written as

$$\lim_{i\to\infty}\int_1^\infty f(t)\int_{\partial\{v_i>t\}}\frac{ds}{|\nabla v_i|}\left[\frac{1}{(2\pi|z_i(t)|)^2}\int_{\partial\{G_{\Omega,x}>G_{D,0}(z_i(t))\}}\frac{ds}{|\nabla G_{\Omega,x}|}\right]dt.$$

The expression in square brackets tends to $r_\Omega^2(x)$ uniformly in $t \geq 1$, since $|z_i(t)| \leq |z_i(1)| \to 0$ as $i \to \infty$. The claim follows Proposition 9. □

4.5. *Functional isoperimetric inequality*

The proof of Theorem 5 is our main application of the conformal rearrangement just introduced. It allows the construction of a function $u \in B_\Omega$ on an arbitrary domain with $F_\Omega(u)$ above the critical level from a function with this property in the radially symmetric case. In the simply connected case choose $v \in B_{\mathrm{rad}}$ and $x \in \Omega$ and set $u := v \circ h_{\Omega,x}^{-1}$. Then

$$\frac{F_\Omega(u)}{F_\Omega^\delta(x)} = \frac{\int_D f(v)|h'_{\Omega,x}|^2\,dx}{r_\Omega^2(x)F_{\mathrm{rad}}^\delta(0)}$$

by Theorem 13. By radial symmetry of v the numerator can be written as

$$\int_0^1 2\pi r(f \circ v)\frac{1}{2\pi r}\int_{\{|z|=r\}} |h'_{\Omega,x}(z)|^2\,ds\,dr \ge r_\Omega^2(x)F_{\mathrm{rad}}(v)$$

using the mean value inequality. The conformal factor cancels and we get

$$\frac{\sup F_\Omega}{F_\Omega^\delta(x)} \ge \frac{F_{\mathrm{rad}}(v)}{F_{\mathrm{rad}}^\delta(0)} = \frac{F_{\mathrm{rad}}(v)}{\sup F_{\mathrm{rad}}^\delta}$$

for every $x \in \Omega$. Theorem 5 follows by taking the infimum over $x \in \bar{\Omega}$ (for $x \in \partial\Omega$ the left side is infinite) and the supremum over all $v \in B_{\mathrm{rad}}$. On general domains set $u := v_{\Omega,x}$ and Theorem 18 yields the same inequality.

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ETH–Zürich
HG G 36.1
Rämistr. 101
CH-8092 Zürich
Switzerland

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