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Relative growth rates of closed geodesics on a surface under varying hyperbolic structures

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This paper has two interrelated goals. The first is to give necessary and sufficient conditions that an action of a surface group on a tree be geometric, that is to say, be dual to a codimension-1 measured lamination on the surface. The second is to study the limiting ratios of lengths of simple closed geodesics under a degenerating sequence of hyperbolic structures on the surface. We begin by explaining in more details each of these goals.

In order to explain the first, let us return to an old result of Stallings. Recall from [5] that if $X \subset M$ is a compact codimension-1 submanifold, then there is an action of $\pi_1(M)$ on a simplicial tree dual to $X \subset M$ (or more precisely dual to $\tilde{X} \subset \tilde{M}$ where $\tilde{M}$ is the universal covering of M).

THEOREM (Stallings). *Let S be a closed surface; let $\Gamma = \pi_1(S)$; and let $\Gamma \times T \to T$ be a minimal action of Γ on a simplicial tree. Then this action is dual to a compact 1-manifold $X \subset S$ if and only if the stabilizer in Γ of every edge of T is cyclic.*

Our first goal is to generalize this result to actions of surface groups on $\mathbb{R}$-trees and on Λ-trees for more general ordered abelian groups Λ. As was proved in [7], associated to any codimension-1 measured lamination in a manifold M, satisfying some mild conditions, there is a dual action of $\pi_1(M)$ on an $\mathbb{R}$-tree which is minimal. If M is a surface then the stabilizers of germs of edges in such a dual action are all cyclic. This leads naturally to the following question.

If Γ, the fundamental group of a closed surface S acts on an $\mathbb{R}$-tree so that the action is minimal and so that the stabilizer of the germs of edges are cyclic, then is the action dual to a measured lamination in S?

We do not know the answer to this question. The first purpose of this paper is to show that, with one additional assumption on the action, the answer is yes.

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It is easy to establish that the answer to the above question is yes if the Euler characteristic $\chi(S)$ is non-negative. We prove the following theorem which provides a partial answer to the question in the hyperbolic case.

THEOREM. *Let $\Gamma \times T \to T$ be a minimal action of a closed hyperbolic surface group $\pi_1(S)$ on an $\mathbb{R}$-tree. Then the action is dual to a measured lamination in S if and only if the following two conditions are satisfied*:

(a) *The stabilizer of every germ of an edge is cyclic.*

(b) *Any time γ_0 and γ_1 are elements in Γ whose axes in the universal covering $\tilde{S}$ cross, then the axes of γ_0 and γ_1 in T, provided that they both exist, also intersect.*

Thus, one of our conditions is purely geometric and the other concerns the stabilizers. To answer the question above is to decide whether (a) implies (b). It seems likely to us, though we have not seriously tried to establish it, that (b) implies (a). When we prove this theorem in Section IV.1, we prove it in the generality of Λ-trees for any ordered abelian group Λ of finite rank.

The second main theme of the paper is the study of growth rates of simple closed geodesics under a degenerating sequence of hyperbolic structures. A hyperbolic structure α on a surface S determines a function from the set $\mathscr{C}(S)$ of conjugacy classes in $\Gamma = \pi_1(S)$ to $\mathbb{R}$. This function associates to each $\gamma \in \mathscr{C}(S)$ the length $\ell_\gamma(\alpha)$ of the closed geodesic representing γ in the hyperbolic structure. If (α_n) is a sequence of hyperbolic structures, we can always extract a subsequence so that for any pair of elements $\gamma, \mu \in C$ the ratio $\lim_{n\to\infty} \ell_\gamma(\alpha_n)/\ell_\mu(\alpha_n)$ exists in $[0, \infty]$. If (α_n) is an unbounded sequence in Teichmüller space, then these limit ratios determine a point in the projective space with homogeneous coordinates indexed by $\mathscr{C}(S)$. This point is the limit point in Thurston's compactification of Teichmüller space.

It is easy to establish that conditions (a) and (b) above hold for the action produced in [6] from unbounded sequences in Teichmüller space. Thus, one corollary of the above theorem is that all limit points of Teichmüller space in the projective spaces are given by intersection numbers with a fixed measured lamination. (In [6] this result was recovered by appealing to Hironaka's result on resolvability of singularities; here we give a more direct, satisfactory proof.) Thus, this part of the paper can be viewed as a natural conclusion to [6].

