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# Classification of compact homogeneous pseudo-Kähler manifolds

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## Introduction

Compact homogeneous Kähler manifolds have been classified by Borel [1] and Matsushima [11] (see also Borel-Remmert [2]). Together with the flat homogeneous Kähler manifolds and the bounded homogeneous domains they form the building blocks of an arbitrary homogeneous Kähler manifold [4]. Since the proof of the Fundamental Conjecture for homogeneous Kähler manifolds [4] the structure of these manifolds is known. We are interested in considering more general classes of homogeneous complex manifolds.

One of the most natural generalizations of Kähler manifolds are pseudo-Kähler manifolds (see 1.1 for a definition).

In [5] and [6] we have classified all homogeneous pseudo-Kähler manifolds admitting a reductive transitive group of automorphisms.

In this note we classify all compact homogeneous pseudo-Kähler manifolds. Note that by an automorphism of a pseudo-Kähler manifold we always mean a biholomorphic map which leaves the pseudo-metric invariant. We prove

THEOREM A. Let M be a compact homogeneous pseudo-Kähler manifold and G an effective transitive group of automorphisms of M. Then G is reductive, and its semisimple part is compact.

This and results from [5] and [6] then yield the main result of this paper.

THEOREM B. Let M be a compact homogeneous pseudo-Kähler manifold and G an effective and transitive group of automorphisms of M. Then

(a)  $G = C \times S$  where C is a complex torus and S is a compact semisimple Lie group with trivial center. In particular, G is compact.

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(b) The isotropy subgroup H of a base point in M is contained in S and we have

 $M = G/H = C \times S/H$ 

as a product of pseudo-Kähler manifolds where S/H is a rational homogeneous space.

(c) The pseudo-Kähler structures on C and S/H are a difference of Kähler structures.

To prove that transitive groups of automorphisms of a compact pseudo-Kähler manifold are reductive we consider two natural fibrations of M, the Huckleberry– Oeljeklaus–Tits fibration and the Hano–Kobayashi fibration (see 1.2 and 1.4 for definitions). We show

THEOREM C. Let M be a compact homogeneous manifold admitting an invariant volume form. Then the Huckleberry–Oeljeklaus–Tits fibration and the Hano– Kobayashi fibration of M are the same.

This last theorem is the main result of  $\S1$ . In  $\S2$  we prove part of the main result of this paper (Theorem B) under the assumption that M is homogeneous under a reductive group of holomorphic transformations. In the last section ( $\S3$ ) we show that transitive groups of automorphisms of a pseudo-Kähler manifold are reductive and prove the main result quoted above.

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# **§1.** Two fibrations

1.1. Let M be a complex manifold and j its complex structure tensor. Let  $\varphi$  be a (real) closed non-degenerate two-form on M, i.e.  $(M, \varphi)$  is a symplectic manifold. If  $\varphi$  is *j*-invariant, the  $(M, \varphi)$  is called a *pseudo-Kähler manifold*. In this case

 $(X, Y) = \varphi(jX, Y) + i\varphi(X, Y)$ 

is a non-degenerate sesqui-linear form on M,  $\mathbb{C}$ -linear in the first argument and  $\mathbb{C}$ -antilinear in the second argument.

A pseudo-Kähler manifold  $(M, \varphi)$  is called *homogeneous* if there exists a Lie group  $G \subset \operatorname{Aut}(M, \varphi)$  that acts transitively on M. Here by  $\operatorname{Aut}(M, \varphi)$  we denote the group of biholomorphic maps of M leaving  $\varphi$  invariant. As usual, if  $(M, \varphi)$  is homogeneous we identify M = G/H and we say that G acts *effectively* if H does not contain any normal subgroup of G. We say G acts *almost effectively* if  $\{g \in G; g \cdot p = p \text{ for all } p \in M\}$  is discrete in G.

1.2. In this section we recall some basic results on a generalization of the Tits fibration, introduced by A. Huckleberry and E. Oeljeklaus [9]. It coincides with a fibration considered by Hano [7] in case the isotropy group is connected. Using the initials of the authors involved in the development of this fibration we will talk about the *HOT-fibration* (instead of the g-anticanonical fibration [9]).

Denoting by  $H_0$  the connected component of the identity in H and by Norm<sub>G</sub> ( $H_0$ ) the normalizer of  $H_0$  in G we have

THEOREM ([9]). Let G be a connected real Lie group acting almost effectively and transitively as a group of holomorphic transformations on the complex manifold M = G/H and let  $G/H \rightarrow G/J$  be the HOT-fibration.

- Then
- (a)  $J = \{k \in \text{Norm}_G(H_0); R(k) : G/H_0 \to G/H_0, gH_0 \to gkH_0, is holomorphic\}$ where  $G/H_0$  carries the complex structure induced by  $G/H_0 \to G/H$ . In particular we have  $J \subset \text{Norm}_G(H_0)$ .
- (b)  $J/H_0$  is a complex Lie group and  $G/H_0 \rightarrow G/J$  is a holomorphic  $J/H_0$ -principal fiber bundle.

In particular, the fibering  $G/H \rightarrow G/J$  is locally holomorphically trivial.

(c) If G is a connected complex Lie group and H a closed complex subgroup, then  $J = \operatorname{Norm}_G(H_0)$ . Thus for a complex Lie group G the HOT-fibration coincides with the Tits fibration.

1.3. For later use we will recall Tit's result on the fibration of compact homogeneous spaces

THEOREM ([13]). Let G be a connected complex Lie group and H a closed complex subgroup such that G/H is compact.

Then  $G/\operatorname{Norm}_G(H_0)$  is a rational homogeneous space and  $\operatorname{Norm}_G(H_0)/H$  is connected and parallelizable. Moreover, if  $G/H \to G/R$  is a holomorphic fibration with parallelizable fiber R/H, then  $R \subset \operatorname{Norm}_G(H_0)$ ; if in addition the base G/R is rational homogeneous, then  $R = \operatorname{Norm}_G(H_0)$ .

For definitions and results on rational homogeneous spaces we refer to the literature cited in [13]. We would like to point out however, that rational homogeneous spaces are simply connected. Moreover, if G is a real Lie group such that G/H is a compact complex manifold with G acting holomorphically, then there exists a connected complex Lie group  $G^{\mathbb{C}}$  such that  $G \subset G^{\mathbb{C}}$  and  $G/H = G^{\mathbb{C}}/H^{\mathbb{C}}$ .

