H1-BMO duality on Riemann surfaces.

Comment. Math. Helvetici 67 (1992) 592–612 0010-2571/92/040592-21$1.50 + 0.20/0

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HIROSHIGE SHIGA

1. Introduction

It is a famous result by Fefferman–Stein that the space of BMO functions (functions of bounded mean oscillation) in $\mathbf{R}^n$ is equal to $H^1(\mathbf{R}^n)^*$, the dual space of the Hardy space $H^1(\mathbf{R}^n)$ ([5]). Recently, it is also shown on simply connected Riemannian manifolds with negative curvatures ([1]). In particular, when $n = 1$, their theorem implies that the duality between certain spaces of holomorphic functions holds on a simple Riemann surface, the unit disk (or the upper half plane). In the previous paper [13], we have established the duality theorem between H^1 and $BMOA$ (the space of analytic BMO functions) on compact bordered Riemann surfaces (another proof is given in [7]). However, as we noted there, the duality does not hold on all Riemann surfaces. Furthermore, if we consider a Riemann surface given by Heins in [8], we verify that the duality does not necessarily hold for any Riemann surface even if the Riemann surface has the small ideal boundary. Thus, it would be a natural and interesting problem to find a class of Riemann surfaces where the duality theorem holds (Metzger also poses the similar problem in [10]).

Hence, to consider the duality, we shall introduce the space of harmonic functions on Riemann surfaces whose lifts on the universal covering surface are real parts of ordinary H^1 functions in the universal covering surface. This space can be said a harmonic version of ordinary H^1 space on Riemann surfaces and it is natural to consider such a space. Actually, in Fefferman–Stein's paper, the space H^1 on $\mathbf{R}^n$ is the space of integrable functions on $\mathbf{R}^n$ whose Riesz transforms are also integrable. They are regarded as boundary functions of harmonic functions on the upper half space $\mathbf{H}^{n+1}$ and when $n = 1$, Riesz transforms correspond to conjugate harmonic functions in the upper half plane which is the universal covering surface of itself.

In this paper, we shall show that the duality theorem for harmonic functions hold on bordered Riemann surfaces with "small' ideal boundaries. Namely, the following theorem will be shown (as for the terminologies, see Sec. 2).

THEOREM 1. *Let R be an open Riemann surface of SO_{HB} end. Then, the duality between $h^1(R)$ and $BMOH(R)$, $(h^1(R))^* = BMOH(R)$ holds. More precisely, for every $l \in h^1(R)^*$ there exists a BMOH function $\tilde{g}_l$ on R such that it induces l. Namely, l is the extension of a bounded linear functional on $HB(R)$, which is a dense subspace of $h^1(R)$, defined by*

$$h \mapsto \int_{\partial R} h\tilde{g}_l \, d\omega^R_{z_0}$$

for every $h \in HB(R)$. Furthermore, there exists a constant $K > 1$ such that it satisfies an inequality:

$$K^{-1}\|\tilde{g}_l\|_{*,R} \leq \|l\| \leq K\|\tilde{g}_l\|_{*,R}, \tag{1.1}$$

where $\|\tilde{g}_l\|_*$ is the BMO norm of $\tilde{g}_l$ and $\|l\|$ is the operator norm of $l \in h^1(R)^*$.

The proof of the theorem is rather long and complicated because of some technical details while the idea is simple. So, we sketch the outline of the proof for convenience of the reader.

Let R be a Riemann surface of SO_{HB} end. We may assume that the relative boundary ∂R consists of a finite number of analytic Jordan curves C_j $(j = 1, \ldots, n)$. First, we observe harmonic functions in $h^1(R)$. Every function $h \in h^1(R)$ has the non-tangential limits on each C_j. Furthermore, the non-tangential boundary function is an integrable function with respect to the harmonic measure and conversely a harmonic function h in $h^1(R)$ is determined by the boundary function. Thus, the space $h^1(R)$ is regarded as a subspace of $L^1(\partial R) \cong \Pi_{j=1}^n L^1(C_j)$. On the other hand, there exist neighbourhoods U_j of C_j $(j = 1, \ldots, n)$ and conformal mappings f_j on $U_j \cup \partial U_j$ such that $f_j(U_j) = \{r_j < |z| < 1\}$ and $f_j(C_j) = \{|z| = 1\}$. Under this identification, we shall confirm that $h^1(R)$ is isomorphic to $\Pi_{j=1}^n h^1(\Delta)$, where Δ is the unit disk (Proposition 3.1). Sec. 3 will be devoted to the proof of the result. To show it, a theorem of Burkholder–Gundy–Silverstein and a consideration of hyperbolic geometry in the unit disk are used. As a byproduct of the argument, we shall show that the space of bounded harmonic functions is dense in h^1 on an SO_{HB} end (Corollary 3.1).

Once the above identification is established, a linear functional $l \in h^1(R)^*$ is regarded as an element in $\Pi_{j=1}^n h^1(\Delta)^*$. Since $(h^1(\partial R))^* = \Pi_{j=1}^n h^1(\Delta)^* = \Pi_{j=1}^n BMO(\Delta)$ (Fefferman–Stein's duality theorem), a BMO function on ∂R is obtained from $l \in h^1(R)^*$. Finally, after slightly long calculation we shall show that the function is really a boundary function of a function in $BMOH(R)$ (Lemma 4.2).

From the view point of automorphic function theory, Theorem 1 implies that if the limit set of a Fuchsian group is of linear measure zero and if the "boundary curves" are compact, then the duality between spaces of certain automorphic functions for the Fuchsian group are valid. In the last section, a characterization of the dual spaces of h^1 on more general Riemann surfaces is established in terms of conditional expectations for Fuchsian groups. As an application of the characterization and the duality theorem, we see that on an SO_{HB} end, the conditional expectation for the Fuchsian group which determines the Riemann surface preserves the *BMO* property (Corollary 5.1).

2. Basic facts and terminologies

First, we shall define *BMO*-functions and Hardy spaces of harmonic functions on a Riemann surface. As for the detail of *BMO*, see [11] or [6].

A measurable function h on the unit circle $\partial\Delta$ is called *BMO* function if there exists a constant $M > 0$ such that for every interval $I \subset \partial\Delta$,

$$\frac{1}{|I|}\int_I |h - h_I|\, d\theta < M,$$

where $|I| = \int_I d\theta$ and $h_I = |I|^{-1}\int_I h\, d\theta$. A harmonic (resp. analytic) function f on the unit disk Δ is called a *BMOH* (resp. *BMOA*) function if it is represented by the Poisson integral of a *BMO*-function on $\partial\Delta$. We denote by $BMOH(\Delta)$ (resp. $BMOA(\Delta)$) the set of *BMOH* (resp. *BMOA*) functions in Δ. Both $BMOH(\Delta)$ and $BMOA(\Delta)$ are Banach spaces with *BMO* norm

$$\|f\|_* = \sup_I \frac{1}{|I|}\int_I |f - f_I|\, d\theta + \int_0^{2\pi} |f(e^{i\theta})|\, d\theta.$$

Similarly, we can define *BMOH* and *BMOA* on a Riemann surface R. Let R be a Riemann surface of hyperbolic type. Then, R is represented by Δ/Γ, where Γ is a torsion free Fuchsian group on Δ. We denote by π the canonical projection of Δ onto R. A harmonic (resp. analytic) function f on R is called a *BMOH* (resp. *BMOA*)-function on R if the lift via π is a *BMOH* (resp. *BMOA*)-function on Δ. We denote by $BMOH(R)$ (resp. $BMOA(R)$) the set of all *BMOH* (resp. *BMOA*)-functions on R. The space $BMOH(R)$ (resp. $BMOA(R)$) is regarded as the set of *BMOH* (resp. *BMOA*) functions on Δ which are automorphic for Γ. Thus, they are also Banach spaces with norm $\|f\|_{*,R} = \|f \circ \pi\|_*$ for $f \in BMOH(R)$ or $BMOA(R)$.

