# Classification of compact homogeneous pseudo-Kähler manifolds.

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# Classification of compact homogeneous pseudo-Kähler manifolds

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## Introduction

Compact homogeneous Kähler manifolds have been classified by Borel [1] and Matsushima [11] (see also Borel–Remmert [2]). Together with the flat homogeneous Kähler manifolds and the bounded homogeneous domains they form the building blocks of an arbitrary homogeneous Kähler manifold [4]. Since the proof of the Fundamental Conjecture for homogeneous Kähler manifolds [4] the structure of these manifolds is known. We are interested in considering more general classes of homogeneous complex manifolds.

One of the most natural generalizations of Kähler manifolds are pseudo-Kähler manifolds (see 1.1 for a definition).

In [5] and [6] we have classified all homogeneous pseudo-Kähler manifolds admitting a reductive transitive group of automorphisms.

In this note we classify all compact homogeneous pseudo-Kähler manifolds. Note that by an automorphism of a pseudo-Kähler manifold we always mean a biholomorphic map which leaves the pseudo-metric invariant. We prove

THEOREM A. *Let $M$ be a compact homogeneous pseudo-Kähler manifold and $G$ an effective transitive group of automorphisms of $M$. Then $G$ is reductive, and its semisimple part is compact.*

This and results from [5] and [6] then yield the main result of this paper.

THEOREM B. *Let $M$ be a compact homogeneous pseudo-Kähler manifold and $G$ an effective and transitive group of automorphisms of $M$. Then*

(a) *$G = C \times S$ where $C$ is a complex torus and $S$ is a compact semisimple Lie group with trivial center. In particular, $G$ is compact.*

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(b) *The isotropy subgroup H of a base point in M is contained in S and we have*

$$M = G/H = C \times S/H$$

*as a product of pseudo-Kähler manifolds where $S/H$ is a rational homogeneous space.*

(c) *The pseudo-Kähler structures on C and $S/H$ are a difference of Kähler structures.*

To prove that transitive groups of automorphisms of a compact pseudo-Kähler manifold are reductive we consider two natural fibrations of $M$, the Huckleberry–Oeljeklaus–Tits fibration and the Hano–Kobayashi fibration (see 1.2 and 1.4 for definitions). We show

THEOREM C. *Let M be a compact homogeneous manifold admitting an invariant volume form. Then the Huckleberry–Oeljeklaus–Tits fibration and the Hano–Kobayashi fibration of M are the same.*

This last theorem is the main result of §1. In §2 we prove part of the main result of this paper (Theorem B) under the assumption that $M$ is homogeneous under a reductive group of holomorphic transformations. In the last section (§3) we show that transitive groups of automorphisms of a pseudo-Kähler manifold are reductive and prove the main result quoted above.

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## §1. Two fibrations

1.1. Let $M$ be a complex manifold and $j$ its complex structure tensor. Let $\varphi$ be a (real) closed non-degenerate two-form on $M$, i.e. $(M, \varphi)$ is a *symplectic manifold.* If $\varphi$ is $j$-invariant, the $(M, \varphi)$ is called a *pseudo-Kähler manifold.* In this case

$$(X, Y) = \varphi(jX, Y) + i\varphi(X, Y)$$

is a *non-degenerate sesqui-linear form* on $M$, $\mathbb{C}$-linear in the first argument and $\mathbb{C}$-antilinear in the second argument.

A pseudo-Kähler manifold $(M, \varphi)$ is called *homogeneous* if there exists a Lie group $G \subset \text{Aut}(M, \varphi)$ that acts transitively on $M$. Here by $\text{Aut}(M, \varphi)$ we denote the group of biholomorphic maps of $M$ leaving $\varphi$ invariant. As usual, if $(M, \omega)$ is homogeneous we identify $M = G/H$ and we say that $G$ acts *effectively* if $H$ does not contain any normal subgroup of $G$. We say $G$ acts *almost effectively* if $\{g \in G; g \cdot p = p \text{ for all } p \in M\}$ is discrete in $G$.

1.2. In this section we recall some basic results on a generalization of the Tits fibration, introduced by A. Huckleberry and E. Oeljeklaus [9]. It coincides with a fibration considered by Hano [7] in case the isotropy group is connected. Using the initials of the authors involved in the development of this fibration we will talk about the *HOT-fibration* (instead of the $g$-anticanonical fibration [9]).

Denoting by $H_0$ the connected component of the identity in $H$ and by $\text{Norm}_G(H_0)$ the *normalizer* of $H_0$ in $G$ we have

THEOREM ([9]). *Let $G$ be a connected real Lie group acting almost effectively and transitively as a group of holomorphic transformations on the complex manifold $M = G/H$ and let $G/H \to G/J$ be the HOT-fibration.*

*Then*

(a) $J = \{k \in \text{Norm}_G(H_0);\ R(k): G/H_0 \to G/H_0,\ gH_0 \to gkH_0,$ *is holomorphic*$\}$ *where $G/H_0$ carries the complex structure induced by $G/H_0 \to G/H$.* *In particular we have $J \subset \text{Norm}_G(H_0)$.*

(b) *$J/H_0$ is a complex Lie group and $G/H_0 \to G/J$ is a holomorphic $J/H_0$-principal fiber bundle.* *In particular, the fibering $G/H \to G/J$ is locally holomorphically trivial.*

(c) *If $G$ is a connected complex Lie group and $H$ a closed complex subgroup, then $J = \text{Norm}_G(H_0)$.* *Thus for a complex Lie group $G$ the HOT-fibration coincides with the Tits fibration.*

1.3. For later use we will recall Tit's result on the fibration of compact homogeneous spaces

THEOREM ([13]). *Let $G$ be a connected complex Lie group and $H$ a closed complex subgroup such that $G/H$ is compact.*

*Then $G/\text{Norm}_G(H_0)$ is a rational homogeneous space and $\text{Norm}_G(H_0)/H$ is connected and parallelizable. Moreover, if $G/H \to G/R$ is a holomorphic fibration with parallelizable fiber $R/H$, then $R \subset \text{Norm}_G(H_0)$; if in addition the base $G/R$ is rational homogeneous, then $R = \text{Norm}_G(H_0)$.*

For definitions and results on rational homogeneous spaces we refer to the literature cited in [13]. We would like to point out however, that rational homogeneous spaces are simply connected. Moreover, if $G$ is a real Lie group such that $G/H$ is a compact complex manifold with $G$ acting holomorphically, then there exists a connected complex Lie group $G^{\mathbb{C}}$ such that $G \subset G^{\mathbb{C}}$ and $G/H = G^{\mathbb{C}}/H^{\mathbb{C}}$.