We would like to point out that in general  $G^{\mathbb{C}}$  is not a complexification of G. But we can – and will – assume from now on that Lie  $G^{\mathbb{C}} = \text{Lie } G + i$  Lie G holds.

From the definition of the HOT-fibration [9; §1.7] it is easy to see that G/H and  $G^{\mathbb{C}}/H^{\mathbb{C}}$  have the same HOT-fibration. We rephrase this more precisely in

**PROPOSITION.** Let G be a connected real Lie group acting almost effectively and transitively as a group of holomorphic transformations on the compact, complex manifold  $G/H \cong G^{\mathbb{C}}/H^{\mathbb{C}}$ .

Let  $G/H \to G/J$  denote the HOT-fibration of G/H. Then the action of  $G^{\mathbb{C}}$  on G/Hpreserves this fibration. Moreover, let  $G^{\mathbb{C}}/H^{\mathbb{C}} \to G^{\mathbb{C}}/J^{\mathbb{C}}$  denote the Tits fibration. Then  $J = J^{\mathbb{C}} \cap G$ , i.e.,  $G/J \cong G^{\mathbb{C}}/J^{\mathbb{C}}$ . Thus for compact G/H the HOT-fibration and the Tits-fibration are the same. In particular, J is connected and G/J is rational homogeneous.

1.4. Next we want to discuss the Hano-Kobayashi fibration. We will call this the *HK*-fibration. Let M be a complex manifold and  $\omega$  a volume form on M. Then locally we have  $\omega = K(z, \bar{z}) dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$ . We also set

$$R_{i\bar{j}} = \frac{\partial^2 \log K}{\partial z^i \, \partial \bar{z}^j}$$

and

$$\chi = i \sum R_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

Then  $\chi$  is called the *Ricci form* of *M*. For later use we recall the main result on the HK-fibration for homogeneous complex manifolds.

THEOREM ([8]). Let M be a connected complex manifold and G a connected real Lie group acting holomorphically on M. Assume moreover that M = G/H admits a G-invariant volume element  $\omega$  and denote by  $\chi$  the associated Ricci form of M.

Then there exists a unique closed subgroup I of G containing H and a non-degenerate closed two-form  $\hat{\chi}$  on G/I such that

(a) G/I is a homogeneous symplectic manifold with respect to  $\hat{x}$  and the projection  $G/H \rightarrow G/I$  is G-invariant.

- (b) The fiber I/H of this fibration is a complex connected submanifold of G/H and  $\chi \mid I/H = 0$ .
- (c) The pull-back of  $\hat{\chi}$  to M is equal to  $\chi$ .
- (d) If I/H is compact, then it is (complex) parallelizable.

The fibration described in this Theorem will be called the HK-fibration.

1.5. In the rest of this paper we will use frequently arguments on the Lie algebra level.

First we recall the following result due to Koszul ([10]).

**PROPOSITION.** Let G be a real Lie group and H a closed subgroup. Then G/H admits a G-invariant complex structure if and only if there exists an endomorphism j of  $\underline{g} = \text{Lie } G$  such that for all  $x, y \in \underline{g}, r \in H$  we have  $(\underline{h} = \text{Lie } H)$ 

$$j\underline{h} \subset \underline{h},\tag{1.5.1}$$

$$j^2 x = -x \pmod{\underline{h}},\tag{1.5.2}$$

$$\operatorname{Ad} r \cdot (jx) = j \operatorname{Ad} r \cdot x (\operatorname{mod} \underline{h}), \tag{1.5.3}$$

$$[jx, jy] = j[jx, y] + j[x, jy] + [x, y] (\text{mod } \underline{h}).$$
(1.5.4)

Note that j is only determined modulo <u>h</u>. In what follows we will always assume  $j\underline{h} = 0$ .

1.6. We retain the notation and the assumptions of Proposition 1.5. In addition we assume that M = G/H has a G-invariant volume form  $\omega$ . We set

$$\psi(x) = \operatorname{trace}_{\underline{g}/\underline{h}} (\operatorname{ad} jx - j \operatorname{ad} x), \qquad x \in \underline{g}.$$
(1.6.1)

Then

THEOREM ([10]). The Ricci form associated with  $\omega$  is given by the formula

$$\chi(x, y) = \psi([x, y]), \qquad x, y \in g.$$
 (1.6.2)

Moreover, the Ricci form satisfies for  $x, y, z \in g$ 

$$\chi(jx, jy) = \chi(x, y), \tag{1.6.3}$$

 $\chi([x, y], z) + \chi([y, z], y) + \chi([z, y], x) = 0, \qquad (1.6.4)$ 

$$\chi(g,\underline{h}) = 0. \tag{1.6.5}$$

*Remark.* If M = G/H is a homogeneous pseudo-Kähler manifold, then M has a G-invariant volume element and the results above apply to the associated Ricci form.

1.7. In the rest of this chapter we will compare the subgroups I and J associated with the HK-fibration (see 1.4) and the HOT-fibration (see 1.2) respectively. To be able to do this we consider a connected complex homogeneous manifold M = G/H, where G is a real Lie group acting holomorphically on M. We also assume that M admits a G-invariant volume form  $\omega$ . We set  $\underline{g} = \text{Lie } G$  and  $\underline{h} = \text{Lie } H$ . From Theorem 1.2 and Theorem 1.4 it is easy to derive

$$\underline{j} = \text{Lie } J = \{x \in \underline{g}; [x, jy] = \underline{j}[x, y] \pmod{\underline{h}} \text{ for all } y \in \underline{g}\},$$
(1.7.1)

$$\underline{i} = \text{Lie } I = \{ x \in g; \, \chi(x, g) = 0 \}.$$
(1.7.2)

From [7] we know that  $\underline{j}$  can also be described as follows: Let  $\underline{g}^{\mathbb{C}}$  denote the complexification of g and set  $g_{-} = \{x + ijx; x \in g\}$ . Then  $\underline{h} = g \cap g_{-}$  and

$$\underline{j} = \underline{g} \cap \operatorname{norm}_{g^{\mathbb{C}}}(\underline{g}_{-}). \tag{1.7.3}$$

Moreover, since we assume  $j\underline{h} = 0$ , (1.7.1) implies

 $\underline{j} \subset \operatorname{norm}_{g}(\underline{h}).$ 

In particular,  $\underline{h}$  is an ideal of j.