Next, we shall define another space of harmonic functions on R. As is well known, Hardy space $H^p(R)$ $(1 \leq p < \infty)$ is the set of all analytic functions f on R such that $|f|^p$ has a harmonic majorant on the Riemann surface R. The space $H^p(R)$ is a complex Banach space with norm

$$\|f\|_p = (L.H.M._R |f|^p(z_0))^{1/p},$$

where $L.H.M._R$ stands for the least harmonic majorant on R and z_0 is a fixed point in R. Let $HB(R)$ denote the set of all (real valued) bounded harmonic functions on r. It is a real Banach space with usual supremum norm. Here, we define a harmonic version of $H^p(R)$.

DEFINITION 2.1. Let R be an open Riemann surface of hyperbolic type, and let $\pi : \Delta \to R$ be the holomorphic universal covering mapping with $\pi(0) = z_0$. A real valued harmonic function h on R is called an h^p-function if the lift $h \circ \pi$ of h is the real part of an H^p function on Δ. We denote by $h^p(R)$ the set of h^p-functions on R.

The space $h^p(R)$ is a real Banach space with norm

$$\|u\|_{(p),R} = \|u \circ \pi + i^*(u \circ \pi)\|_p,$$

where $\|\cdot\|_p$ denote the H^p-norm on Δ and $^*(u \circ \pi)$ is a conjugate harmonic function of $u \circ \pi$ with $^*(u \circ \pi)(0) = 0$. From Riesz' theorem ([8]), for a finite number $p > 1$, $h^p(R)$ is equal to the set of real valued harmonic functions h satisfying that $|h \circ \pi|^p$ has a harmonic majorant on Δ. Furthermore, it is also equal to the space of harmonic functions which are represented by the Poisson integral of Γ-automorphic L^p functions on $\partial\Delta$. Therefore, it is immediately obvious that $(h^p(\Delta))^* = h^q(\Delta)$ if $1 < p < \infty$, where $q = 1/(p-1)$. It is also known that $HB(R) \subset BMOH(R) \subset \bigcup_{1<p<\infty} h^p(R)$.

PROPOSITION 2.1 (Fefferman–Stein [5]). *The dual space of $h^1(\Delta)$ is $BMOH(\Delta)$. More precisely, for every $l \in h^1(\Delta)^*$, there exists a unique BMOH function f such that l is the extension of the linear functional*:

$$h \mapsto \frac{1}{2\pi}\int_0^{2\pi} hf\,d\theta$$

defined on the space of bounded harmonic functions on Δ. Moreover, the assignment from $h^1(\Delta)^*$ to $BMOH(\Delta)$ is isomorphic. Namely, there exists a constant $K > 0$ not depending on l such that an inequality

$$K^{-1}\|f\|_* \leq \|l\| \leq K\|f\|_*$$

holds, where $\|l\|$ is the operator norm of l.

Next, we note a relation between the norm of functions in $h^1(\Delta)$ and the L^1 norm of the maximal functions.

DEFINITION 2.2. For each point $e^{i\theta} \in \partial\Delta$, the Stolz region at $e^{i\theta}$ is defined by

$$A(e^{i\theta}) = \{z \in \Delta : |z - e^{i\theta}| < 2(1 - |z|)\}. \tag{2.1}$$

We define the maximal function of a harmonic function u on Δ by

$$u^*(e^{i\theta}) = \sup_{s \in A(e^{i\theta})} |u(z)|.$$

By using the maximal function, we can characterize harmonic functions in $h^1(\Delta)$.

PROPOSITION 2.2 (Burkholder, Gundy, and Silverstein: [6] Chap. III-3). *A harmonic function u on Δ belongs to $h^1(\Delta)$ if and only if the maximal function u^* belongs to $L^1(\partial\Delta)$. Furthermore, there exists a constant $C > 0$ such that*

$$C^{-1}\|u^*\|_1 \leq \|u\|_{(1),\Delta} \leq C\|u^*\|_1,$$

holds for all $u \in h^1(\Delta)$.

There are some equivalent definitions of BMO (cf. [11]). Here, we mention the following one which will be used later.

PROPOSITION 2.3 ([2], [9]). *Let R be a Riemann surface of hyperbolic type. A harmonic function h in R is in $BMOH(R)$ if and only if*

$$\sup_{p \in R} L.H.M._R|h - h(p)|(p) < \infty.$$

Furthermore,

$$G_R(h) = \sup_{p \in R} L.H.M._R|h - h(p)|(p) + L.H.M._R|h|(z_0)$$

defines an equivalent norm with the BMO norm $\|h\|_{*,R}$.

Let S be an open Riemann surface in O_G, that is, has no Green's function. A subregion R of S is called to be an SO_{HB} end if it is non-compact and the relative boundary ∂R consists of finite number of analytic Jordan curves. Thus, it is easily seen that the class of SO_{HB} ends is an extension of that of compact bordered Riemann surfaces. Roughly speaking, an SO_{HB}-end is a Riemann surface with very small ideal boundary. Because of this reason, it has many nice properties. Here, we note the following one (cf. [14]).

PROPOSITION 2.4. *Let R be an SO_{HB}-end with the compact relative boundary ∂R. For every quasi-bounded harmonic function u, i.e., a harmonic function obtained by difference of monotone limits of boundary harmonic functions, the following is valid:*

(1) *u is bounded near the ideal boundary.*

(2) *The Dirichlet integral of u is finite near the ideal boundary.*

(3) *(The maximum principle) For every subregion G of R whose closure does not intersect with the relative boundary ∂R,*

$$\sup_{s \in G} |u(z)| \leq \sup_{z \in \partial G} |u(z)|$$

holds for every quasi-bounded harmonic function u on R.

(4) *u has a non-tangential limit almost every where in ∂R and every quasi-bounded harmonic function is uniquely determined by the limit function on ∂R.*

(5) *For every $p \in R$*

$$L.H.M._R|u|(p) = \int_{\partial R} |u| \, d\omega_p^R ,$$

where $d\omega_p^R$ is the harmonic measure on R with respect to p.

3. Auxiliary results

In this section, we consider only an SO_{HB} end $R(=\Delta/\Gamma)$ with smooth relative boundary ∂R. Let $C_1, \ldots, C_n$ be the set of analytic Jordan curves of the relative boundary ∂R. We take the closed geodesics $\tilde{C}_j$ which are homotopic to C_j $(j = 1, \ldots, n)$ and annular regions $U_1, \ldots, U_n$ in R bounded by C_j and $\tilde{C}_j$ $(j = 1, \ldots, n)$. Each $U_j \cup \partial U_j$ is conformally equivalent to an annulus $A_j = \{z : 0 < r_j \leq |z| \leq 1\}$ via a conformal mapping f_j from U_j onto A_j and C_j corresponds to the unit circle $\partial\Delta$ under f_j. Let $\Delta_1, \ldots, \Delta_n$ be n copies of the unit disk Δ so that each A_j is regarded as a subregion of Δ_j and $\partial\Delta_j$ corresponds to C_j via f_j.

