# On planarity of graphs in 3-manifolds.

Comment. Math. Helvetici 67 (1992) 635–647 0010-2571/92/040635-13\$1.50 + 0.20/0


# On planarity of graphs in 3-manifolds*

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A graph $\Gamma$ in a 3-manifold $M$ is called *planar* if it is contained in an embedded 2-sphere in $M$. It is *abstractly planar* if it can be embedded into an abstract 2-sphere. In [3] Scharlemann and Thompson gave necessary and sufficient conditions for a graph $\Gamma$ to be planar in $S^3$ (see Theorem 3 in section 3). The special case that $\Gamma$ has a single vertex was proved by Gordon [1], while the generic case was shown [2] to be equivalent to: An abstractly planar graph $\Gamma$ in $S^3$ is planar if and only if both $\Gamma - e$ and $\Gamma/e$ are planar, where $e$ is a noncycle edge of $\Gamma$. Fix an embedding of $\Gamma$ in a 2-sphere $F$. We say that the embedding of $\Gamma$ in $S^3$ is $F$-planar if it can be extended to an embedding of $F$ into $S^3$. It turns out that the above result is equivalent to: If both $\Gamma - e$ and $\Gamma/e$ are $F$-planar, then $\Gamma$ is also $F$-planar.

In this paper, we study the $F$-planarity of a graph $\Gamma$ in a 3-manifold $M$, where $F$ can be an arbitrary surface containing $\Gamma$, or more generally a 2-dimensional cell complex with $\Gamma$ as 1-skeleton. An embedding of $\Gamma$ in a 3-manifold $M$ is called $F$-planar if it can be extended to an embedding of $F$ in $M$. We are interested in the problem of whether the $F$-planarity of $\Gamma$ is determined be that of $\Gamma - e$ and $\Gamma/e$. A statement parallel to the case of $F = S^2$ is not true in this general setting. However we will show it is true if $\Gamma$ is a "regular" graph.

We first study the triviality of cycles. This can be considered a special case of the above problem, when the cell complex has only one 2-cell. A *cycle* of $\Gamma$ is a subgraph $C$ which is homeomorphic to a circle.

DEFINITION. Suppose $\Gamma$ is embedded in a 3-manifold $M$. Then a cycle $C$ of $\Gamma$ is trivial (with respect to $(M, \Gamma)$), if it bounds a disk with interior disjoint from $\Gamma$.

In section 2 we prove a theorem about triviality of simple cycles. Note that if $C$ is a cycle of $\Gamma$, and $e$ is an edge intersecting $C$ at most once, then $C$ remains a cycle in both $\Gamma - e$ and $\Gamma/e$. Therefore it makes sense talking about the triviality of $C$ with respect to $(M, \Gamma - e)$ and $(M/e, \Gamma/e)$.

* Partially supported by NSF Grant DMS 9102633

THEOREM 1. *Suppose $\Gamma$ is a graph embedded in a 3-manifold $M$. Let $C$ be a cycle in $\Gamma$, and let $e$ be an edge of $\Gamma$ with at most one end on $C$. If $C$ is trivial with respect to both $(M, \Gamma - e)$ and $(M/e, \Gamma/e)$, then it is trivial with respect to $(M, \Gamma)$.*

A link $L$ in $S^3$ is the unlink if each component of $L$ is a trivial cycle. It turns out that this is also true for any abstractly planar graphs in a 3-manifold $M$:

THEOREM 2. *An abstractly planar graph $\Gamma$ in $M$ is planar if and only if all cycles of $\Gamma$ are trivial.*

We will prove Theorem 2 in Section 3, and use these theorems to give an alternative proof of the Scharlemann–Thompson Theorem.

In Section 4, we study the $F$-planarity of graphs in arbitrary 3-manifolds $M$. Suppose $\Gamma$ is a graph in a compact surface $F$. We assume that $\partial F$ is either empty or a subgraph of $\Gamma$. An embedding of $\Gamma$ into $M$ is *$F$-planar* if it can be extended to an embedding of $F$ into $M$. We call the closure of a component of $F - \Gamma$ a face of $F$. The graph $\Gamma$ is called a *regular graph* in $F$ if each face of $F$ is a disk, and the intersection of any two faces is connected (or empty). Suppose $e$ is an edge of $\Gamma$ with at least one end in the interior of $F$. Then both $\Gamma - e$ and $\Gamma/e$ can be considered as graphs in $F$ in the natural way, so we can talk about the $F$-planarity of $\Gamma - e$ and $\Gamma/e$. The following theorem is proved in section 4.

THEOREM 5. *Suppose $\Gamma$ is a regular graph on a surface $F$, and suppose $\Gamma$ is embedded in a 3-manifold $M$. Let $e$ be an edge of $\Gamma$ with at least one end in* Int $F$. *If both $\Gamma/e$ and $\Gamma - e$ are $F$-planar, then $\Gamma$ is $F$-planar.*

The regularity condition on $\Gamma$ is necessary. We will give an example of a graph $\Gamma$ on a torus $F$ that can be embedded into $S^3$, so that both $\Gamma - e$ and $\Gamma/e$ are $F$-planar, but $\Gamma$ itself is not $F$-planar.

I would like to thank Marty Scharlemann for some helpful discussion on this topic, and to the referee for many useful comments.

## 1. Definitions and preliminaries

Given a graph $\Gamma$ in a 3-manifold $M$, choose a regular neighborhood for each vertex and each edge of $\Gamma$, so that the disks $\partial N(v) \cap N(e)$ are mutually disjoint for all $v$ and $e$. The union of all such neighborhoods forms a regular neighborhood $N(\Gamma)$ of $\Gamma$ and we define the exterior of $\Gamma$ to be $E(\Gamma) = M - \operatorname{Int} N(\Gamma)$. For each vertex $v$, denote by $\delta(v)$ the punctured sphere $\partial N(v) - \bigcup \operatorname{Int} N(e)$; similarly, for each edge $e$, let $\delta(e)$ be the annulus $\partial N(e) - \bigcup \operatorname{Int} N(v)$. Sometimes the graph may vary, in which case we use $\delta_\Gamma(e)$ and $\delta_\Gamma(v)$ to denote $\delta(e)$ and $\delta(v)$, respectively. If $C$ is a cycle, or more generally a subgraph of $\Gamma$, we use $\delta(C)$ to denote the union of $\delta(t)$ with $t$ ranges over all edges and vertices of $C$.