1.8. We retain the notation and assumptions of the last section.

LEMMA. Under the above assumptions we have  $\underline{j} \subset \underline{i}$ . Proof. Let  $x \in \underline{j}$  and  $y \in \underline{g}$ . Then (1.7.1) implies j[x, y] = [x, jy] + h. Therefore ad (j[x, y]) - j ad [x, y] = ad [x, jy] + ad h - j[ad x, ad y] = [ad x, ad jy] + ad h - [ad x, j ad y] + [ad x, j] ad y= [ad x, ad jy - j ad y] + ad h + [ad x, j] ad y.

We note that  $\operatorname{ad}(jy) - j$  ad y and  $\operatorname{ad} x$  leave <u>h</u> invariant. Therefore the trace of the first summand vanishes on  $\underline{g}/\underline{h}$ . Since M admits an invariant volume form, we know  $\operatorname{trace}_{\underline{g}/\underline{h}}$  ad h = 0 for all  $h \in \underline{h}$ . Finally, (1.7.1) implies  $[\operatorname{ad} x, j]\underline{g} \subset \underline{h}$ , whence the last term vanishes on  $\underline{g}/\underline{h}$ . Altogether this shows  $\chi(\underline{j}, \underline{g}) = 0$ , proving the assertion.

1.9. In this section we prove the first main result of this paper (Theorem C of the introduction).

THEOREM. Let M be a connected complex compact manifold and let G be a connected real Lie group acting transitively and holomorphically on M. Assume that M = G/H admits a G-invariant volume element.

Then the Lie groups I and J defining the HK-fibration and the HOT-fibration are connected and equal.

In particular, the fiber of this fibration is complex parallelizable.

*Proof.* From Proposition 1.3 we know that J is connected. Hence Lemma 1.8 implies  $H \subset J \subset I_0 \subset I$ , where  $I_0$  is the identity component of I. From 1.7 we know that <u>h</u> is an ideal of j and [14; Theorem 1] implies that <u>h</u> is an ideal of <u>i</u>. Hence  $J/H_0$ is a Lie subgroup of the Lie group  $I_0/H_0$ , where  $H_0$  denotes the identity component of H. Moreover, from Theorem 1.4 and [9; §1.7, Corollary 5] it follows that  $J/H_0$ and  $I_0/H_0$  are actually complex Lie groups. Hence  $I_0/J \subset G/J$  is a closed complex submanifold and therefore a projective manifold. Since G/J is projective algebraic it embeds equivariantly into  $\mathbb{P}_{N}$  [9; Chapter I, Theorem 6]. This implies that the maximal solvable subgroups of  $I_0^{\mathbb{C}}$  have a fixed point in  $I_0/J$  by Borel's Fixed Point Theorem [9; Chapter I]. Therefore the stabilizer of  $I_0^{\mathbb{C}}$  at e/J is parabolic and [9; Chapter I, Theorem 6] implies that  $I_0/J$  is a rational homogeneous space. Finally, we consider the two complex fibrations  $I_0/H_0 \rightarrow I_0/J$  and  $I_0/H_0 \rightarrow I_0/I_0$ . Both fibrations have rational homogeneous spaces as bases and parallelizable homogeneous spaces as fibers. Therefore, by the uniqueness of the Tits-fibration (1.2) we get  $J = I_0$ . From Part (b) of Theorem 1.4 we know that I/H is connected. Since  $H \subset I_0$ , this implies  $I = I_0 = J$ .

COROLLARY.  $\underline{i} = \text{Lie } I = \underline{j} = \text{Lie } J.$ 

### §2. The case of a reductive group action

The main goal of this section is to prove

THEOREM. Let  $(M, \varphi)$  be a connected compact symplectic manifold and let G be a connected reductive Lie group acting transitively and effectively on M. Assume moreover that G leaves  $\varphi$  invariant.

Then M = G/H and H is connected and compact. Moreover, Lie G' =[Lie G, Lie G] is a semisimple compact subalgebra of  $\underline{g}$ , Lie  $H \subset$  Lie G' and there exists some  $w \in$  Lie G' such that Lie  $H = \{x \in$  Lie G';  $[x, w] = 0\}$ . *Proof.* Let  $\tilde{G}$  be the universal covering group of G and  $\pi : \tilde{G} \to G$  the covering homomorphism. Set  $\tilde{H} = \pi^{-1}(H)$ . Since  $\tilde{G}/\tilde{H} = G/H = M$  is compact and symplectic, we know that M admits a finite invariant measure. Hence, by a result of Selberg (see e.g. [12; Lemma 5.4]),  $\tilde{H}$  has "property (S) in  $\tilde{G}$ ", i.e. for any neighborhood  $\tilde{M}$  of the identity of  $\tilde{G}$  and for any element  $g \in \tilde{G}$ , there exists an integer n > 0 such that  $g^n \in \tilde{M}\tilde{H}\tilde{M}$ .

Next, since  $\tilde{G}$  is simply connected and reductive, we obtain  $\tilde{G} \cong \tilde{G}_n \times \tilde{C} \times \tilde{G}_c$ , where  $\tilde{G}_n$  corresponds to the sum of the non-compact factors in Lie G,  $\tilde{G}_c$  to the sum of the compact factors and  $\tilde{C}$  to the center in Lie G. Let  $\pi_n: \tilde{G} \to \tilde{G}_n$  be the canonical projection. Then  $\pi_n(\tilde{H})$  is a subgroup of  $\tilde{G}_n$  having property (S) in  $\tilde{G}_n$ . Since  $\tilde{G}_n$  has no compact factors we can apply Borel's Density Theorem (see e.g. [12; Corollary 5.16]) and obtain that the Lie algebra  $\underline{h}_n = d\pi_n$  (Lie  $\tilde{H}$ ) is an ideal of  $g_n = \text{Lie } \tilde{G}_n$ . On the other hand we know  $\text{Lie } G = g = g_n + \underline{c} + g_{\underline{c}} = \text{Lie } G_n + \underline{c}$ Lie  $C + \text{Lie } G_c$ . Moreover, from a result of Matsushima [11; Theorem 1] we know that the identity component  $H_0$  of H is contained in the maximal semisimple subgroup S of G and that there exists an element  $w \in \underline{s} = \text{Lie } S = g_n + g_c$  such that  $\underline{h} = \text{Lie } H = \{x \in \underline{s}; [x, w] = 0\}$ . Therefore, splitting  $w = w_n + w_c, w_n \in g_n, w_c \in g_c$ , we obtain that  $\underline{h}_n = d\pi_n$  (Lie  $\tilde{H}$ ) is the centralizer of  $w_n$  in  $\underline{g}_n$ . From this it is easy to derive, since g is reductive, that  $\underline{h}_n \subset \underline{h}$  is an ideal of g. Since G acts effectively,  $\underline{h}_n = 0$ . This implies  $\underline{g}_n = 0$ . Therefore G itself has no non-compact factor. Matsushima's result thus implies that  $H_0$  is contained in the (maximal) compact factor of G. In particular,  $H_0$  is compact. Hence, again using [11; Theorem 1] we see that H is connected, whence also compact. This finishes the proof of the Theorem.