Now, take a harmonic function $h \in h^1(R)$. Since h is a quasi-bounded harmonic function on R, h has a non-tangential limits almost everywhere on $C_j \cong \partial\Delta_j$ $(j = 1, \ldots, n)$. Furthermore, the boundary functions is an integrable function on ∂R with respect to the harmonic measure on R. Denote by h_{Δ_j} the solution of the Dirichlet problem with respect to the boundary function of h in Δ_j $(j = 1, \ldots, n)$.

More precisely, it is the solution of Dirichlet problem for $h \circ f_j^{-1}$ on Δ_j. Then, h_{Δ_j} belongs to $h^1(\Delta_j)$. Indeed, it is easy to see that the modulus of the conjugate harmonic function of h_{Δ_j} has a harmonic majorant in A_j. Therefore, the conjugate harmonic function has the non-tangential boundary function on the unit circle which is an integrable function with respect to the Lebesgue measure. This implies that h_{Δ_j} belongs to $h^1(\Delta_j)$.

The purpose of this section is to prove that $h^1(R)$ is isomorphic to $\prod_{j=1}^n h^1(\Delta_j)$.

To show it, we shall estimate the L^1 norm of the maximal functions. Since $h_{\Delta_j} \in h^1(\Delta_j)$ for $h \in h^1(R)$, the maximal function $h^*_{\Delta_j}$ of h_{Δ_j} is in L^1 as a function $\bigcup_{j=1}^n \partial\Delta_j \cong \bigcup_{j=1}^n C_j$. Here, we define a 'local' maximal function $h^*_{\Delta_j}$ of h_{Δ_j} is in L^1 as a function on $\bigcup_{j=1}^n \partial\Delta_j \cong \bigcup_{j=1}^n C_j$. Here, we define a 'local' maximal function h^*_{loc} on ∂R as follows:

Let π be the universal covering mapping from Δ onto R. Then π is extended continuously to subarcs of Δ which corresponds to ∂R. Thus, we can take closed arcs $I_1, \ldots, I_n$ in $\partial\Delta$ such that π is an injection on the interior of $\bigcup_{j=1}^n I_j$ and a surjection from I_j onto C_j $(j = 1, \ldots, n)$. For each $p \in \partial R$ we define the local maximal function of h by

$$h^*_{\text{loc}}(p) = (h \circ \pi)^*(e^{i\theta}),$$

where $e^{i\theta}$ is a point in $I = \bigcup_{j=1}^n I_j$ with $\pi(e^{i\theta}) = p$. Note that h^*_{loc} depends on the choice of $I_1, \ldots, I_n$. By using the identification fo C_j and $\partial\Delta_j$, h^*_{loc} is regarded as a measurable function on $\prod_{j=1}^n \partial\Delta_j$.

We use the space $\{h^*_{\text{loc}} : h \in h^1(R)\}$ as an intermediate space between $h^1(R)$ and $\prod_{j=1}^n h^1(\Delta_j)$. First, we compare $\{h^*_{loc} : h \in h^1(R)\}$ with $h^1(\Delta_j)$.

LEMMA 3.1. *Let h be a harmonic function in $h^1(R)$. Then there is a constant $K > 1$ not depending on h such that*

$$K^{-1} \sum_{j=1}^{n} \|h^*_{\Delta_j}\|_1 \le \|h^*_{\text{loc}}\|_1 \le K \sum_{j=1}^{n} \|h^*_{\Delta_j}\|_1,$$

where both $\|h^*_{\Delta_j}\|_1$ and $\|h^*_{\text{loc}}\|_1$ and L^1 norms with respect to the Lebesgue measures of $h^*_{\Delta_j}$ and h^*_{loc} on $\partial\Delta_j$ and $\prod_{j=1}^n \partial\Delta_j$, respectively.

Proof. We take a neighborhood V_j of I_j so small that $\partial\pi(V_j \cap \Delta)$ is a compact subset of U_j. Since π is extended conformally beyond I_j, $(f_j \circ \pi)' \neq 0$ in $(V_j \cap \Delta) \cup I_j$. Hence, there exists an angle $\alpha > 0$ which is greater than that of the Stolz region (2.1) such that for all $p \in C_j$ a Stolz region $\tilde{A}(p) \subset \Delta_j$ at $f_j(p) \in \partial\Delta$ with angle α

contains $f_j(\pi(A(e^{i\theta}) \cap V_j))$, where $A(e^{i\theta})$ is the Stolz region defined by (2.1) at Sec. 2 for $e^{i\theta} \in I_j$ with $\pi(e^{i\theta}) = p$. Obviously, for $e^{i\theta} \in I_j$,

$$\sup_{A(e^{i\theta})} |h \circ \pi| \le \sup_{A(e^{i\theta}) - V_j} |h \circ \pi| + \sup_{A(e^{i\theta}) \cap V_j} |h \circ \pi|$$
$$= \sup_{A(e^{i\theta}) - V_j} |h \circ \pi| + \sup_{\pi(A(e^{i\theta}) \cap V_j)} |h|. \tag{3.1}$$

Set $u_j(z) = h \circ f_j^{-1}(z) - h_{\Delta_j}(z)$ for $z \in f_j(\pi(V_j \cap \Delta))$. The function u_j is harmonic on an annulus $W_j = \{z : 0 < \rho_j < |z| < 1\}$ which contains $f_j(\pi(V_j \cap \Delta))$. Since $h \circ f_j^{-1}$ and h_{Δ_j} have the same boundary value on $\partial\Delta_j$, u_j vanishes identically on $\partial\Delta_j$. R is an SO_{HB} end, so we have

$$L.H.M._R |h|(f_j^{-1}(w_j)) = \int_{\partial R} |g| \, d\omega^R_{f_j^{-1}(w_j)}, \qquad \text{for a fixed } w_j \in W_j.$$

Noting that the ratio of harmonic measures $d\omega^R_{f_j^{-1}(\omega_j)}$ to $d\omega^{\Delta_j}_{w_j}$ is bounded on $C_j \cong \partial\Delta_j$ and vice versa,

$$\int_{C_j} |h| \, d\omega^R_{f_j^{-1}(w_j)} \le K \int_{\partial\Delta_j} |h_{\Delta_j}| \, d\omega^{\Delta_j}_{w_j} \qquad (j = 1, \ldots, n) \tag{3.2}$$

for some constant K not depending on h.

Since $\partial\pi(V_j \cap \Delta)$ is a compact subset of R, Harnack's inequality implies that there exists a constant K' such that for all $z \in f_j(\partial\pi(V_j \cap \Delta))$,

$$|u_j(z)| \le |h \circ f_j^{-1}(z)| + |h_{\Delta_j}(z)| \le L.H.M._R |h|(f_j^{-1}(z)) + L.H.M._{\Delta_j} |h_{\Delta_j}|(z)$$
$$\le K' \sum_{j=1}^{n} \int_{\partial\Delta_j} |h_{\Delta_j}| \, d\omega^{\Delta_j}_{w_j} \tag{3.3}$$

and $|u_j(z)| \le K' \sum_{j=1}^n \int_{\partial\Delta_j} |h_{\Delta_j}| \, d\omega^{\Delta_j}_{w_j}$ in $f_j(V_j)$ because $u_j \equiv 0$ on $\partial\Delta_j$. Therefore, we have

$$\sup_{\pi(A(e^{i\theta}) \cap V_j)} |h| \le \sup_{\tilde{A}(p)} |h_{\Delta_j}| + \sup_{f_j(\pi(V_j \cap \Delta))} |u_j| \le h^{**}_{\Delta_j}(p) + K' \sum_{j=1}^{n} \int_{\partial\Delta_j} |h_{\Delta_j}| \, d\omega^{\Delta_j}_{w_j}, \tag{3.4}$$

where $(\cdot)^{**}$ means the maximal function determined by $\tilde{A}$.