For an edge $e$ in $\Gamma$, denote by $\Gamma - e$ the subgraph obtained from $\Gamma$ by deleting the interior of the edge $e$. If $e$ is not a loop, then $\Gamma/e$ is a graph in $M/e$. Denote by $\bar{e}$ the image of $e$ in $\Gamma/e$. The quotient map $\pi: M \to M/e$ sends $N(\Gamma)$ to a regular neighborhood $N(\Gamma/e)$ of $\Gamma/e$ in $M/e$, so it induces a homeomorphism $E(\Gamma) \cong E(\Gamma/e) = M/e - \operatorname{Int} N(\Gamma/e)$. We identify $E(\Gamma)$ with $E(\Gamma/e)$ by this homeomorphism. Note that $\delta_{\Gamma/e}(\bar{e}) = \delta_\Gamma(v) \cup \delta_\Gamma(e) \cup \delta_\Gamma(v')$ if $\partial e = v \cup v'$.

If $X$ is a subset of $M$, denote the number of components in $X$ by $|X|$.

We define a *simple disk* to be a disk $D$ in $M$ which is bounded by a cycle of $\Gamma$, and has interior disjoint from $\Gamma$. Thus a cycle of $\Gamma$ is a trivial cycle if and only if it bounds a simple disk.

Define a *normal structure* on $N(\Gamma)$ to be a set of line segments $\{l_x \mid x \in \partial N(\Gamma)\}$ as follows: For any vertex $v \in \Gamma$ and any $x \in \delta(v)$, let $l_x$ be the straight line in $D^3 = N(v)$ connecting $x$ to $v$. If $e$ is not a loop, the closure of $N(e) - \bigcup \{l_x \mid x \in \bigcup \delta(v)\}$ has a product structure $e \times D^2$ such that for $x \in \partial e \times \partial D^2$, the $l_x$ defined above is the line between $x$ and a point in $\partial e \times 0$. Now for any $x \in p \times \partial D^2$ with $p \in e$, let $l_x$ be the line connecting $x$ to $p \times 0$. If $e$ is a loop, $N(e) - \bigcup \{l_x \mid x \in \bigcup \delta(v)\}$ is homeomorphic to $e \times D^2/\partial e \times 0$, so we can define $l_x$ in the same way as above. For any $p \in e$, $p \times D^2$ is called a *meridian disk* of $\Gamma$ (or $e$) at $p$, and $p \times \partial D^2$ is called a *meridian* of $\Gamma$.

Suppose $P$ is a surface in $E(\Gamma)$. The normal extension $D$ of $P$ is the union of $P$ and the lines $l_x$ with $x \in P \cap \partial N(\Gamma)$. If $P$ is a properly embedded disk in $E(\Gamma)$, and $C$ is a cycle of $\Gamma$ such that $P$ intersects any meridian of $C$ exactly once, and is disjoint from the other meridians of $\Gamma$, then $D$ is a disc with $\partial D = C$. A surface $S$ in $M$ with $\partial S$ in $\Gamma$ is called in normal position if $S$ is the normal extension of $S \cap E(\Gamma)$. The following lemma is useful in modifying disks to make their interiors disjoint.

LEMMA 1.1. *Suppose $D_1, \ldots, D_n$ are simple disks in $M$ with mutually disjoint interiors. Suppose $C$ is a trivial cycle, and $C \cap D_i$ is connected for all $i$. Then $C$ bounds a simple disk $D$ with interior disjoint from $D_i$ for all $i$.*

*Proof.* By an isotopy we may assume $D_1, \ldots, D_n$ are in normal position. Let $P_i = D_i \cap E(\Gamma)$. Choose a simple disk $D$ in normal position and bounded by $C$ so that $P = D \cap E(\Gamma)$ is transverse to $P_i$, and $|P \cap (\bigcup P_i)|$ is minimal. Let $A$ be the closure of $\operatorname{Int} D \cap (\bigcup D_i)$. Since $A \cap N(\Gamma)$ consists of lines $l_x$ with $x \in \partial P \cap (\bigcup P_i)$, we know that $A$ is the union of some circles which may intersect $\Gamma$ at one point, and some arcs with different endpoints on $\Gamma$. These circles and arcs might intersect on $\Gamma$, but are otherwise disjoint. If $A$ has some circles, choose a circle $\alpha$ which is innermost in some $D_i$, and let $\Delta$ and $\Delta_i$ be the disks it bounds in $D$ and $D_i$ respectively. Then $(D-\Delta)\cup\Delta_i$ can be rel $\partial D$ isotoped into a disk $D'$ with $|D'\cap(\bigcup P_i)|<|P\cap(\bigcup P_i)|$. If $A$ has no circles but has some arcs, let $\beta$ be an arc in $A$ which is outermost in the sense that there is an arc $\gamma$ in some $C\cap\partial D_i$, such that $\beta\cup\gamma$ bounds a disk $\Delta_i$ in $D_i$ with $\operatorname{Int}\Delta_i\cap D=\emptyset$. (This is possible because of the assumption that $C\cap\partial D_i$ is connected for all $i$). Let $\Delta$ be the disk in $D$ with $\partial\Delta=\partial\Delta_i=\beta\cup\gamma$. Then a perturbation of $(D-\Delta)\cup\Delta_i$ produces a disk $D'$ with $|D'\cap(\bigcup P_i)|<|P\cap(\bigcup P_i)|$. By the minimality of $|P\cap(\bigcup P_i)|$, neither case can happen. Therefore $A=\emptyset$. □

In section 3 we will need some handle addition lemmas. Let $F$ be a surface on the boundary of a 3-manifold $M$, and let $J$ be a simple loop on $F$. Denote by $\tau(M, J)$ the manifold obtained from $M$ by attacking a 2-handle along $J$, that is, $\tau(M, J)=M\cup(D^1\times D^2)$, where $D^1\times\partial D^2$ is identified with a regular neighborhood $N(J)$ of $J$ in $F$. Denote by $\sigma(F, J)$ the surface $(F-N(J))\cup(\partial D^1\times D^2)$. We have the following generalized handle addition lemma.