### §3. Reductivity of G

3.1. In this section we consider a compact pseudo-Kähler manifold  $(M, \varphi)$ . We assume that there exists a connected real Lie group G acting holomorphically, effectively and transitively on M.

The goal of this chapter is to prove that G is reductive.

To fix some notation we note that we have M = G/H, where H is some closed subgroup of G.

We set  $\underline{g} = \text{Lie } G$  and  $\underline{h} = \text{Lie } H$ . In what follows we will use intensively the Lie algebras  $\underline{i}$  and j as described in section 1.7.

We also set  $\underline{r} = rad(\underline{g})$  and denote by  $\underline{s}$  a maximal semisimple subalgebra of  $\underline{g}$ . Moreover, by  $\underline{s}_n$  and  $\underline{s}_c$  we denote the sum of all noncompact and all compact summands of  $\underline{s}$  respectively. 3.2. In this section we prove

LEMMA. With the notation and under the assumptions of 3.1 we have (a)  $\underline{i} = \underline{r} + \underline{s}_0 + \underline{i}_c$ , where  $\underline{s} = \underline{s}_0 + \underline{s}'_c$ ,  $\underline{s}_0 = \underline{s}_n + \underline{s}'_c$  and  $\underline{s}'_c$  and  $\underline{s}''_c$  are ideals of  $\underline{s}_c$ . (b)  $\underline{i}_c$  is the centralizer of some  $w_c \in \underline{i}_c$  in  $\underline{s}_c$ .

*Proof.* From Theorem 1.9 we know that the HOT-fibration and the HK-fibration are the same. Therefore G/I is a rational homogeneous, compact, pseudo-Kählerian manifold realtive to  $\hat{\chi}$ , the two-form on G/I induced from the Ricci form  $\chi$  on M = G/H. Moreover, from [13; Theorem 4.1] we know rad (Lie  $G^{\mathbb{C}}) \subset$  Lie  $J^{\mathbb{C}}$ , whence  $\underline{r} \subset \underline{i} = \underline{j}$  holds. Let  $\underline{q}$  denote the maximal ideal of  $\underline{g}$  contained in  $\underline{i}$  and Q the maximal (normal) subgroup of G satisfying Lie  $Q = \underline{q}$ . Then G/Q acts transitively and effectively on G/I. Since  $\underline{r} \subset \underline{q}$ , we know that  $\underline{g}/\underline{q}$  is semisimple. Thus the Theorem in §2 implies that  $\underline{g}/\underline{q}$  is a semisimple and compact Lie algebra. Moreover,  $\underline{h}/q$  is the centralizer of some element  $[w] \in \underline{g}/\underline{q}$ . From this the Lemma follows.

COROLLARY. With the notation and under the assumption of 3.1 the algebra  $\underline{i}_{\underline{c}}$  is reductive, i.e.  $\underline{i}_{\underline{c}} = \underline{c}_{\underline{c}} + \underline{c}_{\underline{s}}$ , where  $\underline{c}_{\underline{s}}$  is semisimple and  $\underline{c}_{\underline{c}}$  is abelian.

3.3. Our assumption always was that G be a real Lie group. In case G is actually a complex Lie group, we have

LEMMA. We retain the notation and the assumptions of 3.1. Moreover we assume that G is a complex Lie group. Then G/H is a complex abelian Lie group.

*Proof.* Let  $\varphi$  denote the pullback of the given pseudo-Kähler form on G/H. This can be written  $\varphi = \sum_{i=1}^{n} c_i \omega_i \wedge \bar{\omega}_i$  where  $\omega_1, \ldots, \omega_n$  is a basis for the Maurer-Cartan forms of  $\underline{g}$ . Let us assume that  $\omega_1, \ldots, \omega_k$  are a basis for the Maurer-Cartan forms of  $\underline{h}$ . Since  $\varphi$  is pseudo-Kählerian, we know  $c_i = 0$  for  $i \le k$ , and  $c_i \ne 0$  for i > k. The closedness condition of  $\varphi$  implies  $0 = d\varphi = \sum c_i (\omega_i \wedge d\bar{\omega}_i + d\omega_i \wedge \bar{\omega}_i)$ . Note that here the first term is of type (1, 2) and the second is of type (2, 1). Therefore  $0 = \sum c_i \omega_i \wedge d\bar{\omega}_i$  and  $0 = \sum c_i d\omega_i \wedge \bar{\omega}_i$ . But  $d\omega_i = \frac{1}{2} \sum_{r,s} c_{rs}^i \omega_r \wedge \omega_s$ , where  $c_{rs}^i$  denotes the structure constants of  $\underline{g}$  (see [3; §IV]). Therefore,  $c_{rs}^i = 0$  for all i > k and all r, s. This implies  $[\underline{g}, \underline{g}] \subset \underline{h}$ , and the assertion follows.