Next, we shall estimate $\sup_{A(e^{i\theta}) - V_j} |h \circ \pi|$. For each $e^{i\theta} \in I$, the set $\pi(A(e^{i\theta}) - V_j)$ is a subset of $R - \bigcup_{j=1}^n V_j$ and $R - \bigcup_{j=1}^n V_j$ is also an SO_{HB} end. From the maximum principle for SO_{HB} ends, we have

$$\sup_{A(e^{i\theta}) - V_j} |h \circ \pi| = \sup_{\pi(A(e^{i\theta}) - V_j)} |h| \le \sup_{\partial\pi(\bigcup_{j=1}^n V_j \cap \Delta)} |h| \le \sup_{\partial\pi(\bigcup_{j=1}^n V_j \cap \Delta)} L.H.M._R |h|.$$

Harnack's inequality and (3.2) show that

$$\sup_{A(e^{i\theta})-V_j} |h \circ \pi| \leq K' \sum_{j=1}^{n} \int_{\partial \Delta_j} |h_{\Delta_j}| \, d\omega_{w_j}^{\Delta_j}. \tag{3.5}$$

From (3.1), (3.4) and (3.5), we have for $p \in C_j$

$$h_{\mathrm{loc}}^*(p) \leq h_{\Delta_j}^{**}(p) + K' \sum_{j=1}^{n} \int_{\partial \Delta_j} |h_{\Delta_j}| \, d\omega_{w_j}^{\Delta_j}.$$

Integrating both sides, we have

$$\|h_{\mathrm{loc}}^*\|_1 \leq \|h_{\Delta_j}^{**}\|_1 + K' \sum_{j=1}^{n} \int_{\partial \Delta_j} |h_{\Delta_j}| \, d\omega_{w_k}^{\Delta_j}. \tag{3.6}$$

From the definition of h^1 norm, the L^1 norm of an h^1 function is less than the h^1 norm. Thus, from Propositon 2.3,

$$\|h_{\mathrm{loc}}^*\|_1 \leq K \sum_{j=1}^{n} \|h_{\Delta_j}^{**}\|_1.$$

It is known that the L^1 norm of the maximal function defined by a 'wider' Stolz region $\tilde{A}(p)$ is equivalent to the L^1 norm of the maximal function defined by the Stolz region (2.1) ([6]). Thus,

$$\|h_{\mathrm{loc}}^*\|_1 \leq K \sum_{j=1}^{n} \|h_{\Delta_j}^{*}\|_1.$$

The similar argument shows that the L^1 norm of the maximal function of h_{Δ_j} is bounded by that of h_{loc}^* from above, and it shows another inequality. □

For $h \in h^1(R)$ the lift $h \circ \pi$ is a Γ-automorphic function on Δ. However, the maximal function $(h \circ \pi)^*$ may not be Γ-automorphic because the Stolz region (2.1) is not invariant under non-trivial transformations in Γ. But the following inequalities are valid.

LEMMA 3.2. *Let $I_1, \ldots, I_n$ be the same ones as before. Then for each point $e^{i\theta} \in I = \bigcup_{j=1}^{n} I_j$ and for each $\gamma \in \Gamma$, inequalities*

$$(h \circ \pi)^*(e^{i\theta}) - (h \circ \pi)_\alpha^*(\gamma(e^{i\theta})) \leq K \int_{\partial R} |h| \, d\omega_{z_0}^R, \tag{3.7a}$$

and

$$(h \circ \pi)_\beta^*(\gamma(e^{i\theta})) - (h \circ \pi)^*(e^{i\theta}) \leq K \int_{\partial R} |h| \, d\omega_{z_0}^R \tag{3.7b}$$

are valid for some constant K which does not depend on h, where $(\cdot)^*$ is the maximal function defined by (2.1) *and $(\cdot)^*_\alpha$ and $(\cdot)^*_\beta$ are the maximal functions defined by Stolz regtions whose angles are certain ones, α and β, respectively* (*see the proof below*).

Proof. Let $r[e^{i\theta}]$ denote the radius from the origin to $e^{i\theta}$ and let $V_\varepsilon(E)$ denote the ε-hyperbolic neighborhood of $E \subset \Delta$. Since $r[\gamma(e^{i\theta})]$ is a hyperbolic geodesic in Δ for every $\gamma \in \Gamma$, so is $\gamma^{-1}(r[\gamma(e^{i\theta})])$ which connects $\gamma^{-1}(0)$ with $e^{i\theta}$. The point $e^{i\theta}$ in I is not in the limit of Γ. Therefore, the Euclidean distance between $e^{i\theta}$ and $\bigcup_{\gamma \in \Gamma} \gamma(0)$ is positive. Moreover, we see that there exists a constant $K > 0$ which does not depend on $e^{i\theta} \in I$ such that $|\gamma^{-1}(0) - e^{i\theta}| > K$ for all $\gamma \in \Gamma$.

From the definition of the Stolz region, we verify that there exists constants $\varepsilon, \varepsilon' > 0$ such that $V_\varepsilon(r[e^{i\theta}]) \subset A(e^{i\theta}) \subset V_{\varepsilon'}(r[e^{i\theta}])$ for every $e^{i\theta} \in \partial\Delta$, where $A(e^{i\theta})$ is the Stolz region for $e^{i\theta}$ defined by (2.1) at Sec. 2. Since $\gamma \in \Gamma$ is an isometry with respect to the hyperbolic metric, we verify that $\gamma(A(e^{i\theta})) \supset V_\varepsilon(r(\gamma(e^{i\theta})))$. Considering that $\gamma^{-1}(0)$, the initial point of $\gamma^{-1}(r[\gamma(e^{i\theta})])$, is far from $e^{i\theta}$, we see that there exists a neighborhood $U(e^{i\theta})$ of $e^{i\theta}$ such that

$$r[e^{i\theta}] \cap U(e^{i\theta}) \subset \gamma^{-1}(V_\varepsilon(r[\gamma(e^{i\theta})]) \cap U(e^{i\theta})$$
$$\subset \gamma^{-1}(A(\gamma(e^{i\theta})) \cap U(e^{i\theta})$$

for all $\gamma \in \Gamma$. Thus, there exists a neighborhood U of I such that for all $e^{i\theta} \in I$ and for all $\gamma \in \Gamma$, $r[e^{i\theta}] \cap U \subset \gamma^{-1}(A(\gamma(e^{i\theta}))) \cap U$. We may take U satisfying that $\bigcup_{\gamma \in \Gamma} \gamma(U) \cap \Delta$ does not contain the origin and any hyperbolic geodesic never goes out from U if once it enters there. It is always possible because $\pi(0) = z_0 \notin \bigcup_{j=1}^n U_j$. And as in the proof of Lemma 3.1, we take an angle α which is greater than the angle of $A(e^{i\theta})$ so that $A(e^{i\theta}) \cap U \subset \gamma^{-1}(A_\alpha(\gamma(e^{i\theta}))) \cap U$ for all $\gamma \in \Gamma$ and for all $e^{i\theta} \in I$, where A_α denotes Stolz region whose angle is α. Similarly, we take an angle β such that $\gamma^{-1}(A_\beta(\gamma(e^{i\theta}))) \cap U \subset A(e^{i\theta}) \cap U$ for all $\gamma \in \Gamma$ and for all $e^{i\theta} \in I$. Since $|h \circ \pi|$ is Γ-automorphic, $\sup_{A(\gamma(e^{i\theta}))} |h \circ \pi| = \sup_{\gamma^{-1}(A(\gamma(e^{i\theta})))} |h \circ \pi|$. Thus, we have

$$(h \circ \pi)^*(e^{i\theta}) = \sup_{\gamma^{-1}(A(e^{i\theta}))} |h \circ \pi| \le \sup_{A(e^{i\theta}) \cap U} |h \circ \pi| + \sup_{A(e^{i\theta}) - A(e^{i\theta}) \cap U} |h \circ \pi|.$$