Because we work with Λ-trees rather than $\mathbb{R}$-trees we are able to generalize the results of [6] from the archimedean context described above to the non-archimedean context.

The idea is that the limit point in the projective space ignores much of the information in the limit ratios. Any time $\lim_{n\to\infty} \ell_{\gamma_i}(X_n)/\ell_\mu(X_n) = 0$ for $i = 0, 1,$

then the value $\lim_{n \to \infty} \ell_{\gamma_0}(X_n)/\ell_{\gamma_1}(X_n)$ is completely undetermined by the point in the projective space. In this sense, taking the point in projective space is like taking a top order term of the ratios and ignoring the rest. As we indicated above, the restriction of the results of this paper to $\mathbb{R}$-trees gives a different proof of Thurston's result that the top order term is represented geometrically by intersection with a measured lamination on the surface. By working with more general ordered abelian groups we extend this result to the full limit ratio: we show that it is represented geometrically by intersection with a non-archimedeanly measured lamination on the surface.

The paper is organized along the following lines. Chapter I is an introductory one, devoted to expanding the theory of Λ-trees introduced in [6] and to developing the theory of codimension-1 measured laminations where the measure takes values in an arbitrary ordered abelian group. In Section I.1 we define the notion of an edge of a Λ-tree. Using this we define a morphism of Λ-trees. In Section I.2 we define transversely measured codimension-1 laminations where the measure takes values in an arbitrary ordered abelian group. In Section I.3 we state axioms for a codimension-1 measured lamination and show that these axioms guarantee that the lamination has a dual Λ-tree. In Section I.4 we give an alternate set of axioms for the case of hyperbolic surfaces, and we show that these alternate axioms imply the original ones.

Chapter II is a one-section chapter which introduces the notion of a transverse map from the complement of a codimension-1 lamination in a manifold to a Λ-tree. Next, we show that if the measured lamination has a dual tree then the transverse map factors through a morphism of the dual tree. Lastly, we show that given a compact manifold M and an action of $\pi_1(M)$ on a Λ-tree T, there is a measured lamination $\mathscr{L} \subset M$ and an equivariant transverse map of $\tilde{M} - \tilde{\mathscr{L}}$ to T.

In Chapter III we prove that if M is a complete hyperbolic surface of finite area and if we have an action $\pi_1(M) \times T \to T$ on a Λ-tree T then there is a codimension-1 measured lamination $\mathscr{L} \subset M$ with a dual tree and a transverse, equivariant map $(\tilde{M} - \tilde{\mathscr{L}}) \to T$. The method of proof is to begin with any measured lamination and transverse, equivariant map as constructed in Chapter II and modify it until the axioms set forth in Section I.4 are satisfied. There are several steps in this process; they occupy the entire chapter.

Chapter IV is devoted to proving the two main results of the paper. The characterization of geometric actions of surface groups on Λ-trees is proved in the following way. Given an action of $\pi_1(M)$ on a tree T, by the results of Chapter III we construct a measured lamination in M with a dual tree T' and a transverse, equivariant map from the complement of the lamination to T. We then show by a direct argument that the natural map $T' \to T$ can have no folding if the action of T satisfies the two conditions given in the theorem above. Hence it is an isomorphism.

Then, we show that all the limiting ratios of lengths of closed geodesics for a degenerating sequence of hyperbolic structures on a surface are given by intersection numbers with a non-archimedeanly measured lamination in the surface. Given the first main result of the paper and the work of [6], this result is easily deduced.

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Since this paper was accepted, its main result has been extended by R. Skora ([9], [10]), who proved that an action of a surface group on a Λ tree is dual to a Λ-measured lamination, once the edge stabilizers are cyclic.

Chapter I. Trees and measured laminations

Section I.1. *Trees and pre-trees*

In this section we expand the theory of trees developed in [6]. Let Λ be an ordered abelian group and let $T, d : T \times T \to \Lambda$ be a Λ-tree. Let $T = T_0 \amalg T_1$ be a division of T. A *spanning segment* for this division is a segment $I \subset T$ with one endpoint in T_0 and the other in T_1. If any two spanning segments for the division have the property that their intersection contains at least one point of both T_0 and T_1, then we say that the division is an *edge* of T.