3.4. Next we want to restrict our attention to the subalgebra  $\underline{i}$  of g. We set

$$\underline{h}' = \{ x \in \underline{i}; \, \varphi(x, \underline{i}) = 0 \}. \tag{3.4.1}$$

It is easy to see that  $\underline{h}'$  is *j*-invariant. From 1.7 it follows that  $\underline{h}'/\underline{h}$  is a complex subalgebra of the complex Lie algebra  $\underline{i}/\underline{h}$ . Moreover, the two form  $\hat{\varphi}$  induced from

 $\varphi$  on  $\underline{i}/\underline{h}$  is non-degenerate and *j*-invariant modulo  $\underline{h}'/\underline{h}$ . Therefore, from Lemma 3.3, we obtain

$$\underline{\hat{v}} = (\underline{i}/\underline{h})/(\underline{h}'/\underline{h}) \text{ is abelian.}$$
(3.4.2)

This implies in particular

$$\underline{h}'$$
 is an ideal of  $\underline{i}$ . (3.4.3)

We set  $\underline{r}' = \operatorname{rad}(\underline{i})$ . Then

$$\underline{r}' = \underline{r} + \underline{c}_c. \tag{3.4.4}$$

Moreover, since  $\underline{h}'$  is an ideal of  $\underline{i}$ , we have

$$\underline{h}' = \underline{r}' \cap \underline{h}' + (\underline{s}_0 + \underline{c}_s) \cap \underline{h}'. \tag{3.4.5}$$

We also know that  $\underline{h}$  is an ideal of  $\underline{i}$ , consequently

$$\underline{h} = \underline{r}' \cap \underline{h} + (\underline{s}_0 + \underline{c}_s) \cap h. \tag{3.4.6}$$

More precisely,  $(\underline{s}_0 + \underline{c}_s) \cap \underline{h} = \underline{s}'_0 + \underline{c}'_s$ , where  $\underline{s}'_0$  and  $\underline{c}'_s$  is a direct summand of  $\underline{s}_0$ and  $\underline{c}_s$  respectively. Therefore,  $\underline{i}/\underline{h} \cong \underline{r}'/\underline{r}' \cap \underline{h} + \underline{s}_0/\underline{s}'_0 + \underline{c}_s/\underline{c}'_s$ . But since  $\underline{i}/\underline{h}$  is a complex Lie algebra and  $\underline{c}_s/\underline{c}'_s$  is a semisimple compact Lie algebra (or =0), we obtain  $\underline{c}_s = \underline{c}'_s \subset \underline{h}$ . Thus

$$\underline{h} = \underline{r}' \cap \underline{h} + \underline{s}'_0 + \underline{c}_s. \tag{3.4.7}$$

By the same argument we see  $\underline{s}'_c = \underline{s}_0 \cap \underline{s}_c \subset \underline{s}'_0$ . Next we look at  $\underline{h}'$ . We know  $(\underline{s}_0 + \underline{c}) \cap \underline{h}' = \underline{s}''_0 + \underline{c}_s$ , where  $\underline{s}''_0$  is an ideal of  $\underline{s}_0$  containing  $\underline{s}'_0$ . Then  $\underline{h}'/\underline{h} \cong \underline{r}' \cap \underline{h}'/\underline{r}' \cap \underline{h} + \underline{s}''_0/\underline{s}'_0$  and  $\underline{i}/\underline{h} \cong \underline{r}'/\underline{r}' \cap \underline{h}' + \underline{s}_0/\underline{s}'_0$ . Therefore  $\underline{\hat{v}} = (\underline{i}/\underline{h})/(\underline{h}'/\underline{h}) \cong \underline{r}'/\underline{r}' \cap \underline{h}' + \underline{s}_0/\underline{s}''_0$ . But  $\underline{\hat{v}}$  is abelian by (3.4.2), whence  $\underline{s}_0 = \underline{s}''_0 \subset \underline{h}'$ . We thus have shown

$$\underline{h}' = \underline{r}' \cap \underline{h}' + \underline{s}_0 + \underline{c}_s. \tag{3.4.8}$$

3.5. In the following sections we will use the decompositions derived above to clarify the structures of  $\underline{i}$ . As usual, by nil ( $\underline{i}$ ) we denote the nilradical of  $\underline{i}$ . We retain the notation and the assumptions used above.

LEMMA. nil ( $\underline{i}$ )  $\subset \underline{r}' \cap \underline{h}'$ .

*Proof.* Consider the action of the semisimple Lie algebra  $\underline{s}_0 + \underline{c}_s$  on  $\underline{i}$ . Then  $\underline{i} = \underline{r}' \cap \underline{h}' + \underline{a} + \underline{s}_0 + \underline{c}_s$ , where  $\underline{a}$  is invariant under  $\underline{s}_0 + \underline{c}_s$ . But since  $\underline{h}'$  is an ideal of  $\underline{i}$  and  $\underline{s}_0 + \underline{c}_s \subset \underline{h}'$ , this implies  $[\underline{s}_0 + \underline{c}_s, \underline{a}] = 0$ . Also, since  $\underline{\hat{v}} \cong \underline{i}/\underline{h}'$  is abelian,  $[\underline{a}, \underline{a}] \subset \underline{h}'$ . From this it follows  $[\underline{i}, \underline{r}'] \subset \underline{h}'$ , thus the claim.

COROLLARY 1.  $[\underline{r}, [\underline{r}, \underline{r}]] = 0.$ 

*Proof.* As usual, by  $\underline{s}$  we denote a maximal semisimple subalgebra of  $\underline{g}$ . Then  $\varphi([\underline{r}, [\underline{r}, \underline{r}]], \underline{s}) \subset \varphi(\underline{r}, [\underline{r}, \underline{r}]) = 0$ , since  $\underline{r} \subset \underline{i}$  and  $[\underline{r}, \underline{r}] \subset \operatorname{nil}(\underline{i}) \subset \underline{h}'$ . Since  $[\underline{r}, [\underline{r}, \underline{r}]] \subset \operatorname{nil}(\underline{i}) \subset \underline{h}'$  and  $\underline{r} \subset \underline{i}$  we also have  $\varphi([\underline{r}, [\underline{r}, \underline{r}]], \underline{r}) = 0$ , therefore  $[\underline{r}, [\underline{r}, \underline{r}]] \subset \underline{h}$ . But  $[\underline{r}, [\underline{r}, \underline{r}]]$  is an ideal of g, whence the claim.

COROLLARY 2. ad <u>r</u> consists of nilpotent endomorphisms of g.

3.6. The goal of this section is to show (still under the usual assumptions of this chapter)

LEMMA.  $\underline{s}_0 = 0$ .