Then, from the relation $\gamma^1(A_\alpha(\gamma(e^{i\theta}))) \cap U \supset A(e^{i\theta}) \cap U$,

$$\sup_{A(e^{i\theta}) \cap U} |h \circ \pi| \le \sup_{\gamma^{-1}(A_\alpha(\phi(e^{i\theta}))) \cap U} |h \circ \pi| \le (h \circ \pi)^*_\alpha(e^{i\theta}). \tag{3.8}$$

On the other hand, $\pi(A(e^{i\theta}) - A(e^{i\theta}) \cap U) \subset R - \pi(U)$. Hence, from the maximum principle for SO_{HB} ends,

$$\sup_{A(e^{i\theta}) - A(e^{i\theta}) \cap U} |h \circ \pi| \leq \sup_{\partial\pi(U \cap \Delta)} |h|.$$

Since $\partial\pi(U \cap \Delta)$ is a compact set in R, from Harnack's inequality,

$$\sup_{\partial\pi(U \cap \Delta)} |h| \leq \sup_{\partial\pi(U \cap \Delta)} L.H.M._R |h| \leq K \int_{\partial R} |h| \, d\omega^R_{z_0} \tag{3.9}$$

for some constant K. From (3.8) and (3.9), we have

$$(h \circ \pi)^*(e^{i\theta}) \leq (h \circ \pi)^*_\alpha(\gamma(e^{i\theta})) + K \int_{\partial R} |h| \, d\omega^R_{z_0}. \tag{3.10}$$

Conversely,

$$\sup_{\gamma^{-1}(A_\beta(\gamma(e^{i\theta})))} |h \circ \pi| \leq \sup_{\gamma^{-1}(A_\beta(\gamma(e^{i\theta}))) \cap U} |h \circ \pi| + \sup_{\gamma^{-1}(A_\beta(\gamma(e^{i\theta}))) - \gamma^{-1}(A_\beta(\gamma(e^{i\theta}))) \cap U} |h \circ \pi|.$$

Considering $\gamma^{-1}(A_\beta(\gamma(e^{i\theta}))) \cap U \subset A(e^{i\theta}) \cap U$ for all $\phi \in \Gamma$, we easily verify that the similar argument as above gives the proof of another inequality

$$(h \circ \pi)^*_\beta(e^{i\theta})) \leq (h \circ \pi)^*(e^{i\theta}) + K \int_{\partial R} |h| \, d\omega^R_{z_0}. \tag{3.11}$$

Thus, we have shown the desired inequalities (3.7a) and (3.7b). □

By using the above lemmas, we show that the h^1 norm of h in R is equivalent to the h^1 norm of $(h_{\Delta_1}, \ldots, h_{\Delta_n})$ in $\bigcup_{j=1}^n \Delta_j$, which is the main result of this section.

PROPOSITION 3.1. *There exists a constant* $C > 1$ *such that*

$$C^{-1} \|h\|_{(1),R} \leq \sum_{j=1}^{n} \|h_{\Delta_j}\|_{(1),\Delta_j} \leq C \|h\|_{(1),R}.$$

Proof. Because of Proposition 2.2, it suffices to show that similar inequalities are valid for maximal functions which correspond to h and h_{Δ_j}.

By taking a conjugation of Γ in $PSL(2, \mathbf{C})$, we may assume that $\pi(0) = z_0 \notin \bigcup_{j=1}^n U_j$, where each U_j is the regions defined at the beginning of this section.

Since R is an SO_{HB} end, the linear measure of the limit set of Γ on $\partial\Delta$ is zero. Indeed, let χ_Λ be the characteristic function of the limit set $\Lambda(\Gamma)$ of Γ. The solution of the Dirichlet problem $H^\Delta_{\chi_\Lambda}$ for χ_Λ is a Γ automorphic function because the limit set $\Lambda(\Gamma)$ is invariant under Γ. Hence it is regarded as a lift of a bounded harmonic function on R which vanishes on the relative boundary ∂R. Since R is an SO_{HB} end, it must be zero and so is $H^\Delta_{\chi_\Lambda}$. Therefore, the linear measure of the limit set is zero.

Thus, we may consider the maximal function $(h \circ \pi)^*$ only on $\Omega(\Gamma) \cap \partial\Delta$, where $\Omega(\Gamma)$ is the region of discontinuity of Γ.

We define a function $\hat{h}$ for h on $\partial\Delta$ by the following way.

For a point $e^{i\theta} \in I$, we set $\hat{h}(e^{i\theta}) = h^*_{\text{loc}}(p)$, where p is a point in ∂R with $\pi(e^{i\theta}) = p$. For a point $e^{i\theta} \in \gamma(I)$ $(\gamma \in \Gamma - \{id.\})$, we set $\hat{h}(e^{i\theta}) = \hat{h}(\gamma^{-1}(e^{i\theta}))$.

Since $\hat{h}$ is an automorphic function on $\partial\Delta$, the solution of the Dirichlet problem $H^\Delta_{\hat{h}}$ is an automorphic harmonic function too. This implies that $H^\Delta_{\hat{h}}$ is a lift of a harmonic function on R whose boundary value on ∂R is h^*_{loc}. Noting that the ratio of the harmonic measure $d\omega^R_{z_0}$ on ∂R to that of $\partial\Delta$ is bounded and vice versa, we have

$$\frac{C^{-1}}{2\pi}\int_0^{2\pi} |\hat{h}|\, d\theta \leq \|h^*_{\text{loc}}\|_1 \leq \frac{C}{2\pi}\int_0^{2\pi} |\hat{h}|\, d\theta, \tag{3.12}$$

for some constant $C > 0$. Since $\bigcup_{\gamma \in \Gamma}\gamma(I) = \partial\Delta - \Lambda(\Gamma)$ and the linear measure of $\Lambda(\Gamma)$ is zero, we have from (3.7a), (3.7b)

$$\hat{h}(z) - (h \circ \pi)^*_\alpha(z) \leq K\int_{\partial R} |h|\, d\omega^R_{z_0},$$

and

$$(h \circ \pi)^*_\beta(z) - \hat{h}(z) \leq K\int_{\partial R} |h|\, d\omega^R_{z_0}$$

for almost all $z \in \partial\Delta$. Integrating both inequalities, we have

$$\|\hat{h}\|_1 \leq \|h^*_\alpha\|_1 + K\int_{\partial R} |h|\, d\omega^R_{z_0}, \tag{3.13a}$$

and

$$\|h^*_\beta\|_1 \leq \|\hat{h}\|_1 + K\int_{\partial R} |h|\, d\omega^R_{z_0}. \tag{3.13b}$$