We would like to point out that in general $G^{\mathbb{C}}$ is *not* a complexification of $G$. But we can – and will – assume from now on that Lie $G^{\mathbb{C}}$ = Lie $G + i$ Lie $G$ holds.

From the definition of the HOT-fibration [9; §1.7] it is easy to see that $G/H$ and $G^{\mathbb{C}}/H^{\mathbb{C}}$ have the same HOT-fibration. We rephrase this more precisely in

PROPOSITION. *Let $G$ be a connected real Lie group acting almost effectively and transitively as a group of holomorphic transformations on the compact, complex manifold $G/H \cong G^{\mathbb{C}}/H^{\mathbb{C}}$.*

*Let $G/H \to G/J$ denote the HOT-fibration of $G/H$. Then the action of $G^{\mathbb{C}}$* on $G/H$ *preserves this fibration. Moreover, let $G^{\mathbb{C}}/H^{\mathbb{C}} \to G^{\mathbb{C}}/J^{\mathbb{C}}$ denote the Tits fibration. Then $J = J^{\mathbb{C}} \cap G$, i.e., $G/J \cong G^{\mathbb{C}}/J^{\mathbb{C}}$. Thus for compact $G/H$ the HOT-fibration and the Tits-fibration are the same. In particular, $J$ is connected and $G/J$ is rational homogeneous.*

1.4. Next we want to discuss the Hano–Kobayashi fibration. We will call this the *HK-fibration*. Let $M$ be a complex manifold and $\omega$ a volume form on $M$. Then locally we have $\omega = K(z, \bar{z})\, dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$. We also set

$$R_{i\bar{j}} = \frac{\partial^2 \log K}{\partial z^i\, \partial \bar{z}^j}$$

and

$$\chi = i \sum R_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

Then $\chi$ is called the *Ricci form* of $M$. For later use we recall the main result on the HK-fibration for homogeneous complex manifolds.

THEOREM ([8]). *Let $M$ be a connected complex manifold and $G$ a connected real Lie group acting holomorphically on $M$. Assume moreover that $M = G/H$ admits a $G$-invariant volume element $\omega$ and denote by $\chi$ the associated Ricci form of $M$.*

*Then there exists a unique closed subgroup $I$ of $G$ containing $H$ and a non-degenerate closed two-form $\hat{\chi}$ on $G/I$ such that*

(a) *$G/I$ is a homogeneous symplectic manifold with respect to $\hat{x}$ and the projection $G/H \to G/I$ is $G$-invariant.*

(b) *The fiber $I/H$ of this fibration is a complex connected submanifold of $G/H$ and* $\chi \mid I/H = 0$.
(c) *The pull-back of $\hat{\chi}$ to $M$ is equal to $\chi$.*
(d) *If $I/H$ is compact, then it is (complex) parallelizable.*

The fibration described in this Theorem will be called the HK-fibration.

1.5. In the rest of this paper we will use frequently arguments on the Lie algebra level.

First we recall the following result due to Koszul ([10]).

PROPOSITION. *Let $G$ be a real Lie group and $H$ a closed subgroup. Then $G/H$ admits a $G$-invariant complex structure if and only if there exists an endomorphism $j$ of $\underline{g} = \mathrm{Lie}\, G$ such that for all $x, y \in \underline{g}$, $r \in H$ we have* ($\underline{h} = \mathrm{Lie}\, H$)

$$j\underline{h} \subset \underline{h}, \tag{1.5.1}$$

$$j^2 x = -x (\mathrm{mod}\ \underline{h}), \tag{1.5.2}$$

$$\mathrm{Ad}\, r \cdot (jx) = j\, \mathrm{Ad}\, r \cdot x (\mathrm{mod}\ \underline{h}), \tag{1.5.3}$$

$$[jx, jy] = j[jx, y] + j[x, jy] + [x, y] (\mathrm{mod}\ \underline{h}). \tag{1.5.4}$$

Note that $j$ is only determined modulo $\underline{h}$. In what follows we *will always assume* $j\underline{h} = 0$.

1.6. We retain the notation and the assumptions of Proposition 1.5. In addition we assume that $M = G/H$ has a $G$-invariant volume form $\omega$. We set

$$\psi(x) = \mathrm{trace}_{\underline{g}/\underline{h}} (\mathrm{ad}\, jx - j\, \mathrm{ad}\, x), \qquad x \in \underline{g}. \tag{1.6.1}$$

Then

THEOREM ([10]). *The Ricci form associated with $\omega$ is given by the formula*

$$\chi(x, y) = \psi([x, y]), \qquad x, y \in \underline{g}. \tag{1.6.2}$$

*Moreover, the Ricci form satisfies for $x, y, z \in \underline{g}$*

$$\chi(jx, jy) = \chi(x, y), \tag{1.6.3}$$

$$\chi([x, y], z) + \chi([y, z], y) + \chi([z, y], x) = 0, \tag{1.6.4}$$

$$\chi(\underline{g}, \underline{h}) = 0. \tag{1.6.5}$$

*Remark.* If $M = G/H$ is a homogeneous pseudo-Kähler manifold, then $M$ has a $G$-invariant volume element and the results above apply to the associated Ricci form.

1.7. In the rest of this chapter we will compare the subgroups $I$ and $J$ associated with the HK-fibration (see 1.4) and the HOT-fibration (see 1.2) respectively. To be able to do this we consider a connected complex homogeneous manifold $M = G/H$, where $G$ is a real Lie group acting holomorphically on $M$. We also assume that $M$ admits a $G$-invariant volume form $\omega$. We set $\underline{g} = \text{Lie}\, G$ and $\underline{h} = \text{Lie}\, H$. From Theorem 1.2 and Theorem 1.4 it is easy to derive

$$\underline{j} = \text{Lie}\, J = \{x \in \underline{g};\ [x, jy] = j[x, y] (\text{mod}\, \underline{h}) \quad \text{for all } y \in \underline{g}\}, \tag{1.7.1}$$

$$\underline{i} = \text{Lie}\, I = \{x \in \underline{g};\ \chi(x, \underline{g}) = 0\}. \tag{1.7.2}$$

From [7] we know that $\underline{j}$ can also be described as follows: Let $\underline{g}^{\mathbb{C}}$ denote the complexification of $\underline{g}$ and set $\underline{g}_- = \{x + ijx;\ x \in \underline{g}\}$. Then $\underline{h} = \underline{g} \cap \underline{g}_-$ and

$$\underline{j} = \underline{g} \cap \text{norm}_{\underline{g}^{\mathbb{C}}} (\underline{g}_-). \tag{1.7.3}$$

Moreover, since we assume $j\underline{h} = 0$, (1.7.1) implies

$$\underline{j} \subset \text{norm}_{\underline{g}} (\underline{h}).$$

In particular, $\underline{h}$ is an ideal of $\underline{j}$.