LEMMA 1.2. *Suppose $S$ is a surface on the boundary of a 3-manifold $M$. Let $\gamma$ be a 1-manifold on $S$ such that $S-\gamma$ is compressible, and let $J$ be a circle in $S$ disjoint from $\gamma$. If $\sigma(S, J)$ is compressible in $\tau(M, J)$ with $D'$ a compressing disk, then $S-J$ has a compressing disk $D$ such that $\partial D\cap\gamma\subset\partial D'\cap\gamma$.*

This was implied in the proof of [4, Thm 1]. It was shown that under the assumption we have $|\partial D\cap\gamma|\le|\partial D'\cap\gamma|$, but the argument there has actually proved that $\partial D\cap\gamma\subset\partial D'\cap\gamma$.

## 2. Trivial cycles in a graph

Given a cycle $C$ in $\Gamma\subset M$, and a noncycle edge $e$ of $\Gamma$, if $e$ does not have both endpoints on $C$, then $C$ remains a cycle in $\Gamma-e$ and $\Gamma/e$. The following theorem shows that the triviality of $C$ with respect to $(M,\Gamma)$ is determined by that with respect to $(M,\Gamma-e)$ and $(M/e,\Gamma/e)$.

THEOREM 1. *Suppose $\Gamma$ is a graph embedded in a 3-manifold $M$. Let $C$ be a simple cycle in $\Gamma$, and let $e$ be an edge of $\Gamma$ with at most one end on $C$. If $C$ is trivial with respect to both $(M,\Gamma-e)$ and $(M/e,\Gamma/e)$, then it is trivial with respect to $(M,\Gamma)$.*

*Proof.* The Theorem is simple when $C$ is disjoint from $e$: Let $\pi : M \to M/e$ be the quotient map. By assumption $C$ bounds a disk $D$ in $M/e$ with interior disjoint from $\Gamma/e$. Since $e$ is disjoint from $C$, $\pi^{-1}(D)$ is a simple disk in $M$ bounded by $C$.

Now we assume $e$ has exactly one end on $C$. Since $C$ is trivial with respect to $(M, \Gamma - e)$, there is a disk $D$ in $M$ such that $\partial D = C$, and $\operatorname{Int} D \cap \Gamma = \operatorname{Int} D \cap e$. Consider $E(\Gamma) = M - \operatorname{Int} N(\Gamma)$. The surface $P = D \cap E(\Gamma)$ is a planar surface satisfying

(*1): $\partial P$ consists of circles $\partial_0, \partial_1, \ldots, \partial_n$, where $\partial_1, \ldots, \partial_n$ are meridians of $e$ on $\partial N(e)$, and $\partial_0$ is a curve on $\delta(C)$ intersecting each meridian of $C$ at a single point.

Conversely, any planar surface $P$ in $E(\Gamma)$ satisfying (*1) can be extended to a disk $D$ in $M$ such that $\partial D = C$ and $\operatorname{Int} D \cap \Gamma = \operatorname{Int} D \cap e$.

Now consider $C$ as a cycle in $\Gamma/e$. Since $C$ is trivial with respect to $(M/e, \Gamma/e)$, there is a disk $D'$ in $M/e$ bounded by $C$ with $\operatorname{Int} D'$ disjoint from $\Gamma/e$. The surface $Q = D' \cap E(\Gamma)$ is a disk satisfying

(*2): $\partial Q$ is a curve on $\partial N(C \cup e)$, which intersects each meridan of $C$ at a single point.

Conversely, any such disk $Q$ can be extended to a disk $D'$ in $M/e$ with $\partial D' = C$ and $\operatorname{Int} D' \cap (\Gamma/e) = \varnothing$.

We choose $P$ and $Q$ to satisfy (*1) and (*2), as well as the following general position and minimality conditions:

(*3): $n = |P \cap \delta(e)|$ is minimal, and $k = |Q \cap \delta(e)|$ is minimal.

(*4): $P$ intersects $Q$ transversely, and $|P \cap Q|$ is minimal subject to (*3).

(*5): $P \cap Q \cap \delta(e') = \varnothing$ for each edge $e'$ in $C$.

(*5) is possible because by (*1) and (*2) each of $P \cap \delta(e')$ and $Q \cap \delta(e')$ is an essential arc in $\delta(e')$, so we can isotop $Q$ to make them disjoint. Since $k$ is minimal, $Q \cap \delta(e)$ consists of parallel essential arcs. So we may further assume

(*6): each component of $Q \cap \delta(e)$ intersects each $\partial_j$ at a single point, $j = 1, \ldots, n$.

If either $n = 0$ or $k = 0$, then an extension of $P$ or $Q$ is a disc $D$ in $M$ with $\partial D = C$ and $\operatorname{Int} D \cap \Gamma = \varnothing$, so $C$ is trivial with respect to $(M, \Gamma)$, as required. Hence we assume both $n$ and $k$ are positive. Label the components of $\partial P$ so that, beginning with a point on $\partial N(C)$, an arc of $\partial Q \cap \delta(e)$ intersects $\partial_1, \partial_2, \ldots, \partial_n$ successively.