If $T = T_0 \amalg T_1$ is an edge of T, then an element $v_i \in T_i$ is a *vertex of the edge* if v_i belongs to every spanning segment for the edge. There is at most one vertex of the edge in each T_i, and hence at most two vertices in all. In general, an edge can have none, one, or two vertices. If e is an edge with vertex v, then e is called a *direction* at v. The *order of* $v \in T$ is the number of distinct directions at v.

If I is a segment in a Λ-tree, then a Dedekind cut of I is a division $I = I_0 \amalg I_1$, with neither I_0 nor I_1 empty, such that there is an isometric embedding $\varphi : I \hookrightarrow \Lambda$ with $\varphi(\alpha_0) < \varphi(\alpha_1)$ for all $\alpha_0 \in I_0$ and $\alpha_1 \in I_1$. A point $p \in I$ is an end point of the Dedekind cut if either $I_0 = \{\alpha \in I \mid \varphi(\alpha) < \varphi(p)\}$ or $I_0 = \{\alpha \in I \mid \varphi(\alpha) \leq \varphi(p)\}$.

If e is an edge of T and if I is a spanning segment of e, then e determines a Dedekind cut of I. Any endpoint of this Dedekind cut is a vertex of e and conversely. Conversely, if $I \subset T$ is a segment, then a Dedekind cut of I determines an edge of T. If $I = I_0 \amalg I_1$ is the Dedekind cut, then we define $T_i \subset T$ to be the set of points in T which can be joined by a segment which misses I_{1-i} to I_i. One sees easily that $T = T_0 \amalg T_1$ is an edge of T and that I is a spanning segment for this edge.

Examples. (i) A $\mathbb{Z}$-tree is the same thing as the set of vertices of a simplicial tree. An edge of a $\mathbb{Z}$-tree is the same thing as an edge of the corresponding simplicial tree. Each edge of a $\mathbb{Z}$-tree has two vertices.

(ii) If Λ is a dense subgroup of $\mathbb{R}$, then any Λ-tree T can be embedded isometrically as a dense subset of an $\mathbb{R}$-tree $\bar{T}$. An edge of T corresponds to a maximal class of arcs in $\bar{T}$ with the property that any two members of the class have intersection which contains a third member. An edge e of T has at most one vertex; and it has v as a vertex if and only if all the arcs in $\bar{T}$ belonging to the corresponding class contain the point v.

The notion of an edge is important in order to define a morphism between Λ-trees. A *morphism* from T to T' is a function $\varphi : T \to T'$ with the property that for each edge e of T there is a spanning segment $I_e \subset T$ for e with $\varphi \mid I_e$ an isometry.

We claim that if $\varphi : T \to T'$ is a morphism then φ induces a map from the set of edges of T to those of T'. To define the induced map on the set of edges, let e be an edge of T. Choose a spanning segment I_e for e for which $\varphi \mid I_e$ is an isometry. The edge e is equivalent to a Dedekind cut $(I_e)_0 \cup (I_e)_1$ of I_e. The image under φ is a Dedekind cut of $\varphi(I_e)$. Hence, it determines an edge e' of T'. One sees easily that e' is independent of the choice of I_e. We define $\varphi_*(e) = e'$.

The map on edges and vertices is compatible: if $v \in T$ is a vertex of an edge e, then $\varphi(v)$ is a vertex of $\varphi_*(e)$. Thus, φ induces a map from the set of directions at $v \in T$ to the set of directions at $\varphi(v)$ in T'. This map need not be $1-1$. If it is not $1-1$, then we say that φ *folds* at v. Along the same lines, we say that φ *folds along a segment* $I \subset T$ at $v \in \mathring{I}$ if the two directions at v in I are identified under φ. Suppose that φ folds along I at v. Parametrize I as $\{\alpha \in \Lambda \mid a \le \alpha \le b\}$. Set a_0 equal to the parameter value at v. Then there are a_-, a_+, with $a \le a_- < a_0 < a_+ \le b$ such that for all $\alpha \in \Lambda, 0 \le \alpha \le a_0 - a_-$, we have $\varphi(a_0 + \alpha) = \varphi(a_0 - \alpha)$.