1.8. We retain the notation and assumptions of the last section.

LEMMA. *Under the above assumptions we have* $\underline{j} \subset \underline{i}$.

*Proof.* Let $x \in \underline{j}$ and $y \in \underline{g}$. Then (1.7.1) implies $j[x, y] = [x, jy] + h$. Therefore

$$\begin{aligned} \text{ad}\,(j[x, y]) - j\, \text{ad}\,[x, y] &= \text{ad}\,[x, jy] + \text{ad}\, h - j[\text{ad}\, x, \text{ad}\, y] \\ &= [\text{ad}\, x, \text{ad}\, jy] + \text{ad}\, h - [\text{ad}\, x, j\, \text{ad}\, y] + [\text{ad}\, x, j]\, \text{ad}\, y \\ &= [\text{ad}\, x, \text{ad}\, jy - j\, \text{ad}\, y] + \text{ad}\, h + [\text{ad}\, x, j]\, \text{ad}\, y. \end{aligned}$$

We note that $\text{ad}\,(jy) - j\, \text{ad}\, y$ and $\text{ad}\, x$ leave $\underline{h}$ invariant. Therefore the trace of the first summand vanishes on $\underline{g}/\underline{h}$. Since $M$ admits an invariant volume form, we know $\text{trace}_{\underline{g}/\underline{h}}\, \text{ad}\, h = 0$ for all $h \in \underline{h}$. Finally, (1.7.1) implies $[\text{ad}\, x, j]\underline{g} \subset \underline{h}$, whence the last term vanishes on $\underline{g}/\underline{h}$. Altogether this shows $\chi(\underline{j}, \underline{g}) = 0$, proving the assertion.

1.9. In this section we prove the first main result of this paper (Theorem C of the introduction).

THEOREM. *Let M be a connected complex compact manifold and let G be a connected real Lie group acting transitively and holomorphically on M. Assume that* $M = G/H$ *admits a G-invariant volume element.*

*Then the Lie groups I and J defining the HK-fibration and the HOT-fibration are connected and equal.*

*In particular, the fiber of this fibration is complex parallelizable.*

*Proof.* From Proposition 1.3 we know that $J$ is connected. Hence Lemma 1.8 implies $H \subset J \subset I_0 \subset I$, where $I_0$ is the identity component of $I$. From 1.7 we know that $\underline{h}$ is an ideal of $\underline{j}$ and [14; Theorem 1] implies that $\underline{h}$ is an ideal of $\underline{i}$. Hence $J/H_0$ is a Lie subgroup of the Lie group $I_0/H_0$, where $H_0$ denotes the identity component of $H$. Moreover, from Theorem 1.4 and [9; §1.7, Corollary 5] it follows that $J/H_0$ and $I_0/H_0$ are actually complex Lie groups. Hence $I_0/J \subset G/J$ is a closed complex submanifold and therefore a projective manifold. Since $G/J$ is projective algebraic it embeds equivariantly into $\mathbb{P}_N$ [9; Chapter I, Theorem 6]. This implies that the maximal solvable subgroups of $I_0^{\mathbb{C}}$ have a fixed point in $I_0/J$ by Borel's Fixed Point Theorem [9; Chapter I]. Therefore the stabilizer of $I_0^{\mathbb{C}}$ at $e/J$ is parabolic and [9; Chapter I, Theorem 6] implies that $I_0/J$ is a rational homogeneous space. Finally, we consider the two complex fibrations $I_0/H_0 \to I_0/J$ and $I_0/H_0 \to I_0/I_0$. Both fibrations have rational homogeneous spaces as bases and parallelizable homogeneous spaces as fibers. Therefore, by the uniqueness of the Tits-fibration (1.2) we get $J = I_0$. From Part (b) of Theorem 1.4 we know that $I/H$ is connected. Since $H \subset I_0$, this implies $I = I_0 = J$.

COROLLARY. $\underline{i} = \text{Lie}\, I = \underline{j} = \text{Lie}\, J$.

## §2. The case of a reductive group action

The main goal of this section is to prove

THEOREM. *Let* $(M, \varphi)$ *be a connected compact symplectic manifold and let G be a connected reductive Lie group acting transitively and effectively on M. Assume moreover that G leaves* $\varphi$ *invariant.*

*Then* $M = G/H$ *and H is connected and compact. Moreover,* $\text{Lie}\, G' = [\text{Lie}\, G, \text{Lie}\, G]$ *is a semisimple compact subalgebra of* $\underline{g}$, $\text{Lie}\, H \subset \text{Lie}\, G'$ *and there exists some* $w \in \text{Lie}\, G'$ *such that* $\text{Lie}\, H = \{x \in \text{Lie}\, G';\ [x, w] = 0\}$.

*Proof.* Let $\tilde{G}$ be the universal covering group of $G$ and $\pi : \tilde{G} \to G$ the covering homomorphism. Set $\tilde{H} = \pi^{-1}(H)$. Since $\tilde{G}/\tilde{H} = G/H = M$ is compact and symplectic, we know that $M$ admits a finite invariant measure. Hence, by a result of Selberg (see e.g. [12; Lemma 5.4]), $\tilde{H}$ has "property ($S$) in $\tilde{G}$", i.e. for any neighborhood $\tilde{M}$ of the identity of $\tilde{G}$ and for any element $g \in \tilde{G}$, there exists an integer $n > 0$ such that $g^n \in \tilde{M}\tilde{H}\tilde{M}$.