A point of $\partial P \cap \partial Q$ is labeled $i$ if it is a point on $\partial_i$. Thus any arc on $P \cap Q$ has a label on each of its end points.

LEMMA 2.1. *A component of $P \cap Q$ in $P$ is an arc which is either essential or has both ends on $\partial_0$.*

*Proof.* If $P \cap Q$ has some circle components, a 2-surgery of $P$ along some disk in $Q$ bounded by an innermost circle will reduce $|P \cap Q|$. Therefore $P \cap Q$ consists of arcs only.

If $P \cap Q$ has some arc which is inessential in $P$ and has both ends on some $\partial_j$ with $j \neq 0$, let $\alpha$ be an outermost one, so there is an arc $\beta$ on $\partial_j$ such that $\alpha \cup \beta$ bounds a disk $\Delta$ in $P$ with interior disjoint from $Q$. A boundary compression of $Q$ along $\Delta$ produces two disks, one of which satisfies (*2), but has less components of intersection with $\delta(e)$, contradicting the minimality of $k$. □

LEMMA 2.2. *There is a label $i_0 > 0$ such that no arc of $P \cap Q$ has both ends labeled $i_0$.*

*Proof.* Otherwise choose an $\alpha_i$ for each $i = 1, \ldots, n$, with $\partial\alpha_i$ on $\partial_i$. Then the innermost such $\alpha_i$ will be an inessential arc on $P$. □

Examine the order in which the indices appear on $\partial Q$. By (*6), if we delete all the 0 indices, the sequence is $1, 2, \ldots, n, n, \ldots, 2, 1$ repeated $k/2$ times. The 0 indices appear only possibly between two successive 1's.

LEMMA 2.3. *An arc $\alpha$ of $P \cap Q$ which is outermost in $Q$ is of one of the following types.*

Type (i): *$\alpha$ has both ends labeled 1 or both ends labeled $n$.*

Type (ii): *$\alpha$ has one end labeled 1 and the othere labeled 0.*

*Proof.* Note that if $i, j$ are successive labels on $\partial Q$, then $|i - j| \leq 1$. Therefore if $\alpha$ is not of Type (i) or (ii), then the labels of $\alpha$ are either $\{0, 0\}$ or $\{i, i+1\}$ for some $i > 0$. Let $\beta$ be the arc on $\partial Q$ so that $\alpha \cup \beta$ bounds a disk $\Delta$ in $Q$ with interior disjoint from $P$.

Suppose $\alpha$ has label 0 on both endpoints. Then $\partial\alpha$ divides $\partial_0 \subset \partial P$ into two arcs $\partial_0'$ and $\partial_0''$, one of which, say $\partial_0''$, has the property that it intersects a meridian of $C$ if and only if $\beta$ does. So $\partial_0' \cup \beta$ intersects any meridian of $C$ at a single point. Let $P_1$ be the part of $P$ bounded by $\partial_0' \cup \alpha$. Then $P' = P_1 \cup \Delta$ satisfies (*1). Moreover, $|\partial P'| \leq |\partial P|$, and a perturbation of $P'$ has less components of intersection with $Q$ than $P$ does. This is impossible by (*4).

Now suppose $\alpha$ has labels $\{i, i+1\}$ for some $i > 0$. Then the normal extension of $\Delta$ is a disk $\Delta'$ in $M$ such that $\partial\Delta' = \alpha' \cup \beta'$, where $\alpha' \subset D$, $\beta' \subset e$, and $\operatorname{Int} \Delta' \cap \Gamma = \varnothing$. So we can isotop $\beta'$ through $\Delta'$ to reduce $|D \cap e|$. This contradicts the minimality of $n$. □

Note that the proof does not apply to the case when the labels of $\alpha$ are $\{0, 1\}$, since part of $\beta'$ may be on $C$.

LEMMA 2.4. *There are at least two outermost edges* $\alpha_1$, $\alpha_2$ *of Type* (ii).

*Proof.* By Lemma 2.2, there is an index $i_0$ such that no arc in $P \cap Q$ has both ends labeled $i_0$. Let $A$ be the set of arcs in $P \cap Q$ with one end labeled $i_0$. Let $\Delta$ be a disk in $Q$ such that $\gamma = \partial\Delta - \partial Q$ is an arc in $A$, and $\Delta$ contains no other arcs in $A$. Note that there are at least two such $\Delta$'s. So we need only to show that there is at least one type (ii) outermost edge in $\Delta$.

Suppose there is no outermost arc of type (ii) in $\Delta$. Then by Lemma 2.3, each outermost arc in $\Delta$ is of type (i), so the labels of the arc are either $\{1, 1\}$ or $\{n, n\}$. If there are two such outermost arcs, then the index $i_0$ appears between them, which is impossible by the definition of $\Delta$. So there is only one outermost arc on $\Delta$. This implies that the arcs of $P \cap Q$ are all parallel in $\Delta$. It is now clear that every arc in $\Delta$ has the same index on both ends. Especially, both ends of $\gamma$ are labeled $i_0$, contradicting the choice of $i_0$. □

Now let $\Delta_1$, $\Delta_2$ be two disks in $Q$ such that $\partial\Delta_i - \partial Q$ is an outermost arc of type (ii). Then the normal extension of $\Delta_i$ is a disk $\Delta_i'$ in $M$ with $\partial\Delta_i' = \alpha_i \cup \beta_i \cup \gamma_i$, where $\alpha_i$ is an arc in $D$ connecting a vertex $v_i$ of $C$ to the first intersection $x$ of $e$ with Int $D$, $\beta_i$ is an arc on $e$ connecting $x$ to $v_0 = e \cap C$, and $\gamma_i$ is an arc on $C$ connecting $v_0$ to $v_i$. ($\gamma_i$ may degenerate to a single point.) Since $\partial Q$ intersects a meridan of $C$ at a single point, the two arcs $\gamma_1$ and $\gamma_2$ cannot have an edge in common, and hence intersect only at $v_0$. Thus $\Delta_1' \cap \Delta_2' = \beta_1 = \beta_2$, so $\Delta = \Delta_1' \cup \Delta_2'$ is a disk in $M$. Let $D_1$ be the part of $D$ bounded by $\partial\Delta$, and let $D_2$ be $(D - D_1) \cup \Delta$ pushed off $\beta_1 - v_0$. Then $D_2$ is a disk in $M$ with $\partial D_2 = C$, and $|\text{Int } D_2 \cap e| \le n - 1$. This contradicts the minimality of $n = |\text{Int } D \cap e|$. □