LEMMA I.1.1. *Let $\varphi : T \to T'$ be a morphism of Λ-trees. Either there is a vertex $v \in T$ at which φ folds or φ is an isometry onto a sub-Λ-tree of T'.*

Proof.

STEP I. *For all $a, b \in T$ we have $d(\varphi(a), \varphi(b)) \le d(a, b)$.*

Proof of Step I. Suppose that there is a, b with $d(\varphi(a), \varphi(b)) > d(a, b)$. Let I be the segment in T from a to b. Let $I_0 \subset I$ be $\{t \mid d(\varphi(a), \varphi(s)) \le d(a, s)$ for all s, $a \le s \le t\}$. Clearly, if $t_1 < t_2$ and if $t_2 \in I_0$, then $t_1 \in I_0$. Also $a \in I_0$ and $b \notin I_0$. Hence, if we set $I_1 = I - I_0$, (I_0, I_1) is a Dedekind cut of I. This determines an edge e of T. Let $J \subset I$ be a spanning segment for e with the property that $\varphi \mid J$ is an isometry. Let j_0 and j_1 be the endpoints of J with $j_i \in I_i$. For any $j \in J$, we have

$d(a, j) = d(a, j_0) + d(j_0, j))$. Since $j_0 \in I_0$; $d(a, j_0) \geq d(\varphi(a), \varphi(j_0))$. Since $\varphi \mid J$ is an isometry, $d(\varphi(j_0), \varphi(j)) = d(j_0, j)$. Hence, for all $j \in J$, we have

$$d(a, j) \geq d(\varphi(a), \varphi(j_0)) + d(\varphi(j_0), \varphi(j)).$$

By the triangle inequality,

$$d(\varphi(a), \varphi(j_0)) + d(\varphi(j_0), \varphi(j)) \geq d(\varphi(a), \varphi(j)).$$

Hence, the required inequality holds for all $j \in J$. Hence, $j_1 \in I_0$. This contradicts the fact that $j_1 \in I_1$.

STEP II. *For any $a, b \in T$ let I be the segment with endpoints a and b. Then $d(\varphi(a), \varphi(b)) = d(a, b)$ if and only if φ has no folding along I.*

Proof of Step II. If $\varphi \mid I$ folds at $t \in I$ then there are $t_1 < t < t_2$ with $\varphi(t_1) = \varphi(t_2)$. Hence

$$\begin{aligned} d(\varphi(a), \varphi(b)) &= d(\varphi(a), \varphi(t_1)) + d(\varphi(t_2), \varphi(b)) \\ &\leq d(a, t_1) + d(t_2, b) = d(a, b) - d(t_2, t_1) < d(a, b). \end{aligned}$$

Conversely, suppose that $\varphi \mid I$ has no folds. Let $I_0 \subset I$ be the set of t for which $d(\varphi(a), \varphi(t)) = d(a, t)$. Clearly, $a \in I_0$. By step I, if $a \leq s \leq t$, and $t \in I_0$, then $s \in I_0$. Hence, either $I_0 = I$, in which case $d(\varphi(a), \varphi(b)) = d(a, b)$, or $(I_0, I - I_0)$ is a Dedekind cut of I. Let us assume the latter. Let e be the corresponding edge of T, and let $J \subset I$ be a spanning segment for e with the property that $\varphi \mid J$ is an isometry, say $J = [j_0, j_1]$ with $j_0 \in I_0$ and $j_1 \in I - i_0$. We have $\varphi([a, j_0])$ is a segment of length $d(a, j_0)$, and $\varphi([j_0, j_1])$ is a segment of length $d(j_0, j_1)$. Thus, either $\varphi([a, j_1])$ is a segment of length $d(a, j_0) + d(j_0, j_1) = d(a, j_1)$ or $\varphi([a, j_0]) \cap \varphi([j_0, j_1])$ is a non-degenerate segment. In the former case $j_1 \in I_0$ which is a contradiction; in the latter φ folds along I at j_0.