Next, since $\tilde{G}$ is simply connected and reductive, we obtain $\tilde{G} \cong \tilde{G}_n \times \tilde{C} \times \tilde{G}_c$, where $\tilde{G}_n$ corresponds to the sum of the non-compact factors in Lie $G$, $\tilde{G}_c$ to the sum of the compact factors and $\tilde{C}$ to the center in Lie $G$. Let $\pi_n : \tilde{G} \to \tilde{G}_n$ be the canonical projection. Then $\pi_n(\tilde{H})$ is a subgroup of $\tilde{G}_n$ having property ($S$) in $\tilde{G}_n$. Since $\tilde{G}_n$ has no compact factors we can apply Borel's Density Theorem (see e.g. [12; Corollary 5.16]) and obtain that the Lie algebra $\underline{h}_n = d\pi_n(\text{Lie } \tilde{H})$ is an ideal of $\underline{g}_n = \text{Lie } \tilde{G}_n$. On the other hand we know $\text{Lie } G = \underline{g} = \underline{g}_n + \underline{c} + \underline{g}_c = \text{Lie } G_n + \text{Lie } C + \text{Lie } G_c$. Moreover, from a result of Matsushima [11; Theorem 1] we know that the identity component $H_0$ of $H$ is contained in the maximal semisimple subgroup $S$ of $G$ and that there exists an element $w \in \underline{s} = \text{Lie } S = \underline{g}_n + \underline{g}_c$ such that $\underline{h} = \text{Lie } H = \{x \in \underline{s};\ [x, w] = 0\}$. Therefore, splitting $w = w_n + w_c$, $w_n \in \underline{g}_n$, $w_c \in \underline{g}_c$, we obtain that $\underline{h}_n = d\pi_n(\text{Lie } \tilde{H})$ is the centralizer of $w_n$ in $\underline{g}_n$. From this it is easy to derive, since $\underline{g}$ is reductive, that $\underline{h}_n \subset \underline{h}$ is an ideal of $\underline{g}$. Since $G$ acts effectively, $\underline{h}_n = 0$. This implies $\underline{g}_n = 0$. Therefore $G$ itself has no non-compact factor. Matsushima's result thus implies that $H_0$ is contained in the (maximal) compact factor of $G$. In particular, $H_0$ is compact. Hence, again using [11; Theorem 1] we see that $H$ is connected, whence also compact. This finishes the proof of the Theorem.

## §3. Reductivity of $G$

3.1. In this section we consider a compact pseudo-Kähler manifold $(M, \varphi)$. We assume that there exists a connected real Lie group $G$ acting holomorphically, effectively and transitively on $M$.

The goal of this chapter is to prove that $G$ is reductive.

To fix some notation we note that we have $M = G/H$, where $H$ is some closed subgroup of $G$.

We set $\underline{g} = \text{Lie } G$ and $\underline{h} = \text{Lie } H$. In what follows we will use intensively the Lie algebras $\underline{i}$ and $\underline{j}$ as described in section 1.7.

We also set $\underline{r} = \text{rad}\ (\underline{g})$ and denote by $\underline{s}$ a maximal semisimple subalgebra of $\underline{g}$. Moreover, by $\underline{s}_n$ and $\underline{s}_c$ we denote the sum of all noncompact and all compact summands of $\underline{s}$ respectively.

3.2. In this section we prove

LEMMA. *With the notation and under the assumptions of* 3.1 *we have*
(a) $\underline{i} = \underline{r} + \underline{s}_0 + \underline{i}_c$, *where* $\underline{s} = \underline{s}_0 + \underline{s}_c''$, $\underline{s}_0 = \underline{s}_n + \underline{s}_c'$ *and* $\underline{s}_c'$ *and* $\underline{s}_c''$ *are ideals of* $\underline{s}_{\underline{c}}$.
(b) $\underline{i}_c$ *is the centralizer of some* $w_c \in \underline{i}_c$ *in* $\underline{s}_{\underline{c}}$.

*Proof.* From Theorem 1.9 we know that the HOT-fibration and the HK-fibration are the same. Therefore $G/I$ is a rational homogeneous, compact, pseudo-Kählerian manifold realtive to $\hat{\chi}$, the two-form on $G/I$ induced from the Ricci form $\chi$ on $M = G/H$. Moreover, from [13; Theorem 4.1] we know rad (Lie $G^{\mathbb{C}}$) $\subset$ Lie $J^{\mathbb{C}}$, whence $\underline{r} \subset \underline{i} = \underline{j}$ holds. Let $\underline{q}$ denote the maximal ideal of $\underline{g}$ contained in $\underline{i}$ and $Q$ the maximal (normal) subgroup of $G$ satisfying Lie $Q = \underline{q}$. Then $G/Q$ acts transitively and effectively on $G/I$. Since $\underline{r} \subset \underline{q}$, we know that $\underline{g}/\underline{q}$ is semisimple. Thus the Theorem in §2 implies that $\underline{g}/\underline{q}$ is a semisimple and compact Lie algebra. Moreover, $\underline{h}/\underline{q}$ is the centralizer of some element $[w] \in \underline{g}/\underline{q}$. From this the Lemma follows.

COROLLARY. *With the notation and under the assumption of* 3.1 *the algebra* $\underline{i}_{\underline{c}}$ *is reductive, i.e.* $\underline{i}_{\underline{c}} = \underline{c}_c + \underline{c}_s$, *where* $\underline{c}_{\underline{s}}$ *is semisimple and* $\underline{c}_{\underline{c}}$ *is abelian.*

3.3. Our assumption always was that $G$ be a real Lie group. In case $G$ is actually a complex Lie group, we have

LEMMA. *We retain the notation and the assumptions of* 3.1. *Moreover we assume that* $G$ *is a complex Lie group. Then* $G/H$ *is a complex abelian Lie group.*

*Proof.* Let $\varphi$ denote the pullback of the given pseudo-Kähler form on $G/H$. This can be written $\varphi = \Sigma_{i=1}^{n} c_i\omega_i \wedge \bar{\omega}_i$ where $\omega_1, \ldots, \omega_n$ is a basis for the Maurer–Cartan forms of $\underline{g}$. Let us assume that $\omega_1, \ldots, \omega_k$ are a basis for the Maurer–Cartan forms of $\underline{h}$. Since $\varphi$ is pseudo-Kählerian, we know $c_i = 0$ for $i \leq k$, and $c_i \neq 0$ for $i > k$. The closedness condition of $\varphi$ implies $0 = d\varphi = \Sigma\, c_i(\omega_i \wedge d\bar{\omega}_i + d\omega_i \wedge \bar{\omega}_i)$. Note that here the first term is of type $(1, 2)$ and the second is of type $(2, 1)$. Therefore $0 = \Sigma\, c_i\omega_i \wedge d\bar{\omega}_i$ and $0 = \Sigma\, c_i\, d\omega_i \wedge \bar{\omega}_i$. But $d\omega_i = \frac{1}{2}\Sigma_{r,s}\, c^i_{rs}\omega_r \wedge \omega_s$, where $c^i_{rs}$ denotes the structure constants of $\underline{g}$ (see [3; §IV]). Therefore, $c^i_{rs} = 0$ for all $i > k$ and all $r, s$. This implies $[\underline{g}, \underline{g}] \subset \underline{h}$, and the assertion follows.