## 3. Planar graphs in manifolds

In this section we will discuss the planarity of graphs in a 3-manifold. Suppose $\Gamma$ is a graph embedded in $M$. An edge $e$ of $\Gamma$ is called a free edge if it is not a cycle, and one of its endpoints is not incident to any other edges. Clearly, if $e$ is a free edge, then $\Gamma$ is planar in $M$ if and only if $\Gamma - e$ is planar. Therefore, without loss of generality we will always assume that $\Gamma$ has no free edges.

We need the following definitions: A graph $\Gamma$ in $M$ is called *split* if there is a 2-sphere $S$ in $M$ which is disjoint from $\Gamma$, and separates $M$ into $M_1$ and $M_2$, such that both $M_i$ contain part of $\Gamma$. It is called *decomposable* if there is a vertex $v \in \Gamma$ such that $\delta(v)$ has a compressing disk $D$ in $E(\Gamma)$ which is separating. The following lemma and its proof is similar to that of [3, Lemma 1.3].

LEMMA 3.1. *Let* $\Gamma$ *be a split or decomposable graph in a* 3*-manifold. If all proper subgraphs of* $\Gamma$ *are planar, then* $\Gamma$ *is planar.*

*Proof.* First assume $\Gamma$ is split. Let $S$ be a 2-sphere disjoint from $\Gamma$, separating $M$ into $M_1$ and $M_2$, such that $\Gamma_i = M_i \cap \Gamma$ are proper subgraphs of $\Gamma$. By assumption, there are 2-spheres $S_i \subset M$ such that $\Gamma_i \subset S_i$. By 2-surgery along disks bounded by innermost circles of $S_i \cap S$, we can delete all intersections of $S_i$ with $S$, and get $S_i \subset M_i$. Tubing $S_1$ to $S_2$ gives a 2-sphere containing $\Gamma$.

Now suppose $\Gamma$ is decomposable, and let $D$ be a separating compressing disk of $\delta(v)$ in $E(\Gamma)$. It can be extended to a 2-sphere $S$ in $M$ so that $S \cap \Gamma = \{v\}$, and $S$ separates $M$ into $M_1$ and $M_2$. Let $\Gamma_i = \Gamma \cap M_i$. Since $\Gamma_i$ is planar, there is a 2-disk $D_i$ in $E = M - \operatorname{Int} N(v)$ which contains $\Gamma_i \cap E$. By surgery along disks bounded by innermost circles or outermost arcs of $D \cap D_i$ in $D$, we can assume $D_i \cap D = \varnothing$. Gluing a band on $\delta(v)$ to $D_1 \cup D_2$ produces a single disk containing $\Gamma \cap E$, which can be extended to a sphere in $M$ containing $\Gamma$. □

Define a cut point of $\Gamma$ to be a vertex $v$ such that $\Gamma - v$ has more components than $\Gamma$. Let $\{v_1, \dots, v_k\}$ be the cut points of $\Gamma$. Then there is a component $X$ of $\Gamma - \{v_1, \dots, v_k\}$ which has the property that $\Gamma_1 =$ (the closure of $X$) contains at most one of these $v_i$; for otherwise one can find a simple loop in $\Gamma$ passing through some of the $v_j$'s, contradicting the definition of cutting points. This subgraph $\Gamma_1$ is connected, and has no cut point of its own. (It is possible that $\Gamma_1 = \Gamma$.)

Suppose $\Gamma$ is abstractly planar. Embed $\Gamma_1$ into a 2-sphere $S$. Since $\Gamma_1$ is connected and has no cut points of its own, the closure of each components of $S - \Gamma_1$ is a disk. Let $D_0, D_1, \dots, D_n$ be these disks. If $\Gamma_1$ contains a cut point $v$ of $\Gamma$, choose $D_0$ to contain $v$. Let $D = S - \operatorname{Int} D_0$.

Denote by $\Gamma_1^c$ the closure of $\Gamma - \Gamma_1$. Then $\Gamma_1^c \cap \Gamma_1 = \{v\}$ or $\varnothing$, depending on whether $\Gamma_1$ contains a cut point $v$ of $\Gamma$. Embed $\Gamma_1^c$ into a disk $D'$ so that $\partial D' \cap \Gamma_1^c = \{v\}$ or $\varnothing$ accordingly. Glue $D$ and $D'$ together, we get an embedding of $\Gamma$ into $S^2 = D \cup D'$. We fix this embedding.

Recall that $D_1, \dots, D_n$ are the closures of the components of $D - \Gamma_1$.

LEMMA 3.2. *We can number the disks so that $B_k = D_1 \cup \cdots \cup D_k$ is a disk for all $k$.*

*Proof.* If $D_i$ is a disk such that $D_1 \cap D_i$ is not connected, then $D_1 \cup D_i$ is not simply connected, so there is a region $\Omega$ in $D$ bounded by a boundary component of $D_1 \cup D_i$. $\Omega$ is the union of some $D_j$'s. Choose $i$ so that $\Omega$ contains a minimal number of these disks. Since $D_i$ is a disk, $\partial\Omega$ is not completely contained in $\partial D_i$, so there is a disk $D_j$ in $\Omega$ which has an edge in common with $D_1$. If $D_1 \cup D_j$ is not simply connected, then it bounds a region $\Omega' \subset \Omega$, contradicting the choice of $D_i$. Hence we can name this $D_j$ as $D_2$. Generally, if $B_k = D_1 \cup \cdots \cup D_k$ is a disk, then by the same argument we can find $D_{k+1}$ so that $B_k \cap D_{k+1}$ is an arc, and hence $B_k \cup D_{k+1}$ is a disk. The Lemma now follows by induction. □

A link in $S^3$ is a trivial link if and only if all of its components are trivial. The following theorem shows that this is also true for graphs in 3-manifolds.