Proof of Lemma. If φ is not $1-1$, then there are points $a \neq b$ in T with $\varphi(a) = \varphi(b)$. Let I be the segment in T connecting a to b. Clearly, according to Step II φ folds along I at some point. This proves that either φ is $1-1$ or φ has folding. If φ is $1-1$, then it follows immediately from Step II that φ is an isometry onto its image. □

COROLLARY I.1.2. *Suppose that T and T' are Λ-trees each with a Γ-action. Suppose that T' is minimal with respect to its Γ-action, and suppose that $\varphi : T \to T'$ is a Γ-equivariant morphism. Then either φ is an isometry or φ has folding.*

Proof. $\varphi(T) \subset T'$ is a Γ-invariant subtree. Since T' is minimal for the Γ-invariant subtree. Since T' is minimal for the Γ-action, this implies that $\varphi(T) = T'$. The result is now immediate from Lemma I.1.1. □

(I.1.3). It is convenient to work with slightly more general objects than Λ-trees. These were introduced by Bass [1] and Gillet–Shalen [2]. We follow Gillet–Shalen and call them *pre-trees*. A Λ pre-tree is a Λ-metric space (T, d) such that:

(i) for any $x, y \in T$ the set $[x, y]$ consisting of all $z \in T$ with $d(x, z) + d(y, z) = d(x, y)$ is isometric to a subset of a closed interval in Λ, with x and y corresponding to the endpoints of that interval;

(ii) for any $x, y, z \in T$ we have $[x, y] \cap [x, z] = [x, w]$ for some $w \in T$; and

(iii) if $[x, y] \cap [x, z] = \{x\}$ then $[x, y] \cup [x, z] = [y, z]$. The set $[x, y]$ is the *pre-segment with endpoints x and y*. Gillet–Shalen [2] show that any Λ-pre-tree can be embedded isometrically in a Λ-tree ΛT in such a way that every point of ΛT lies in a segment with endpoints in T. Furthermore, ΛT is unique up to an isometry which is the identity on T. It is called the *Λ-completion* of T. Every point of $\Lambda T - T$ is an edge point of ΛT.

Section I.2. *Non-Archimedean measured laminations*

In [7] the basic theory of codimension-1, measured laminations is presented. These objects have a transverse measure which is real-valued, or equivalently complete and archimedean. In order to measure the non-archimedean growth rates discussed in the previous sections we need non-archimedean measures. These will not be complete but rather will have values in an ordered abelian group.

Recall that a codimension-1 measured lamination $\mathscr{L}$ has at most countably many complementary regions each bounded by at most countably many leaves of ℓ. These leaves are said to be *isolated on one side*.

(I.2.1). Let $\mathscr{L} \subset M$ be a codimension-1 lamination and let $V = U \times I$ be a flow box for $\mathscr{L}$. Then the local leaf space of $\mathscr{L} \mid V$ is identified with a closed subset $X \subset I$ and the support of $\mathscr{L}$, $|\mathscr{L}|$, meets V in $U \times X$. The set Y of complementary regions of $\mathscr{L}$ in V is then identified with the set of components of $I - X$. These complementary regions are ordered (up to sign) by the order on I. By a *segment in Y* we mean any subset of the form $S_a^b = \{y \in S \mid a \leq y \leq b\}$, for $a \leq b$ arbitrary elements of y. Given a segment $S_a^b \subset Y$ and an element $y_0 \in S_a^b$ we can subdivide S_a^b into two segments $S_a^{y_0} = \{y \in S \mid a \leq y \leq y_0\}$ and $S_{y_0}^b = \{y \in S \mid y_0 \leq y \leq b\}$.

A *measure* on Y with values in an ordered abelian group Λ is a symmetric function

$$\mu : Y \times Y \to \Lambda^{\geq 0}$$

such that for any segment $S_a^b \subset Y$ the function

$$S_a^b \to [0, \mu(a, b)] \text{ defined by } y \to \mu(y, a)$$

is an order preserving bijection. We define the measure $\mu(S_a^b)$, to be $\mu(a, b)$. This measure is an additive function on segments in the sense that if $y_0 \in S_a^b$, then $\mu(S_a^{y_0}) + \mu(S_{y_0}^b) = \mu(S_a^b)$. A segment S is degenerate if and only if $\mu(S) = 0$; if S is a subsegment of S, then $\mu(S) \leq \mu(S')$. Lastly, if $S = S_1 \cup \cdots \cup S_n$ is a finite partition of S, then $\mu(S) = \Sigma_{i=1}^n \mu(S_i)$. Since we do not assume that Λ is complete we cannot hope to extend this to a countably additive measure, nor can we hope to define the integral over an arbitrary sub-interval of *i*.