3.4. Next we want to restrict our attention to the subalgebra $\underline{i}$ of $\underline{g}$. We set

$$\underline{h}' = \{x \in \underline{i};\ \varphi(x, \underline{i}) = 0\}. \tag{3.4.1}$$

It is easy to see that $\underline{h}'$ is $j$-invariant. From 1.7 it follows that $\underline{h}'/\underline{h}$ is a complex subalgebra of the complex Lie algebra $\underline{i}/\underline{h}$. Moreover, the two form $\hat{\varphi}$ induced from

$\varphi$ on $\underline{i}/\underline{h}$ is non-degenerate and $j$-invariant modulo $\underline{h}'/\underline{h}$. Therefore, from Lemma 3.3, we obtain

$$\hat{\underline{v}} = (\underline{i}/\underline{h})/(\underline{h}'/\underline{h}) \text{ is abelian.} \tag{3.4.2}$$

This implies in particular

$$\underline{h}' \text{ is an ideal of } \underline{i}. \tag{3.4.3}$$

We set $\underline{r}' = \operatorname{rad}(\underline{i})$. Then

$$\underline{r}' = \underline{r} + \underline{c}_c. \tag{3.4.4}$$

Moreover, since $\underline{h}'$ is an ideal of $\underline{i}$, we have

$$\underline{h}' = \underline{r}' \cap \underline{h}' + (\underline{s}_0 + \underline{c}_s) \cap \underline{h}'. \tag{3.4.5}$$

We also know that $\underline{h}$ is an ideal of $\underline{i}$, consequently

$$\underline{h} = \underline{r}' \cap \underline{h} + (\underline{s}_0 + \underline{c}_s) \cap h. \tag{3.4.6}$$

More precisely, $(\underline{s}_0 + \underline{c}_s) \cap \underline{h} = \underline{s}_0' + \underline{c}_s'$, where $\underline{s}_0'$ and $\underline{c}_s'$ is a direct summand of $\underline{s}_0$ and $\underline{c}_s$ respectively. Therefore, $\underline{i}/\underline{h} \cong \underline{r}'/\underline{r}' \cap \underline{h} + \underline{s}_0/\underline{s}_0' + \underline{c}_s/\underline{c}_s'$. But since $\underline{i}/\underline{h}$ is a complex Lie algebra and $\underline{c}_s/\underline{c}_s'$ is a semisimple compact Lie algebra (or $=0$), we obtain $\underline{c}_s = \underline{c}_s' \subset \underline{h}$. Thus

$$\underline{h} = \underline{r}' \cap \underline{h} + \underline{s}_0' + \underline{c}_s. \tag{3.4.7}$$

By the same argument we see $\underline{s}_c' = \underline{s}_0 \cap \underline{s}_c \subset \underline{s}_0'$. Next we look at $\underline{h}'$. We know $(\underline{s}_0 + \underline{c}) \cap \underline{h}' = \underline{s}_0'' + \underline{c}_s$, where $\underline{s}_0''$ is an ideal of $\underline{s}_0$ containing $\underline{s}_0'$. Then $\underline{h}'/\underline{h} \cong \underline{r}' \cap \underline{h}'/\underline{r}' \cap \underline{h} + \underline{s}_0''/\underline{s}_0'$ and $\underline{i}/\underline{h} \cong \underline{r}'/\underline{r}' \cap \underline{h}' + \underline{s}_0/\underline{s}_0'$. Therefore $\hat{\underline{v}} = (\underline{i}/\underline{h})/(\underline{h}'/\underline{h}) \cong \underline{r}'/\underline{r}' \cap \underline{h}' + \underline{s}_0/\underline{s}_0''$. But $\hat{\underline{v}}$ is abelian by (3.4.2), whence $\underline{s}_0 = \underline{s}_0'' \subset \underline{h}'$. We thus have shown

$$\underline{h}' = \underline{r}' \cap \underline{h}' + \underline{s}_0 + \underline{c}_s. \tag{3.4.8}$$

3.5. In the following sections we will use the decompositions derived above to clarify the structures of $\underline{i}$. As usual, by nil $(\underline{i})$ we denote the nilradical of $\underline{i}$. We retain the notation and the assumptions used above.

LEMMA. $\operatorname{nil}(\underline{i}) \subset \underline{r}' \cap \underline{h}'$.

*Proof.* Consider the action of the semisimple Lie algebra $\underline{s}_0 + \underline{c}_s$ on $\underline{i}$. Then $\underline{i} = \underline{r}' \cap \underline{h}' + \underline{a} + \underline{s}_0 + \underline{c}_s$, where $\underline{a}$ is invariant under $\underline{s}_0 + \underline{c}_s$. But since $\underline{h}'$ is an ideal of $\underline{i}$ and $\underline{s}_0 + \underline{c}_s \subset \underline{h}'$, this implies $[\underline{s}_0 + \underline{c}_s, \underline{a}] = 0$. Also, since $\hat{\underline{v}} \cong \underline{i}/\underline{h}'$ is abelian, $[\underline{a}, \underline{a}] \subset \underline{h}'$. From this it follows $[\underline{i}, \underline{r}'] \subset \underline{h}'$, thus the claim.

COROLLARY 1. $[\underline{r}, [\underline{r}, \underline{r}]] = 0$.

*Proof.* As usual, by $\underline{s}$ we denote a maximal semisimple subalgebra of $\underline{g}$. Then $\varphi([\underline{r}, [\underline{r}, \underline{r}]], \underline{s}) \subset \varphi(\underline{r}, [\underline{r}, \underline{r}]) = 0$, since $\underline{r} \subset \underline{i}$ and $[\underline{r}, \underline{r}] \subset \operatorname{nil}(\underline{i}) \subset \underline{h}'$. Since $[\underline{r}, [\underline{r}, \underline{r}]] \subset \operatorname{nil}(\underline{i}) \subset \underline{h}'$ and $\underline{r} \subset \underline{i}$ we also have $\varphi([\underline{r}, [\underline{r}, \underline{r}]], \underline{r}) = 0$, therefore $[\underline{r}, [\underline{r}, \underline{r}]] \subset \underline{h}$. But $[\underline{r}, [\underline{r}, \underline{r}]]$ is an ideal of $\underline{g}$, whence the claim.