THEOREM 2. *An abstractly planar graph $\Gamma$ in $M$ is planar if and only if all cycles of $\Gamma$ are trivial.*

*Proof.* We want to show that the inclusion $\Gamma \to M$ can be extended to an embedding of $\Gamma \cup B_n$ into $M$. This is done by induction. By assumption, $\partial D_1$ bounds a disk with interior disjoint from $\Gamma$, so we have an embedding of $\Gamma \cup D_1$ into $M$. Generally, suppose we have extended $\Gamma \to M$ to an embedding $i_k : \Gamma \cup B_k \to M$. By Lemma 3.2, $B_k$ is a disk, and $B_k \cap D_{k+1}$ is an arc. Consider the graph $\Gamma' = \Gamma - \operatorname{Int} B_k \subset M$. Then $\partial B_k$ and $\partial D_{k+1}$ are cycles in $\Gamma'$ which are trivial with respect to $(M, \Gamma')$. So by Lemma 1.1, $\partial D_{k+1}$ bounds a disk $\Delta_{k+1}$ which has interior disjoint from $\Gamma' \cup B_k$. Now we can define $i_{k+1} : \Gamma \cup D_1 \cup \cdots \cup D_{k+1} \to M$ so that $D_{k+1}$ is mapped to $\Delta_{k+1}$. This completes the induction.

It follows that the image of $B_n$ is an embedded disk $\Delta$ in $M$ so that $\Delta \cap \Gamma = \Gamma_1$, and $\partial\Delta \subset \Gamma_1$. When $\Gamma = \Gamma_1$ this implies $\Gamma$ is planar. When $\Gamma \neq \Gamma_1$, the set $\Gamma_1 \cup \Gamma_1^c$ is either empty or a cut point, which implies $\Gamma$ is split or decomposable. By induction we may assume that all proper subgraphs of $\Gamma$ are planar. The theorem now follows from Lemma 3.1. □

As an application of the above theorems, we give an alternative proof of a theorem of Scharlemann and Thompson [3].

THEOREM 3. *A finite graph $\Gamma \subset S^3$ is planar if and only if*

(a) *$\Gamma$ is abstractly planar;*

(b) *every graph properly contained in $\Gamma$ is planar;*

(c) *$\pi_1(E(\Gamma))$ is a free group.*

*Proof.* Since $\pi_1(E(\Gamma))$ is free, $E(\Gamma)$ is the connected sum of some handlebodies. If $\Gamma$ is not connected, then it is split, and the theorem follows from Lemma 3.1. So we assume $\Gamma$ is connected. When $\Gamma$ has only one vertex, the theorem was proved in [1], so we assume $\Gamma$ has some noncycle edge $e$. By induction on the number of edges in $\Gamma$, we may assume that $\Gamma/e$ is planar for all such $e$.

According to Theroem 2, we need only to show that each cycle of $\Gamma$ is trivial. Let $C$ by a cycle in $\Gamma$. There are several cases.

CASE 1 ($C$ does not contain all vertices of $\Gamma$). In this case there is some noncycle edge $e$ which has at most one endpoint on $C$. Since both $\Gamma - e$ and $\Gamma/e$ are planar, $C$ is trivial with respect to both $(S^3, \Gamma - e)$ and $(S^3/e, \Gamma/e)$. By Theorem 1, $C$ is also trivial with respect to $(S^3, \Gamma)$.

CASE 2 ($\Gamma$ has some cycle edges). A cycle edge cannot contain all vertices of $\Gamma$ because $\Gamma$ has more than one vertex. By Case 1, a cycle edge is a trivial cycle, so it bounds a simple disk. It follows that $\Gamma$ is decomposable, and the Theorem follows from Lemma 3.1.

In the remaining cases, all edges not in $C$ are noncycle edges with both ends on $C$. Let $e$ be such an edge. Its endpoints divide $C$ into two arcs $C_1$ and $C_2$.

CASE 3 (There is an edge $e'$ which has one endpoint on each of Int $C_i$). Consider the cycle $C_i \cup e$. It is incident to just one endpoint of $e'$. By Case 1, $C_i \cup e$ bounds a disk $D_i$ with interior disjoint from $\Gamma$. By Lemma 1.1, we can choose the $D_i$ to have disjoint interiors. Thus $D = D_1 \cup D_2$ can be modified off $e$ to become a simple disk bounded by $C$.

CASE 4 (No such edges $e'$ as in Case 3 exist). Note that in this case $\bar{e}$, the image of $e$ in $\Gamma/e$, is a cut point of $\Gamma/e$, and hence a decomposing point because $\Gamma/e$ is planar. We want to apply Lemma 1.2 to our situation. To do this, let $M = E(\Gamma)$, and let $F = \partial N(C \cup e) - \operatorname{Int} N(\Gamma)$. This is a punctured genus 2 surface, with one hole for each end of each edge which is not in $C \cup e$. Let $e_1, \ldots, e_k$ be the edges and $v_1, \ldots, v_k$ the vertices of $C$. Denote by $m_i$ a meridian of $e_i$, and by $J$ a meridian of $e$. Let $\gamma = m_1 \cup \cdots \cup m_k$.