*Proof.* Since  $\underline{s}_0 \subset \underline{h}'$  and  $\underline{r} \subset \underline{i}$ , we have  $\varphi(\underline{s}_0, \underline{r}) = 0$ . Moreover, using the notation of 3.1 we have  $\varphi(\underline{s}_0, \underline{s}_c) = \varphi(\underline{s}_0, [\underline{s}_c, \underline{s}_c]) = 0$ . This shows that  $\varphi$  is nondegenerate on  $\underline{s}_0/\underline{s}'_0$ . From the closedness condition of  $\varphi$  we obtain  $\varphi(x, y) = \beta(b, [x, y])$  for all  $x, y \in \underline{s}_0$ , where  $\beta$  denotes the Killing form of  $\underline{s}_0$ . From this we derive  $\underline{s}'_0 = \{x \in \underline{s}_0; [x, b] = 0\}$ . But  $\underline{s}'_0$  is an ideal of  $\underline{s}_0$ , hence  $\underline{s}_0 = \underline{s}'_0$ . Since we know now  $\underline{s}_0 \subset \underline{h}$  and  $\underline{r} \subset \underline{i}$ , clearly  $[\underline{s}_0, \underline{r}] \subset \underline{h} \cap \underline{r}$ . It is easy to see that  $[\underline{s}_0, \underline{r}]$  is invariant under  $\underline{s} = \underline{s}_0 + \underline{i}_c$ . Therefore, the ideal of  $\underline{g}$  generated by  $[\underline{s}_0, \underline{r}]$  is contained in  $\underline{h}$ , whence  $[\underline{s}_0, \underline{r}] = 0$ . Thus  $\underline{s}_0$  is an ideal of  $\underline{g}$ , but  $\underline{s}_0 \subset \underline{h}$  and  $\underline{s}_0 = 0$  follows.

3.7. In this section we prove a result that will be used frequently in the rest of this chapter. We retain the notation and the assumptions of this chapter.

LEMMA. Let  $x_0 \in \underline{g}$  and assume  $[x_0, \underline{r}] \subset \underline{h}$ . Moreover assume that  $\operatorname{ad} x_0$  is semisimple on  $g/\underline{r}$ . Then  $S_{\underline{r}} = 0$ , where S denotes the semisimple part of  $\operatorname{ad} x_0$ .

*Proof.* Let ad  $x_0 = S + N$  the decomposition of ad  $x_0$  into its semisimple part S and its nilpotent part N. We can assume that S leaves <u>s</u> invariant [4; Appendix]. Moreover, since S and N are polynomials in ad  $x_0$  without constant term,  $S\underline{r} \subset \underline{h} \cap \operatorname{nil}(\underline{g})$  and  $N\underline{r} \subset \underline{h} \cap \operatorname{nil}(\underline{g})$ . Let  $\underline{r}^{\mathbb{C}} = \bigoplus \underline{r}_{\alpha}^{\mathbb{C}}$  be the decomposition of  $\underline{r}^{\mathbb{C}}$ , the complexification of  $\underline{r}$ , into eigenspaces relative to S. Then

 $\underline{r}_{\alpha}^{\mathbb{C}} \subset (\underline{h} \cap \operatorname{nil}(\underline{g}))^{\mathbb{C}} \quad \text{for all } \alpha \neq 0.$ (3.7.1)

Suppose there exists some  $\alpha \neq 0$ . In what follows we fix such an  $\alpha$ . Let  $\underline{s}_{\beta}^{C}$  be any eigenspace of S in  $\underline{s}^{C}$ . Then

$$[\underline{s}^{\mathbb{C}}_{\beta}, \underline{r}^{\mathbb{C}}_{\alpha}] \subset \underline{r}^{\mathbb{C}}_{\alpha+\beta} \subset \underline{h}^{\mathbb{C}} \qquad \text{if } \alpha + \beta \neq 0.$$
(3.7.2)

If  $\beta = -\alpha$ , then

$$\varphi(\underline{s}_{\gamma}^{\mathbb{C}}, [\underline{s}_{-\alpha}^{\mathbb{C}}, \underline{r}_{-\alpha}^{\mathbb{C}}]) = 0 \qquad \text{if } \gamma + \alpha \neq 0.$$
(3.7.3)

Indeed,  $\varphi(x_{\gamma}, [y_{-\alpha}, z_{\alpha}]) = -\varphi([x_{\gamma}, z_{\alpha}], y_{-\alpha}) = 0$  if  $x_{\gamma} \in \underline{s}_{\gamma}^{\mathbb{C}}, y_{-\alpha} \in \underline{s}_{-\alpha}^{\mathbb{C}}, z_{\alpha} \in \underline{r}_{\alpha}^{\mathbb{C}} \subset \underline{h}^{\mathbb{C}}$ , and  $\gamma + \alpha \neq 0$  since in this case  $[\underline{s}_{\gamma}^{\mathbb{C}}, \underline{r}_{\alpha}^{\mathbb{C}}] \subset \underline{r}_{\alpha+\gamma}^{\mathbb{C}} \subset \underline{h}^{\mathbb{C}}$  by (3.7.1).

Consider now the case  $\gamma = -\alpha$ . From our assumption we obtain  $\underline{s}_{-\alpha}^{\mathbb{C}} = S\underline{s}_{-\alpha}^{\mathbb{C}}$   $\subset [x_0, \underline{s}_{-\alpha}^{\mathbb{C}}] + \operatorname{nil}(\underline{g})^{\mathbb{C}}$ . Hence,  $\varphi(\underline{s}_{-\alpha}^{\mathbb{C}}, [\underline{s}_{-\alpha}^{\mathbb{C}}, \underline{r}_{\alpha}^{\mathbb{C}}]) \subset \varphi([x_0, \underline{s}_{-\alpha}^{\mathbb{C}}] + \operatorname{nil}(\underline{g})^{\mathbb{C}}, [\underline{s}_{-\alpha}^{\mathbb{C}}, \underline{r}_{\alpha}^{\mathbb{C}}])$   $\subset \varphi(\underline{s}_{-\alpha}^{\mathbb{C}}, [x_0, \operatorname{nil}(\underline{g})^{\mathbb{C}}]) + \varphi([\operatorname{nil}(\underline{g})^{\mathbb{C}}, \underline{r}_{\alpha}^{\mathbb{C}}], \underline{s}_{-\alpha}^{\mathbb{C}}) = 0$ , since  $[x_0, \operatorname{nil}(\underline{g})^{\mathbb{C}}] \subset \underline{h}^{\mathbb{C}}$  and  $[\operatorname{nil}(\underline{g})^{\mathbb{C}}, \underline{r}_{\alpha}^{\mathbb{C}}] \subset \underline{h}^{\mathbb{C}}$ . Therefore we have

$$\varphi(\underline{s}_{-\alpha}^{\mathbb{C}}, [\underline{s}_{-\alpha}^{\mathbb{C}}, \underline{r}_{\alpha}^{\mathbb{C}}]) = 0.$$
(3.7.4)