COROLLARY 2. $\operatorname{ad} \underline{r}$ *consists of nilpotent endomorphisms of* $\underline{g}$.

3.6. The goal of this section is to show (still under the usual assumptions of this chapter)

LEMMA. $\underline{s}_0 = 0$.

*Proof.* Since $\underline{s}_0 \subset \underline{h}'$ and $\underline{r} \subset \underline{i}$, we have $\varphi(\underline{s}_0, \underline{r}) = 0$. Moreover, using the notation of 3.1 we have $\varphi(\underline{s}_0, \underline{s}_c) = \varphi(\underline{s}_0, [\underline{s}_c, \underline{s}_c]) = 0$. This shows that $\varphi$ is nondegenerate on $\underline{s}_0/\underline{s}_0'$. From the closedness condition of $\varphi$ we obtain $\varphi(x, y) = \beta(b, [x, y])$ for all $x, y \in \underline{s}_0$, where $\beta$ denotes the Killing form of $\underline{s}_0$. From this we derive $\underline{s}_0' = \{x \in \underline{s}_0; [x, b] = 0\}$. But $\underline{s}_0'$ is an ideal of $\underline{s}_0$, hence $\underline{s}_0 = \underline{s}_0'$. Since we know now $\underline{s}_0 \subset \underline{h}$ and $\underline{r} \subset \underline{i}$, clearly $[\underline{s}_0, \underline{r}] \subset \underline{h} \cap \underline{r}$. It is easy to see that $[\underline{s}_0, \underline{r}]$ is invariant under $\underline{s} = \underline{s}_0 + \underline{i}_c$. Therefore, the ideal of $\underline{g}$ generated by $[\underline{s}_0, \underline{r}]$ is contained in $\underline{h}$, whence $[\underline{s}_0, \underline{r}] = 0$. Thus $\underline{s}_0$ is an ideal of $\underline{g}$, but $\underline{s}_0 \subset \underline{h}$ and $\underline{s}_0 = 0$ follows.

3.7. In this section we prove a result that will be used frequently in the rest of this chapter. We retain the notation and the assumptions of this chapter.

LEMMA. *Let* $x_0 \in \underline{g}$ *and assume* $[x_0, \underline{r}] \subset \underline{h}$. *Moreover assume that* $\operatorname{ad} x_0$ *is semisimple on* $\underline{g}/\underline{r}$. *Then* $S_{\underline{r}} = 0$, *where* $S$ *denotes the semisimple part of* $\operatorname{ad} x_0$.

*Proof.* Let $\operatorname{ad} x_0 = S + N$ the decomposition of $\operatorname{ad} x_0$ into its semisimple part $S$ and its nilpotent part $N$. We can assume that $S$ leaves $\underline{s}$ invariant [4; Appendix]. Moreover, since $S$ and $N$ are polynomials in $\operatorname{ad} x_0$ without constant term, $S\underline{r} \subset \underline{h} \cap \operatorname{nil}(\underline{g})$ and $N\underline{r} \subset \underline{h} \cap \operatorname{nil}(\underline{g})$. Let $\underline{r}^{\mathbb{C}} = \bigoplus \underline{r}_\alpha^{\mathbb{C}}$ be the decomposition of $\underline{r}^{\mathbb{C}}$, the complexification of $\underline{r}$, into eigenspaces relative to $S$. Then

$$\underline{r}_\alpha^{\mathbb{C}} \subset (\underline{h} \cap \operatorname{nil}(\underline{g}))^{\mathbb{C}} \qquad \text{for all } \alpha \neq 0. \tag{3.7.1}$$

Suppose there exists some $\alpha \neq 0$. In what follows we fix such an $\alpha$. Let $\underline{s}_\beta^{\mathbf{C}}$ be any eigenspace of $S$ in $\underline{s}^{\mathbf{C}}$. Then

$$[\underline{s}_\beta^{\mathbf{C}}, \underline{r}_\alpha^{\mathbf{C}}] \subset \underline{r}_{\alpha+\beta}^{\mathbf{C}} \subset \underline{h}^{\mathbf{C}} \qquad \text{if } \alpha + \beta \neq 0. \tag{3.7.2}$$

If $\beta = -\alpha$, then

$$\varphi(\underline{s}_\gamma^{\mathbf{C}}, [\underline{s}_{-\alpha}^{\mathbf{C}}, \underline{r}_{-\alpha}^{\mathbf{C}}]) = 0 \qquad \text{if } \gamma + \alpha \neq 0. \tag{3.7.3}$$

Indeed, $\varphi(x_\gamma, [y_{-\alpha}, z_\alpha]) = -\varphi([x_\gamma, z_\alpha], y_{-\alpha}) = 0$ if $x_\gamma \in \underline{s}_\gamma^{\mathbf{C}}$, $y_{-\alpha} \in \underline{s}_{-\alpha}^{\mathbf{C}}$, $z_a \in \underline{r}_\alpha^{\mathbf{C}} \subset \underline{h}^{\mathbf{C}}$, and $\gamma + \alpha \neq 0$ since in this case $[\underline{s}_\gamma^{\mathbf{C}}, \underline{r}_\alpha^{\mathbf{C}}] \subset \underline{r}_{\alpha+\gamma}^{\mathbf{C}} \subset \underline{h}^{\mathbf{C}}$ by (3.7.1).

Consider now the case $\gamma = -\alpha$. From our assumption we obtain $\underline{s}_{-\alpha}^{\mathbf{C}} = S\underline{s}_{-\alpha}^{\mathbf{C}} \subset [x_0, \underline{s}_{-\alpha}^{\mathbf{C}}] + \operatorname{nil}(\underline{g})^{\mathbf{C}}$. Hence, $\varphi(\underline{s}_{-\alpha}^{\mathbf{C}}, [\underline{s}_{-\alpha}^{\mathbf{C}}, \underline{r}_\alpha^{\mathbf{C}}]) \subset \varphi([x_0, \underline{s}_{-\alpha}^{\mathbf{C}}] + \operatorname{nil}(\underline{g})^{\mathbf{C}}, [\underline{s}_{-\alpha}^{\mathbf{C}}, \underline{r}_\alpha^{\mathbf{C}}]) \subset \varphi(\underline{s}_{-\alpha}^{\mathbf{C}}, [x_0, \operatorname{nil}(\underline{g})^{\mathbf{C}}]) + \varphi([\operatorname{nil}(\underline{g})^{\mathbf{C}}, \underline{r}_\alpha^{\mathbf{C}}], \underline{s}_{-\alpha}^{\mathbf{C}}) = 0$, since $[x_0, \operatorname{nil}(\underline{g})^{\mathbf{C}}] \subset \underline{h}^{\mathbf{C}}$ and $[\operatorname{nil}(\underline{g})^{\mathbf{C}}, \underline{r}_\alpha^{\mathbf{C}}] \subset \underline{h}^{\mathbf{C}}$. Therefore we have

$$\varphi(\underline{s}_{-\alpha}^{\mathbf{C}}, [\underline{s}_{-\alpha}^{\mathbf{C}}, \underline{r}_\alpha^{\mathbf{C}}]) = 0. \tag{3.7.4}$$