$F - \gamma$ is isotopic to $\delta(e) \cup \delta(v_1) \cup \cdots \cup \delta(v_k) = \delta_{\Gamma/e}(\bar{e}) \cup (\bigcup \{\delta(v_i) \mid v_i \notin \partial e\})$. Since $\bar{e}$ is a decomposing point of $\Gamma/e$, $\delta_{\Gamma/e}(\bar{e})$ is compressible in $E(\Gamma)$. So $F - \gamma$ is compressible.

Consider $\tau(E(\Gamma), J)$. This is the manifold obtained from $E(\Gamma)$ by attaching a 2-handle along a meridian of $e$, so it is actually the exterior of $\Gamma - e$. The surface $\sigma(F, J)$ is the punctured torus $\partial N(C) - \operatorname{Int} N(\Gamma - e)$. Since $\Gamma - e$ is planar, $C$ bounds a disk in $M$, which gives rise to a compressing disk $D'$ of $\sigma(F, J)$ in $E(\Gamma - e)$, so that $D'$ intersects each $m_j$ at a single point. By Lemma 1.2, $F - J$ has a compressing disk $D$ in $E(\Gamma)$ intersecting each $m_j$ at most once. Since $F - J$ is a punctured torus, and $m_j$ are meridians, if $D$ is disjoint from some $m_j$, it is disjoint from all $m_j$, so it will be a compressing disk of some $\delta(v_i)$, which implies $\Gamma$ is decomposable, and the Theorem follows. So we assume $D$ intersects each $m_j$ at one point. Then we can modify $D$ so that $\partial D$ intersects any meridian of $C$ at a single point. The normal extension $\Delta$ of $D$ is now a simple disk bounded by $C$. □

## 4. $F$-planarity of graphs

Let $F$ be a finite 2-dimensional cell complex with a connected graph $\Gamma$ as its 1-skeleton. $\Gamma$ is called a regular graph in $F$ if the attaching map of each face (i.e.

2-cell) is a cycle in $\Gamma$, and the intersection of any two faces is connected. Suppose $\Gamma$ is embedded in a 3-manifold $M$. Then $\Gamma$ is called $F$-planar if it can be extended to an embedding of $F$ in $M$. Suppose $e$ is an edge of $\Gamma$ which is not contained in the boundary of any faces of $F$. Then $f - \operatorname{Int} e$ has $\Gamma - e$ as 1-skeleton, and $F/e$ has $\Gamma/e$ as 1-skeleton. To simplify notations, we call $\Gamma/e$ (resp. $\Gamma - e$) $F$-planar if it is $(F/e)$-planar (resp. $(F - \operatorname{Int} e)$-planar). The following is a generalization of Theorem 1.

THEOREM 4. *Suppose $F$ is a regular 2-complex with $\Gamma$ as its 1-skeleton, and suppose $\Gamma$ is embedded in a 3-manifold $M$. Let $e$ be a noncycle edge of $\Gamma$ such that both $\Gamma - e$ and $\Gamma/e$ are $F$-planar. If $e$ intersects each face of $F$ at most at one of its endpoints, then $G$ is $F$-planar.*

*Proof.* This follows from Theorem 1 and Lemma 1.1 by induction on the number of faces in $F$. $\square$

The most interesting case of $F$-planarity is when $F$ is a surface. It was shown in [2] that Theorem 3 is equivalent to the following:

THEOREM 3′. *Let $\Gamma$ be an abstractly planar graph in $S^3$ (or $R^3$). If $\Gamma$ has a noncycle edge $e$ such that both $\Gamma - e$ and $\Gamma/e$ are planar, then $\Gamma$ is planar.*

The following is a similar result for regular graphs in an arbitrary compact surface $F$. Suppose $\Gamma$ is such a graph, and $e$ is a noncycle edge which has at least one endpoint in the interior of $F$. Since $F/e \cong F$, both $\Gamma - e$ and $\Gamma/e$ can be considered naturally as a graph in $F$.

THEOREM 5. *Suppose $\Gamma$ is a regular graph on a surface $F$, and suppose $\Gamma$ is embedded in a 3-manifold $M$. Let $e$ be a noncycle edge of $\Gamma$ with at least one end in* $\operatorname{Int} F$. *If both $\Gamma/e$ and $\Gamma - e$ are $F$-planar, then $\Gamma$ is $F$-planar.*

*Proof.* We may assume that each end of $e$ has valence at least 3, otherwise $\Gamma$ is hemeomorphic to $\Gamma/e$, and the planarity of $\Gamma$ follows from that of $\Gamma/e$. Especially, an end of $e$ in $\operatorname{Int} F$ is incident to at least 3 faces of $F$.

Denote by $D'$, $D''$ the two disks incident to $e$. Consider the 2-complex $G = F - \operatorname{Int} D' \cup \operatorname{Int} D''$. First suppose $e$ has both ends on some face $D$ of $G$, then $D$ contains $\partial D - e$ because $D \cap D'$ is connected. Similarly, $D$ contains $\partial D'' - e$. By assumption $\partial D$ is a cycle, so $\partial D = \partial(D' \cup D'')$. This is now a very special case: $\Gamma$ has 3 edges and 2 vertices, and $F$ is a 2-sphere. Since $\Gamma/e$ is $F$-planar, $\Gamma/e$, and hence $\Gamma$, is contained in a 3-ball. Therefore the theorem follows from Theorem 3′.