Since $\|h_\alpha^*\|_1$, $\|h_\beta^*\|_1$, and $\|h^*\|_1$ are comparable to each other and since $\|h\|_1 \le \|h\|_{(1),R} \le C\|h^*\|_1$, from (3.13a) and (3.13b) is follows that

$$\|\hat{h}\|_1 \le K'\|h^*\|_1, \tag{3.14a}$$

and

$$\|h^*\|_1 \le K'\|\hat{h}\|_1 \tag{3.14b}$$

for some constant $K' > 0$. Thus, we have from (3.12)

$$(CK')^{-1}\|h^*\|_1 \le \|h^*_{\text{loc}}\|_1 \le CK'\|h^*\|_1.$$

On the other hand, we know that the L^1 norm of h^*_{loc} and the L^1 norms of the maximal functions of h_{Δ_j} $(j = 1, \ldots, n)$ are equivalent to each other (Lemma 3.1). Hence, we conclude that the L^1 norm of the maximal function of h and the sum of the L^1 norms of the maximal functions of h_{Δ_j} are equivalent and we obtain the desired inequality for maximal functions. □

Since $HB(\Delta)$ is a dense subset of $h^1(\Delta)$, we have immediately the following from the above lemma:

COROLLARY 3.1. *Let R be an SO_{HB}-end with the relative boundary ∂R. Then, $HB(R)$, the set of bounded harmonic functions on R is dense in $h^1(R)$.*

4. Proof of Theorem 1

Let l be an element in $h^1(R)^*$. Each $h \in h^1(R)$ has a boundary function on ∂R. We use the same letter h for it. Take a function h_{Δ_j} which belongs to $h^1(\Delta_j)$ as in Sec. 3. Then, we define a linear mapping L of $h^1(R)$ to a Banach space $\Pi_{j=1}^n h^1(\Delta_j)$ by

$$L : h \mapsto (h_{\Delta_1}, \ldots, h_{\Delta_n}).$$

Obviously, the mapping L is injective. And Lemma 3.2 implies L is bounded and L^{-1} is also bounded of $L(h^1(R))$ onto $h^1(R)$. Hence, $l \circ L^1$ is an element in $L(h^1(R))^*$. Therefore, by the Hahn–Banach theorem, $l \circ L^{-1}$ extends to an element in $(\Pi_{j=1}^n h^1(\Delta_j))^*$. Thus, from Proposition 2.1 we obtain a function

$g_l = (g_{l,1}, \ldots, g_{l,n})$ in $\Pi_{j=1}^n BMOH(\Delta_j)$ which corresponds to $l \circ L^{-1}$. In particular, for every $h \in HB(R)$

$$l(h) = l \circ L^{-1}(L(h)) = \sum_{j=1}^{n} \int_{\partial \Delta_j} (h \circ f_j^{-1}) g_{l,j}\, d\theta$$

$$= \sum_{j=1}^{n} \int_{C_j} h(g_{l,j} \circ f_j)\left(\frac{f_j^*(d\theta)}{d\omega_{z_0}^R}\right) d\omega_{z_0}^R,$$

where $f_j^*(d\theta)$ is the pull back of the measure $d\theta$, and f_j is the conformal mapping give in Sec. 3.

Since $HB(R)$ is a dense subset of $h^1(R)$ (Corollary 3.1), we verify that a solution of the Dirichlet problem $H_{\tilde{g}_l}^R$ for $\tilde{g}_l \in L^1(\partial R)$ induces l, where $\tilde{g}_l$ is a measurable function on ∂R with

$$\tilde{g}_l(p) = (g_{l,j} \circ f_j)(p)\left(\frac{f_j^*(d\theta)}{d\omega_{z_0}^R}\right)(p) \qquad \text{if } p \in C_j \ (j = 1, \ldots, n).$$

Therefore, in order to prove Theorem 1, we must show that $H_{\tilde{g}_l}^R \in BMOH(R)$ if $\tilde{g}_l \in \Pi_{j=1}^n BMOH(\Delta_j)$. The function $(f_j^*(d\theta)/d\omega_{z_0}^R)(p)$ is continuous and it is equal to an analytic function $(-if_j^*(d \log z))/(d\omega_{z_0}^R + i^*\, d\omega_{z_0}^R)$ on ∂R. Thus, from a theorem by Stegenga [12] we see that $H_{\tilde{g}_l}^R|_{U_j}$ is a BMOH(U_j) function ($j = 1, \ldots, n$), where U_j is the annular region defined at the beginning of Sec. 3.

Indeed, $(-if_j^*(d \log z))/(d\omega_{z_0}^R + i^*\, d\omega_{z_0}^R)$ is a sufficiently smooth function on $\partial\Delta$. Hence, a mapping

$$g \mapsto g \frac{-if_j^*(d \log z)}{d\omega_{z_0}^R + i^*\, d\omega_{z_0}^R}(f_j^{-1}(z)) = g\frac{f_j^*(d\theta)}{d\omega_{z_0}^R}(f_j^{-1}(z))$$

defines a BMO multipler and it defines a bounded mapping on $BMOH(U_j)$. We can take a constant $K_0 > 0$ which does not depend on l so that

$$\sum_{j=1}^{n} \|\tilde{g}_{l,j} \circ f_j^{-1}\|_{*,\Delta_j} \le K_0 \sum_{j=1}^{n} \|g_{l,j}\|_*, \tag{4.1}$$

where $\|\tilde{g}_{l,j} \circ f_j^{-1}\|_{*,\Delta}$ is the BMO norm of

$$\tilde{g}_{l,j} \circ f_j^{-1} = g_{l,j}\left(\frac{f_j^*(d\theta)}{d\omega_{z_0}^R}\right) \circ f_j^{-1}$$

as a function on $\partial\Delta_j$.

Therefore, $H^{\Delta_j}_{\tilde{g}_l \circ f_j^{-1}}$ is $BMOH$ in Δ_j. On the other hand, $H^R_{\tilde{g}_l} - H^{\Delta_j}_{\tilde{g}_l \circ f_j^{-1}} \circ f_j$ is a bounded harmonic function in U_j. Therefore, $H^R_{\tilde{g}_l}|_{U_j}$ is a $BMOH$ function in U_j.

By the same argument we can show that for every $(g_1, \ldots, g_n) \in \Pi_{j=1}^n BMOH(\Delta_j)$, $(H^R_g|_{U_1}, \ldots, H^R_g|_{U_n})$ belongs to $\Pi_{j=1}^n BMOH(U_j)$, where g is a measurable function on ∂R with $g(p) = g_j(p)$ if $p \in C_j$ $(j = 1, \ldots, n)$. Moreover,

LEMMA 4.1. *The mapping*

$$(g_1, \ldots, g_n) \mapsto (H^R_g|_{U_1}, \ldots, H^R_g|_{U_n})$$

defines a bounded mapping of $\Pi_{j=1}^n BMOH(\Delta_j)$ *to* $\Pi_{j=1}^n BMOH(U_j)$.

Proof. For simplicity, we assume that $n = 1$, but the following proof also works when $n > 1$. Suppose that the above mapping is not bounded. Then, there exists a sequence $\{g_m\}_{m=1}^\infty$ such that the BMO norms in Δ_1 converge to zero, but the BMO norms of $H^R_{g_m}|_{U_1}$ in U_1 is one. Let l_m $(\in h^1(\Delta_1)^*)$ be linear functionals determined by $g_m (m = 1, 2, \ldots)$. Then, we have

$$\|g_m\|_2 = \int_{\partial \Delta_1} |g_m|^2 \, d\theta = |l_m(g_m)| \leq \|l_m\| \, \|g_m\|_{(1)}.$$

Therefore, the L_2 norms of $\{g_m\}_{m=1}^\infty$ on $\partial\Delta_1$ also converges to zero. Hence, $\{H^R_{g_m}\}_{m=1}^\infty$ converges to zero uniformly on every compact subset of R. Indeed, for every point $p \in R$

$$\int_{\partial R} |g_m| \, d\omega^R_p = \int_{\partial R} |g_m| \left(\frac{d\omega^R_p}{f_1^*(d\theta)} \right) f_1^* \, d\theta,$$

and a function $d\omega^R_p / f_1^*(d\theta)$ is continuous on ∂R. Thus, $\lim_{m \to \infty} \int_{\partial R} |g_m| \, d\omega^R_p = 0$ and we verify that $H^R_{g_m} \to 0$ uniformly on every compact subset of R as $m \to \infty$.