As a consequence of the above results we obtain

$$\varphi(\underline{s}^{\mathbf{C}}, [\underline{s}^{\mathbf{C}}, \underline{r}_\alpha^{\mathbf{C}}]) = 0 \qquad \text{if } \alpha \neq 0. \tag{3.7.5}$$

Since $\underline{r}_\gamma^{\mathbf{C}} \subset \underline{h}^{\mathbf{C}}$ for $\gamma \neq 0$, we clearly have $\varphi(\underline{r}_\gamma^{\mathbf{C}}, \underline{g}) = 0$ in this case. If $\gamma = 0$, then $\varphi(\underline{r}_0^{\mathbf{C}}, [\underline{s}^{\mathbf{C}}, \underline{r}_\alpha^{\mathbf{C}}]) = \varphi([\underline{r}_0^{\mathbf{C}}, \underline{r}_\alpha^{\mathbf{C}}], \underline{s}^{\mathbf{C}}]) = 0$, since $[\underline{r}_0^{\mathbf{C}}, \underline{r}_\alpha^{\mathbf{C}}] \subset \underline{r}_\alpha^{\mathbf{C}} \subset \underline{h}^{\mathbf{C}}$. Thus, altogether we have shown

$$\varphi(\underline{r}^{\mathbf{C}}, [\underline{s}^{\mathbf{C}}, \underline{r}_\alpha^{\mathbf{C}}]) = 0. \tag{3.7.6}$$

Equations (3.7.5) and (3.7.6) together imply

$$[\underline{s}^{\mathbf{C}}, \underline{r}_\alpha^{\mathbf{C}}] \subset \underline{h}^{\mathbf{C}}. \tag{3.7.7}$$

Next we consider the vector space $\underline{q}^{\mathbf{C}} \subset \underline{h}^{\mathbf{C}} \cap \underline{r}^{\mathbf{C}}$ spanned by the subspaces $\underline{r}_\alpha^{\mathbf{C}}$ and $[\underline{s}^{\mathbf{C}}, \underline{r}_\alpha^{\mathbf{C}}]$, $\alpha \neq 0$. It is easy to see that $\underline{q}^{\mathbf{C}}$ is invariant under complex conjugation relative to $\underline{q}$.

$$\underline{q}^{\mathbf{C}} \text{ is an } \underline{s}\text{-module.} \tag{3.7.8}$$

Indeed, consider $A = [\underline{s}_\gamma^{\mathbf{C}}, [\underline{s}_\beta^{\mathbf{C}}, \underline{r}_\alpha^{\mathbf{C}}]$. If $\beta + \alpha \neq 0$, then the inner commutator is contained in $\underline{r}_{\alpha+\beta}^{\mathbf{C}}$, whence $A \subset \underline{q}^{\mathbf{C}}$. If $\beta + \alpha = 0$, then we use $A = [[\underline{s}_{-\gamma}^{\mathbf{C}}, \underline{s}_{-\alpha}^{\mathbf{C}}], \underline{r}_\alpha^{\mathbf{C}}] + [\underline{s}_{-\alpha}^{\mathbf{C}}, [\underline{s}_{-\gamma}^{\mathbf{C}}, \underline{r}_\alpha^{\mathbf{C}}]$. Clearly, the first summand is in $\underline{q}^{\mathbf{C}}$. In the second summand we have $[\underline{s}_\gamma^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}] \subset \underline{r}_{\alpha+\beta}^{\mathbb{C}} \subset \underline{h}^{\mathbb{C}}$ if $\alpha + \gamma \neq 0$. If $\alpha + \gamma = 0$, then the whole second summand is contained in $\underline{r}_{-\alpha}^{\mathbb{C}}$, finishing the proof of (3.7.8). Now it is straight forward to verify that the ideal of $\underline{g}$ generated by $\underline{q}^{\mathbb{C}} \cap \underline{g}$ is actually contained in $\underline{h}$. But since the transitive group $G$ in question acts effectively, this ideal is trivial. In particular we have $\underline{r}_\alpha^{\mathbb{C}} = 0$ for all $\alpha \neq 0$. Therefore $S\underline{r} = 0$, proving the assertion.

3.8. In this section we continue our investigation of $\underline{s}$. Since we know from 3.6 that $\underline{s}_0 = 0$ holds, $\underline{s}$ is compact. We split $\underline{s} = \underline{s}_a + \underline{s}_b$, where

$$\underline{s}_a = \{x \in \underline{s}; [x, r] = 0\} \tag{3.8.1}$$

and $\underline{s}_b$ is a complementary ideal of $\underline{s}_a$ in $\underline{s}$. Since $\underline{i}_c$ is the centralizer of some element in $\underline{s}$,

$$\underline{i}_c = \underline{i}_a + \underline{i}_b, \qquad \text{where } \underline{i}_* = \underline{i}_c \cap s_*, \; * = a, b. \tag{3.8.2}$$

Since $\underline{i}_a$ and $\underline{i}_b$ are reductive, with obvious notation we have

$$\underline{i}_a = \underline{c}_a + \underline{c}_{sa} \quad \text{and} \quad \underline{i}_b + \underline{c}_b + \underline{c}_{sb}. \tag{3.8.3}$$

LEMMA. (a) $\underline{s}_a$ *and* $\underline{s}_b + \underline{r}$ are ideals of $\underline{g}$.
(b) $\underline{h} = \underline{h} \cap \underline{r} + \underline{c}_b + \underline{c}_a + \underline{c}_{sa}$.