As a consequence of the above results we obtain

$$\varphi(\underline{s}^{\mathbb{C}}, [\underline{s}^{\mathbb{C}}, \underline{r}_{\alpha}^{\mathbb{C}}]) = 0 \quad \text{if } \alpha \neq 0.$$
(3.7.5)

Since  $\underline{r}_{\gamma}^{\mathbb{C}} \subset \underline{h}^{\mathbb{C}}$  for  $\gamma \neq 0$ , we clearly have  $\varphi(\underline{r}_{\gamma}^{\mathbb{C}}, \underline{g}) = 0$  in this case. If  $\gamma = 0$ , then  $\varphi(\underline{r}_{0}^{\mathbb{C}}, [\underline{s}^{\mathbb{C}}, \underline{r}_{\alpha}^{\mathbb{C}}]) = \varphi([\underline{r}_{0}^{\mathbb{C}}, \underline{r}_{\alpha}^{\mathbb{C}}], \underline{s}^{\mathbb{C}}]) = 0$ , since  $[\underline{r}_{0}^{\mathbb{C}}, \underline{r}_{\alpha}^{\mathbb{C}}] \subset \underline{r}_{\alpha}^{\mathbb{C}} \subset \underline{h}^{\mathbb{C}}$ . Thus, altogether we have shown

$$\varphi(\underline{r}^{\mathbb{C}}, [\underline{s}^{\mathbb{C}}, \underline{r}^{\mathbb{C}}_{\alpha}]) = 0.$$
(3.7.6)

Equations (3.7.5) and (3.7.6) together imply

$$[\underline{s}^{\mathbb{C}}, \underline{r}_{\alpha}^{\mathbb{C}}] \subset \underline{h}^{\mathbb{C}}.$$
(3.7.7)

Next we consider the vector space  $\underline{q}^{\mathbb{C}} \subset \underline{h}^{\mathbb{C}} \cap \underline{r}^{\mathbb{C}}$  spanned by the subspaces  $\underline{r}_{\alpha}^{\mathbb{C}}$  and  $[\underline{s}^{\mathbb{C}}, \underline{r}_{\alpha}^{\mathbb{C}}], \alpha \neq 0$ . It is easy to see that  $\underline{q}^{\mathbb{C}}$  is invariant under complex conjugation relative to q.

$$q^{c}$$
 is an s-module. (3.7.8)

Indeed, consider  $A = [\underline{s}_{\gamma}^{\mathbb{C}}, [\underline{s}_{\beta}^{\mathbb{C}}, \underline{r}_{\alpha}^{\mathbb{C}}]$ . If  $\beta + \alpha \neq 0$ , then the inner commutator is contained in  $\underline{r}_{\alpha+\beta}^{\mathbb{C}}$ , whence  $A \subset \underline{q}^{\mathbb{C}}$ . If  $\beta + \alpha = 0$ , then we use  $A = [[\underline{s}_{-\gamma}^{\mathbb{C}}, \underline{s}_{-\alpha}^{\mathbb{C}}], \underline{r}_{\alpha}^{\mathbb{C}}] + [\underline{s}_{-\alpha}^{\mathbb{C}}, [\underline{s}_{-\gamma}^{\mathbb{C}}, \underline{r}_{\alpha}^{\mathbb{C}}]$ . Clearly, the first summand is in  $\underline{q}^{\mathbb{C}}$ . In the second summand we

have  $[\underline{s}_{\gamma}^{\mathbb{C}}, \underline{r}_{\alpha}^{\mathbb{C}}] \subset \underline{r}_{\alpha+\beta}^{\mathbb{C}} \subset \underline{h}^{\mathbb{C}}$  if  $\alpha + \gamma \neq 0$ . If  $\alpha + \gamma = 0$ , then the whole second summand is contained in  $\underline{r}_{-\alpha}^{\mathbb{C}}$ , finishing the proof of (3.7.8). Now it is straight forward to verify that the ideal of  $\underline{g}$  generated by  $\underline{q}^{\mathbb{C}} \cap \underline{g}$  is actually contained in  $\underline{h}$ . But since the transitive group G in question acts effectively, this ideal is trivial. In particular we have  $\underline{r}_{\alpha}^{\mathbb{C}} = 0$  for all  $\alpha \neq 0$ . Therefore  $S\underline{r} = 0$ , proving the assertion.

3.8. In this section we continue our investigation of <u>s</u>. Since we know from 3.6 that  $\underline{s}_0 = 0$  holds, <u>s</u> is compact. We split  $\underline{s} = \underline{s}_a + \underline{s}_b$ , where

$$\underline{s}_a = \{ x \in \underline{s}; [x, r] = 0 \}$$
(3.8.1)

and  $\underline{s}_b$  is a complementary ideal of  $\underline{s}_a$  in  $\underline{s}$ . Since  $\underline{i}_c$  is the centralizer of some element in  $\underline{s}$ ,

$$\underline{i}_c = \underline{i}_a + \underline{i}_b, \quad \text{where } \underline{i}_* = \underline{i}_c \cap s_*, * = a, b. \tag{3.8.2}$$

Since  $\underline{i}_a$  and  $\underline{i}_b$  are reductive, with obvious notation we have

$$\underline{i}_a = \underline{c}_a + \underline{c}_{sa} \quad \text{and} \quad \underline{i}_b + \underline{c}_b + \underline{c}_{sb}. \tag{3.8.3}$$

LEMMA. (a)  $\underline{s}_a$  and  $\underline{s}_b + \underline{r}$  are ideals of  $\underline{g}$ . (b)  $\underline{h} = \underline{h} \cap \underline{r} + \underline{c}_b + \underline{c}_a + \underline{c}_{sa}$ .