Therefore, $H^R_{\tilde{g}_m} - H^{\Delta_1}_{\tilde{g}_m \circ f_1^{-1}} \circ f_1 \to 0$ uniformly on U_1 as $m \to \infty$ and so do the BMO norms, because they vanish on ∂R. Hence, the BMO norms of $\{H^R_{\tilde{g}_m}|_{U_1} = H^R_{\tilde{g}_m} - H^{\Delta_1}_{\tilde{g}_m \circ f_1^{-1}} + H^{\Delta_1}_{\tilde{g}_m \circ f_1^{-1}}\}_{m=1}^\infty$ converge to zero as $m \to \infty$ and it is contradiction. □

We have shown that the function $\tilde{g}_l$ induces $l \in (h^1(R))^*$ and $H^R_{\tilde{g}_l}|_{U_j}$ are in $BMOH(U_j)$ $(j = 1, \ldots, n)$. To complete the proof of Theorem 1, we shall show in Lemma 4.2 below that the mapping

$$(H^R_{\tilde{g}_l}|_{U_1}, \ldots, H^R_{\tilde{g}_l}|_{U_n}) \mapsto H^R_{\tilde{g}_l}$$

is a bounded mapping of $\Pi_{j=1}^n BMOH(U_j)$ to $BMOH(R)$. Then, from Lemma 4.1 we verify that the mapping $g_l \mapsto H_{\tilde{g}_l}^R$ is bounded from $\Pi_{j=1}^n BMOH(\Delta_j)$ to $BMOH(R)$. On the other hand, from the construction of g_l and from Proposition 2.1, we verify that there exists a constant $c > 0$ such that

$$\sum_{j=1}^n \|g_{l,j}\|_* \le c\|l\|.$$

Therefore, there exists a constant $K_1 > 0$ such that

$$\|\tilde{g}_l\|_{*,R} \le K_1\|l\|.$$

On the other hand, $\tilde{g}_l \in BMOH(R)$ induces an element $\tilde{l} \in h^1(\Delta)^*$ whose operator norm on $h^1(\Delta)$ is greater than that of l on $h^1(R)$ because $h^1(R)$ is regarded as a subset of $h^1(\Delta)$. Hence, from Proposition 2.1,

$$\|l\| \le \|\tilde{l}\| \le K_2\|\tilde{g}_l\|_{*,R},$$

for some constant $K_2 > 0$ and we obtain the desired inequality (1.1).

Conversely, an argument similar as in [13] shows that every $BMOH$ function on R induces an element in $h^1(R)^*$ uniquely. And the statement of Theorem 1 is proved.

Hence, we must show the following lemma which implies that BMO property is a boundary property in SO_{HB} ends.

LEMMA 4.2. *Let h be a quasi-bounded harmonic function on an SO_{HB} end R. Suppose that $h|_{U_j}$ $(j = 1, \ldots, n)$ belong to $BMOH(U_j)$. Then, h is a $BMOH$ function on R. Furthermore, the mapping*

$$(h|_{U_1}, \ldots, h|_{U_n}) \mapsto h$$

defines a bounded mapping of $\Pi_{j=1}^n BMOH(U_j)$ to $BMOH(R)$.

Proof of Lemma 4.2. From Proposition 2.3,

$$G_{U_j}(h) < \infty \qquad (j = 1, \ldots, n),$$

where

$$G_{U_j}(h) = \sup_{p \in U_j} L.H.M._{U_j}|h - h(p)|p) + L.H.M._{U_j}|h|(z_j). \tag{4.2}$$

In order to show that h belongs to $BMOH(R)$ and that the above mapping is bounded, it suffices to prove that

$$G_R(h) \leq K \sum_{j=1}^{n} G_{U_j}(h) \tag{4.3}$$

for some constant $K > 0$ which does not depend on h (Proposition 2.3).

Considering the absolute continuity of the harmonic measure on R to harmonic measures on U_j $(j = 1, \ldots, n)$, we confirm that

$$L.H.M._R|h|(z_0) \leq \sum_{j=1}^{n} L.H.M._{U_j}|h|(z_j)$$

is valid for some constant $K > 0$.

We denote by $g_R(z, w)$ (resp. $g_j(z, \omega)$) Green's function on R (resp. U_j) with pole at w. To estimate $L.H.M._R|h - h(p)|(p)$. we take doubly connected subregions V_j and V_j' of R $(j = 1, \ldots, n)$ satisfying the following conditions:

(1) One of their boundary curves is C_j, and other boundary curves are analytic Jordan curves in R which are homotopic to C_j.

(2) $V_j' \cup \partial V_j' \subset V_j \subset V_j \cup \partial V_j \subset U_j$.

We will show that there exists a constant $k_1 > 0$ such that

$$k_1 g_j(z, w) \geq g_R(z, w) > 0, \tag{4.4}$$

for all $(z, w) \in V_j \times V_j'$.

To show (4.4), put $s_w(z) = k_1 g_j(z, w) - g_R(z, w)$ for some $k_1 > 1$. The function $s_w(\cdot)$ is a superharmonic funtion and vanishes on C_j. Hence, if $s_w > 0$ on ∂V_j for every $w \in V_j'$ then $s_w > 0$ in V_j from the minimum principle. For $z \in \partial V_j$, $s_w(z) = s_z(w)$ because of the symmetric property of Green's functions. When we fix a point z on ∂V_j, it is easily seen that for sufficiently large $k_1 > 1$, $s_z > 0$ on a compact subset K_j of V_j. Since $s_z(w)$ is a continuous function for (z, w), we verify that there exists a sufficiently large $k_1 > 1$ such that $s_z(w) = k_1 g_j(w, z) - g_R(w, z) > 0$ on $\partial V_j \times K_j$.