*Proof.* Clearly, $\underline{s}_a$ and $\underline{s}_b + \underline{r}$ are ideals of $\underline{g}$. Moreover, we have $\varphi(\underline{s}_a, \underline{r}) = \varphi([\underline{s}_a, \underline{s}_a], \underline{r}) = 0$ and similarly $\varphi(\underline{s}_a, \underline{s}_b) = 0$. Therefore, $\underline{s}_a$ and $\underline{s}_b + \underline{r}$ are perpendicular. This implies $\underline{h} = \underline{h} \cap \underline{s}_a + \underline{h} \cap (\underline{s}_b + \underline{r})$. From Lemma 3.1 it follows that $\underline{h} \cap \underline{s}_a$ is the centralizer of some $w_a \in \underline{s}_a$.

Now let $x_0 \in \underline{h} \cap (\underline{r} + \underline{c}_a + \underline{c}_b)$. Clearly, $[x_0, \underline{r}] \subset \underline{h}$, since $\underline{h}$ is an ideal of $\underline{i}$ and $\underline{r} \subset \underline{i}$. Moreover, ad $x_0$ is semisimple on $\underline{g}/\underline{r}$. Therefore, by the last lemma $S\underline{r} = 0$, where $S$ denotes the semisimple part of ad $x_0$. In view of Corollary 3.7.1 we can write $\underline{r} = \underline{a} + [\underline{a}, \underline{a}]$ where $[\underline{s}, \underline{a}] \subset \underline{a}$. Hence $x_0 = c + a + n$ with $c \in \underline{c}_a + \underline{c}_b$, $a \in \underline{a}$, and $n \in [\underline{a}, \underline{a}]$. Note $[n, \underline{r}] = 0$ by Corollary 3.5.1. Therefore ad $x_0 \mid \underline{r} = \text{ad}\,(c + a) \mid \underline{r}$. Since we know that the semisimple part of ad $x_0$ vanishes on $\underline{r}$, the endomorphism $A = \text{ad}\,(c + a) \mid \underline{r}$ is nilpotent. But ad $c \mid \underline{r}$ is semisimple and leaves $\underline{a}$ and $[\underline{a}, \underline{a}]$ invariant, while ad $\underline{a}$ maps $\underline{a}$ into $[\underline{a}, \underline{a}]$ and annihilates $[\underline{a}, \underline{a}]$. This shows ad $c \mid \underline{a} = 0$ and ad $c \mid [\underline{a}, \underline{a}] = 0$, whence $[c, \underline{r}] = 0$. Therefore, $c \in \underline{c}_a$ and the assertion follows.

3.9. Clearly, to show that $\underline{g}$ is reductive, we have to prove $\underline{s}_b = 0$. This is the goal of this section.

LEMMA. $\underline{s}_b = 0$.

*Proof.* From Corollary 3.7 we know that ad $\underline{r}$ consists of nilpotent endomorphisms of $\underline{g}$. Moreover, ad $c$, $c \in \underline{c}_b$, is semisimple on $\underline{g}$ and has only purely imaginary eigenvalues. Restricting ad $(\underline{r} + \underline{c}_b)$ to the complex Lie algebra $\underline{i}/\underline{h}$ we obtain the radical of $\underline{i}/\underline{h}$. But this is a complex solvable Lie algebra, whence ad $\underline{c}_b \mid \underline{i}/\underline{h} = 0$. In particular we get $[\underline{c}_b, \underline{r}] \subset \underline{h}$. From Lemma 3.7 we thus obtain $[\underline{c}_b, \underline{r}] = 0$, i.e. $\underline{c}_b = 0$. From Lemma 3.7 it follows easily that $\underline{c}_{bs} = 0$ holds. Thus $\underline{s}_b = 0$.

3.10. With the results of the previous sections it will be easy now to prove (Theorem A of the introduction).

THEOREM. *Let $(M, \varphi)$ be a compact connected pseudo-Kähler manifold and $G$ an effective transitive group of automorphisms of $M$. Then $G$ is reductive and its semisimple part is compact.*

*Proof.* From Lemma 3.9 it follows that $\underline{g} = \underline{r} + \underline{s}_a$, where $[\underline{r}, \underline{s}_a] = 0$ and $\underline{s}_a$ is semisimple. Moreover, $\underline{h} = \underline{h} \cap \underline{r} + \underline{h} \cap \underline{s}_a$. Therefore, the radical of $\underline{i}/\underline{h}$ is $\underline{r}/\underline{h} \cap \underline{r}$. Since this is $j$-invariant we can assume $j\underline{r} \subset \underline{r}$. Also, $\underline{h} \cap \underline{r}$ is an ideal of $\underline{g}$ contained in $\underline{h}$, hence $\underline{h} \cap \underline{r} = 0$. This implies $\underline{h} = \underline{c}_a + \underline{c}_{as}$, by Lemma 3.8, and $\underline{i}/\underline{h} \cong \underline{r}$. In particular, $\underline{r}$ is a complex Lie algebra and $\varphi(\underline{r}, \underline{s}_a) = 0$ shows that $(\underline{r}, 0, j, \varphi)$ is a pseudo-Kähler algebra. Thus Lemma 3.3 shows that $\underline{r}$ is abelian. Therefore $\underline{g}$ is reductive, proving the assertion.

3.11. In this section we will give the *proof of Theorem B* of the introduction.

First we note that Theorem A (see 3.10) shows that $G$ is reductive and its semisimple part $S$ is compact. From the Theorem in §2 we thus obtain that the isotropy subgroup $H$ of $G$ is connected, compact and contained in the maximal semisimple Lie subgroup $S$ of $G$. From [11; Theorem 1] it thus follows that $S$ has trivial center and that $G = C \times S$ holds. Clearly, $G/H = C \times S/H$. Since $G/H$ is compact, we see that $C$ is a complex torus. In particular, $G$ is compact. It is easy to see that Lie $C$ and Lie $S$ are perpendicular relative to the given pseudo-Kählerian structure. Thus $G/H = C \times S/H$ is the product of pseudo-Kähler manifolds.

Therefore it only remains to prove that $S/H$ is a rational homogeneous manifold and that the given pseudo-Kähler structures on $C$ and $S/H$ are a difference of Kähler structures. The first statement follows from 3.9, since $\underline{i} = \underline{r} + \underline{c}_{as}$, where $\underline{h} = \underline{c}_a = \underline{c}_{as}$ and $\underline{r} =$ Lie $C$. The second statement follows from [6].

*Added in proof.* Recently we received the preprint: A. T. Huckleberry, Homogeneous pseudo-Kählerian manifolds: A hamiltonian viewpoint. In this paper Theorem B is proven by a different method.

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