Let $D_1, \ldots, D_n$ be the faces of $G$ and consider $G$ as a subset of $M$. Since $D_i$ intersects $e$ at most once, it remains a disk in $M/e$. By assumption $\Gamma/e$ is $F$-planar in $M/e$, so $\partial D'/e$ bounds a disk $\Delta$ in $M/e$ with $\operatorname{Int}\Delta \cap \Gamma/e = \emptyset$. Since $\partial D' \cap D_i$ is connected, $\partial\Delta \cap D_i$ is connected for all $i = 1, \ldots, n$. By Lemma 1.1 we can choose $\Delta$ so that $\operatorname{Int}\Delta \cap D_i = \emptyset$ for $i = 1, \ldots, n$. Let $Q$ be the disk $\Delta \cap E(\Gamma/e) = \Delta \cap E(\Gamma)$ in $E(\Gamma)$. $Q$ is disjoint from $\bigcup D_i$, and $\partial Q$ intersects each meridian of $\partial D' - e$ at a single point. Let $v$ be an end of $e$ in $\operatorname{Int} F$. Isotop $Q$ so that $|\partial Q \cap \delta(e)|$ is minimal. Then $A = \partial Q \cap \delta(v)$ consists of arcs on the punctured sphere $\delta(v)$ which are all essential. As the circle $\partial Q$ intersects a meridian of $\partial D' - e$ at a single point, there is an arc $\alpha \in A$ with exactly one end on the circle $J = \delta(v) \cap \delta(e)$, while all the other arcs in $A$ have both ends on $J$. The arcs in $A$, being part of $\partial Q$, are disjoint from the disks $D_1, \ldots, D_n$. Because $F$ is a surface, and $v$ is in $\operatorname{Int} F$, these disks cut $\delta(v)$ into an annulus. It follows that all arcs in $A - \{\alpha\}$ are inessential, which is absurd unless $\alpha$ is the only arc in $A$. Therefore $\partial Q$ intersects a meridian of $e$ at a single point. The normal extension $\Delta'$ of $Q$ is now a disk bounded by $\partial D'$, with interior disjoint from $G$. Similarly, there is a disk $\Delta''$ bounded by $\partial D''$, such that $\operatorname{Int}\Delta'' \cap G = \emptyset$, and by Lemma 1.1, it can be chosen so that $\Delta' \cap \Delta'' = e$. The surface $G \cup \Delta' \cup \Delta''$ is now an embedding of $F$ in $M$. $\square$

The regularity condition in Theorem 4 is necessary. Consider the graph $\Gamma$ on a torus $F$ as shown in Figure 1. Embedding $F$ into $S^3$ in the trivial way, we get a graph $\Gamma_1$ which is $F$-planar in $S^3$. Let $\Gamma_2$ be the embedding of $\Gamma$ in $S^3$ as shown in Figure 2, obtained from $\Gamma_1$ by interchanging a crossing in Figure 1. Let $e$ be the edge shown in the figure. It is easy to see that $\Gamma_2 - e$ and $\Gamma_2/e$ are isotopic to $\Gamma_1 - e$ and $\Gamma_1/e$ respectively, so they are $F$-planar in $S^3$. One can also isotop $\Gamma_2$ so that it lies on the trivial torus. But $\Gamma_2$ is not $F$-planar. To see this, one may need the following fact.

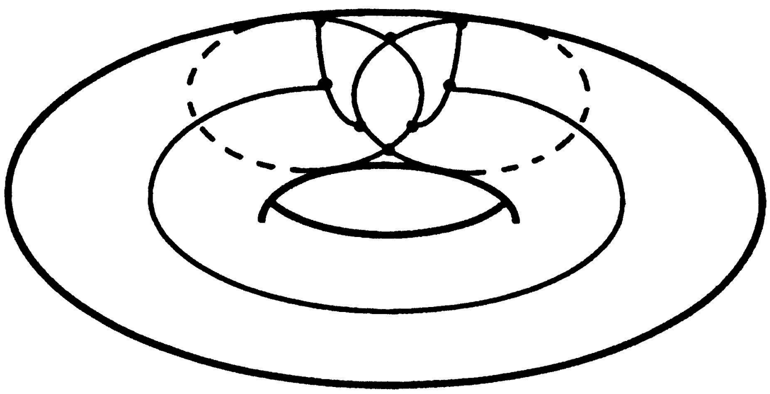

Figure 1

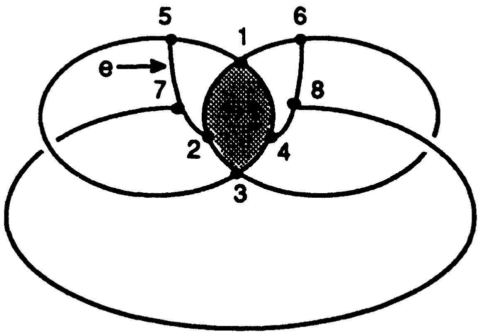


Figure 2

LEMMA 4.1. *Suppose $\Gamma$ is a graph in $S^3$, and $C$ is a trivial cycle with respect to $(S^3, \Gamma)$. If $\Gamma \cap E(C)$ is connected, then the simple disk $D$ bounded by $C$ is unique up to ambient isotopy fixing $\Gamma$.*

Label the vertices of $\Gamma_2$ as in Figure 2. Denote by $C(i_1, \ldots, i_k)$ the cycle successively passing through the vertices labeled $i_1, \ldots, i_k$. Suppose $\Gamma_2$ is $F$-planar. Then $C(1, 2, 3, 4)$ and $C(1, 5, 3, 6)$ should bound simple disks with disjoint interiors. By Lemma 4.1, the disks are unique up to isotopy, so we can take the disk $D$ bounded by $C(1, 2, 3, 4)$ to be the shaded region in Figure 2. Now $C(1, 5, 3, 6)$ cannot bound a disk with interior disjoint from $\Gamma_2 \cup D$, because it has linking number 1 with some curve in $\Gamma_2 \cup D - C(1, 5, 3, 6)$.

REFERENCES

[1] C. GORDON, *On primitive sets of loops in the boundary of a handlebody*, Topology and its Appl. *27* (1987), 285–299.

[2] M. SCHARLEMANN, *Planar graphs, family trees, and braids*, preprint.

[3] M. SCHARLEMANN and A. THOMPSON, *Detecting unknotted graphs in 3-space*, preprint.

[4] Y-Q. WU, *A generalization of the handle addition theorem*, Proc. Amer. Math. Soc. *114* (1992), 237–242.

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Received September 17, 1991