Next, we take a point $w \in C_j$. In a neighborhood of w, $Z = g_R + i^* g_R = X + iY$ and $W = g_j + i^* g_j = U + iV$ are local coordinates. The function U vanishes along C_j which is the Y-axis. Thus,

$$\frac{\partial U}{\partial Y} = 0 \quad \text{on } C_j.$$

Therefore,

$$\frac{\partial U}{\partial X} \neq 0 \quad \text{on } C_j.$$

Thus, there exists a constant $k > 0$ and a neighborhood $A_j(z)$ of C_j, which may depend on z, such that $g_j(w, z) = U \geq kX = kg_R(w, z) > 0$ for every $w \in A_j(z)$. Considering the continuity of Green's functions for (z, w) again, we see that there exists a neighborhood A_j of C_j such that

$$k_{g_j}(z, w) - g_R(z, w) \geq 0$$

for $(z, w) \in \partial V_j \times A_j$. Since $V_j' - A_j$ is a compact subset of V_j, the above inequality holds for every $(z, w) \in \partial V_j \times (V_j' - A_j)$ because of the previous argument. Therefore,

$$s_w(z) = k_1 g_j(z, w) - g_R(z, w) > 0$$

for $(z, w) \in \partial V_j \times V_j'$ and (4.4) holds. It is known that for $p \in U_j$

$$L.H.M._{U_j}|h - h(p)|(p) = \frac{1}{2\pi}\int_{\partial U_j} |h(z) - h(p)| \frac{\partial}{\partial n_z} g_j(z, p)\, ds$$

and

$$L.H.M._R|h - h(p)|(p) = \frac{1}{2\pi}\int_{\partial R} |h(z) - h(p)| \frac{\partial}{\partial n_z} g_R(z, p)\, ds,$$

where $s(z)$ is the length function and n_z is the inner normal vector at $z \in \partial R \cap \partial U_j$. Hence,

$$\sup_{V_j} L.H.M._{U_j}|h - h(p)|(p) \geq \frac{1}{k} \sup_R L.H.M._R|h - h(p)|(p).$$

From the maximum principle, we have for $p \in R - \bigcup_{j=1}^n V_j'$

$$\begin{aligned} L.H.M._R|h - h(p)|(p) &\leq L.H.M._R|h|(p) + |h(p)| \\ &\leq 2 \sup_{\bigcup_{j=1}^n \partial V_j'} L.H.M._R|h|. \end{aligned}$$

Harnack's principle gives inequalities

$$\sup_{\cup_{j=1}^{n}\partial V_j} L.H.M._R|h| \le cL.H.M._{U_j}|h|(z_j)$$

for some constant $c > 0$. We complete the proof of the lemma. □

The construction of $\tilde{g}_l$ implies that $\tilde{g}_l$ induces $l \in h^1(R)^*$ for every function in $HB(R)$. Since $HB(R)$ is dense in $h^1(R)$ (Corollary 3.1), we conclude that the boundary function $\tilde{g}_l$ of a $BMOH$ function $H^R_{\tilde{g}_l}$ induces $l \in h^1(R)^*$.

REMARK. As mentioned in the introduction, the relation $(H^1(R))^* = BMOA(R)$ does not hold on all Riemann surfaces in SO_{HB}. In fact, there is a counter example which is given by Heins [8]. His interesting example means that the ideal boundary of harmonic measure zero is not negligible for certain kinds of (quasi-bounded) analytic functions.

5. Conditional expectation and *BMO*

Let R be a Riemann surface which does not belong to the class O_G, namely the Riemann surface R has a Green's function. The Riemann surface R has non-constant positive superharmonic functions. So, the universal covering surface of R is (conformally equivalent to) and unit disk Δ. Therefore, R is represented by a Fuchsian group Γ as Δ/Γ. Here, we consider a Borel σ-field $\Sigma(\Gamma)$ for Γ. A Borel subset $U \subset \partial\Delta$ belongs to $\Sigma(\Gamma)$ if for each γ, $|U \ominus \gamma(U)| = 0$, where $|\cdot|$ means the Lebesgue measure on $\partial\Delta$ and $A \ominus B$ is the symmetric difference of the sets A and B. We denote by $L^p(\Gamma)$ $(1 \le p \le \infty)$ the set of all $\Sigma(\Gamma)$ measurable functions which are in $L^p(\partial\Delta)$, the L^p space with respect to the Lebesgue measure on $\partial\Delta$.

DEFINITION 5.1. For each measurable function f in $L^p(\partial\Delta)$, there exists a unique function $E_\Gamma[f] \in L^p(\Gamma)$ so that

$$\int_U f(e^{i\theta})\, d\theta = \int_U E_\Gamma[f]\, d\theta$$

for all $U \in \Sigma(\Gamma)$. We call $E_\Gamma[f]$ *the conditional expectation of f.*

The existence of $E_\Gamma[f]$ is guaranteed by the measure theory (see Fisher [3] for details). It is easily seen that $E_\Gamma[f]$ is Γ-automorphic, that is,

$$E_\Gamma[f](\gamma(e^{i\theta})) = E_\Gamma[f](e^{i\theta})$$

for all $\gamma \in \Gamma$ and for almost all $e^{i\theta} \in \partial\Delta$.

Now, we consider to characterize the dual space of $h^1(R)$ in terms of the conditional expectation E_Γ. Take any element $l \in h^1(R)^*$. Considering the lifts of functions in $h^1(R)$, we see that $h^1(R)$ is regarded as a subspace of $h^1(\Delta) = \mathrm{Re}\, H^1(\Delta)$. Hence, by Hahn–Banach theorem, l is extended to an element l^* of $(\mathrm{Re}\, H^1(\Delta))^*$ and it is given by a BMO-function φ on $\partial\Delta$. Take any function $h \in L^\infty(\partial\Delta)$. Then we have

$$l^*(h) = \int_{\partial\Delta} h\varphi \, d\theta.$$

If h is Γ-automorphic, namely it is a lift of some $HB(R)$ function, then

$$\begin{aligned} l(h) = l^*(h) &= \int_{\partial\Delta} h\varphi \, d\theta \\ &= \int_{\partial\Delta} E_\Gamma[h\varphi] \, d\theta = \int_{\partial\Delta} hE_\Gamma[\varphi] \, d\theta. \end{aligned}$$

The last equality is due to a property of the conditional expectation. Thus, l is induced by $E_\Gamma[\varphi]$. Conversely, it is also seen that $E_\Gamma[\varphi]$ ($\varphi \in BMO$) induces an element in $h^1(R)^*$, and we have established the following.

PROPOSITION 5.1. *Let R be a Riemann surface represented by Δ/Γ. If $HB(R)$ is a dense subspace of $h^1(R)$, then $h^1(R)^* = E_\Gamma[BMOH(\Delta)]$.*

The conditional expectation gives automorphic functions, like the Poincaré operator. From the elementary view of the measure theory, we verify that E_Γ is a bounded mapping from $L^p(\partial\Delta)$ onto $L^p(\Gamma)$ $(1 \le p \le \infty)$. From Theorem 1 and Proposition 5.1, the similar results holds for BMO when the Riemann surface is an SO_{HB} end.

COROLLARY 5.1. *Let $R = \Delta/\Gamma$ be an SO_{HB} end. Then, the conditional expectation E_Γ is bounded linear mapping of $BMOH(\Delta)$ onto $BMOH(R) \circ \pi$.*

Proof. From Corollary 3.1, the hypothesis of Proposition 5.1 is satisfied. Hence, $E_\Gamma[BMOH(\Delta)] = (h^1(R))^*$. Thus, Theorem 1 implies that for each $\varphi \in BMO(\partial\Delta)$ there exists a function $f_\varphi \in BMOH(R)$ such that

$$\int_{\partial\Delta} hE_\Gamma[\varphi] \, d\theta = \int_{\partial\Delta} h(f_\varphi \circ \pi) \, d\theta$$

holds for every $h \in h^2(R) \circ \pi = L^2(\Gamma)$. Therefore, $E_\Gamma[\varphi] = f_\varphi \circ \pi$ $(\in BMO(\partial\Delta))$ almost everywhere in $\partial\Delta$. □

REMARK. Earle–Marden [2] gives an explicit form of E_Γ on $L^p(R)$ $(p > 1)$ for a compact bordered Riemann surface R in terms of the Poincaré series for Γ. But, to the best knowledge of the author, no explicit form of E_Γ on $L^p(\partial R)$ is known for a general Riemann surface.

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Oh-Okayama Meguro-ku Tokyo 152 Japan

Received June 19, 1991