*Proof.* Clearly,  $\underline{s}_a$  and  $\underline{s}_b + \underline{r}$  are ideals of  $\underline{g}$ . Moreover, we have  $\varphi(\underline{s}_a, \underline{r}) = \varphi([\underline{s}_a, \underline{s}_a], \underline{r}) = 0$  and similarly  $\varphi(\underline{s}_a, \underline{s}_b) = 0$ . Therefore,  $\underline{s}_a$  and  $\underline{s}_b + \underline{r}$  are perpendicular. This implies  $\underline{h} = \underline{h} \cap \underline{s}_a + \underline{h} \cap (\underline{s}_b + \underline{r})$ . From Lemma 3.1 it follows that  $\underline{h} \cap \underline{s}_a$  is the centralizer of some  $w_a \in \underline{s}_a$ .

Now let  $x_0 \in \underline{h} \cap (\underline{r} + \underline{c}_a + \underline{c}_b)$ . Clearly,  $[x_0, \underline{r}] \subset \underline{h}$ , since  $\underline{h}$  is an ideal of  $\underline{i}$  and  $\underline{r} \subset \underline{i}$ . Moreover, ad  $x_0$  is semisimple on  $\underline{g}/\underline{r}$ . Therefore, by the last lemma  $S\underline{r} = 0$ , where S denotes the semisimple part of ad  $x_0$ . In view of Corollary 3.7.1 we can write  $\underline{r} = \underline{a} + [\underline{a}, \underline{a}]$  where  $[\underline{s}, \underline{a}] \subset \underline{a}$ . Hence  $x_0 = c + a + n$  with  $c \in \underline{c}_a + \underline{c}_b$ ,  $a \in \underline{a}$ , and  $n \in [\underline{a}, \underline{a}]$ . Note  $[n, \underline{r}] = 0$  by Corollary 3.5.1. Therefore ad  $x_0 | \underline{r} = ad (c + a) | \underline{r}$ . Since we know that the semisimple part of ad  $x_0$  vanishes on  $\underline{r}$ , the endomorphism  $A = ad (c + a) | \underline{r}$  is nilpotent. But  $dc | \underline{r}$  is semisimple and leaves  $\underline{a}$  and  $[\underline{a}, \underline{a}]$  invariant, while  $d\underline{a}$  maps  $\underline{a}$  into  $[\underline{a}, \underline{a}]$  and annihilates  $[\underline{a}, \underline{a}]$ . This shows  $ad c | \underline{a} = 0$  and  $ad c | [\underline{a}, \underline{a}] = 0$ , whence  $[c, \underline{r}] = 0$ . Therefore,  $c \in \underline{c}_a$  and the assertion follows.

3.9. Clearly, to show that  $\underline{g}$  is reductive, we have to prove  $\underline{s}_b = 0$ . This is the goal of this section.

LEMMA.  $\underline{s}_b = 0$ .

*Proof.* From Corollary 3.7 we know that ad <u>r</u> consists of nilpotent endomorphisms of <u>g</u>. Moreover, ad  $c, c \in \underline{c}_b$ , is semisimple on <u>g</u> and has only purely imaginary eigenvalues. Restricting ad  $(\underline{r} + \underline{c}_b)$  to the complex Lie algebra  $\underline{i}/\underline{h}$  we obtain the radical of  $\underline{i}/\underline{h}$ . But this is a complex solvable Lie algebra, whence ad  $\underline{c}_b | \underline{i}/\underline{h} = 0$ . In particular we get  $[\underline{c}_b, \underline{r}] \subset \underline{h}$ . From Lemma 3.7 we thus obtain  $[\underline{c}_b, \underline{r}] = 0$ , i.e.  $\underline{c}_b = 0$ . From Lemma 3.7 it follows easily that  $\underline{c}_{bs} = 0$  holds. Thus  $\underline{s}_b = 0$ .

3.10. With the results of the previous sections it will be easy now to prove (Theorem A of the introduction).

THEOREM. Let  $(M, \varphi)$  be a compact connected pseudo-Kähler manifold and G an effective transitive group of automorphisms of M. Then G is reductive and its semisimple part is compact.

*Proof.* From Lemma 3.9 it follows that  $\underline{g} = \underline{r} + \underline{s}_a$ , where  $[\underline{r}, \underline{s}_a] = 0$  and  $\underline{s}_a$  is semisimple. Moreover,  $\underline{h} = \underline{h} \cap \underline{r} + \underline{h} \cap \underline{s}_a$ . Therefore, the radical of  $\underline{i}/\underline{h}$  is  $\underline{r}/\underline{h} \cap \underline{r}$ . Since this is *j*-invariant we can assume  $\underline{jr} \subset \underline{r}$ . Also,  $\underline{h} \cap \underline{r}$  is an ideal of  $\underline{g}$  contained in  $\underline{h}$ , hence  $\underline{h} \cap \underline{r} = 0$ . This implies  $\underline{h} = \underline{c}_a + \underline{c}_{as}$ , by Lemma 3.8, and  $\underline{i}/\underline{h} \cong \underline{r}$ . In particular,  $\underline{r}$  is a complex Lie algebra and  $\varphi(\underline{r}, \underline{s}_a) = 0$  shows that  $(\underline{r}, 0, j, \varphi)$  is a pseudo-Kähler algebra. Thus Lemma 3.3 shows that  $\underline{r}$  is abelian. Therefore  $\underline{g}$  is reductive, proving the assertion.

3.11. In this section we will give the proof of Theorem B of the introduction.

First we note that Theorem A (see 3.10) shows that G is reductive and its semisimple part S is compact. From the Theorem in §2 we thus obtain that the isotropy subgroup H of G is connected, compact and contained in the maximal semisimple Lie subgroup S of G. From [11; Theorem 1] it thus follows that S has trivial center and that  $G = C \times S$  holds. Clearly,  $G/H = C \times S/H$ . Since G/H is compact, we see that C is a complex torus. In particular, G is compact. It is easy to see that Lie C and Lie S are perpendicular relative to the given pseudo-Kählerian structure. Thus  $G/H = C \times S/H$  is the product of pseudo-Kähler manifolds.

Therefore it only remains to prove that S/H is a rational homogeneous manifold and that the given pseudo-Kähler structures on C and S/H are a difference of Kähler structures. The first statement follows from 3.9, since  $\underline{i} = \underline{r} + \underline{c}_{as}$ , where  $\underline{h} = \underline{c}_{a} = \underline{c}_{as}$ and  $\underline{r} = \text{Lie } C$ . The second statement follows from [6].

Added in proof. Recently we received the preprint: A. T. Huckleberry, Homogeneous pseudo-Kählerian manifolds: A hamiltonian viewpoint. In this paper Theorem B is proven by a different method.

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