If $J : [0, 1] \to N$ is transverse to $\mathscr{L}$ (see [7]) and if the endpoints of J lie in $V - |\mathscr{L}|$, then $\mu(J)$ is defined as an element of Λ. Namely, we partition J into finitely many closed measure intervals with endpoints in $V - |\mathscr{L}|$, each of which is either contained in a complementary component or is monotone in the I-direction. Each such interval has a natural image in Y which is a segment. By definition $\mu(J)$ is the sum of the values of μ along these segments in Y.

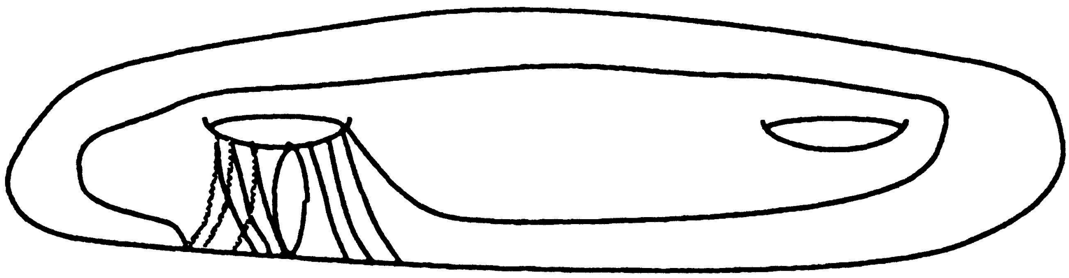
If V_1 and V_2 are two flow boxes and if μ_1 and μ_2 are Λ-valued transverse measures for V_1 and V_2, then μ_1 and μ_2 are *compatible* provided that for any closed interval J in $V_1 \cap V_2$ which is transverse to $\mathscr{L}$ and has endpoints in $V_1 \cap V_2 - |\mathscr{L}|$ we have $\mu_1(J) = \mu_2(J)$.

DEFINITION I.2.2. A transverse measure with values in Λ for $\mathscr{L}$, is a compatible set of transverse measures with values in Λ, one for each flow box of $\mathscr{L}$.

Notice that if μ is a transverse measure and if $J : [0, 1] \to M$ is a path transverse to $\mathscr{L}$ with $\{J(0) \cup J(1)\} \subset M - |\mathscr{L}|$, then $\mu(J) \in \Lambda$ is defined.

Examples I.2.3. Let S be a surface and let $\tau \subset S$ be a train track. Associated to this train track we have the real vector space with basis the branches of τ, $\mathbb{R}^{\mathfrak{B}}$, and the integral linear branch equations (see [7]). A non-negative solution $w \in \mathbb{R}^{\mathfrak{B}}$ to these equations gives rise to a measured lamination carried by τ with the given weights. One can show that any closed interval transverse to the lamination has measure in the free abelian subgroup of $\mathbb{R}$ generated by the weights $w(b)$, $b \in \mathfrak{B}$. Thus, in fact the measured lamination has value group contained in this free abelian group of finite rank. Of course, this group is ordered via its inclusion into $\mathbb{R}$. This order is archimedean. It thus can be completed to a $\mathbb{R}$-measure. This completion is the usual (countably additive) transverse measure.

Example. Let Λ be $\mathbb{Z} \times \mathbb{Z}$ with the lexicographic order. Let $\mathscr{L}$ be the lamination on a surface with two leaves, indicated below.



The transverse measure is determined by defining $\mu(A) = (0, 1)$ and $\mu(B) = (1, 0)$. This lamination does not admit an archimedean measure of full support, for the integral of B must be infinitely large with respect to the integral of A.

Section I.3. *Trees dual to measured laminations*

In [5] the notion of a tree dual to an (archimedeanly) measured lamination was introduced and necessary and sufficient conditions were given for the existence of such a dual tree. In this section we generalize this to the non-archimedean case.

Let $(\mathscr{L}, \mu)$ be a codimension-1 lamination in a compact n-manifold M with a transverse measure taking values in the ordered abelian group Λ. We denote by $(\tilde{\mathscr{L}}, \tilde{\mu})$ the induced measured lamination in the universal covering $\tilde{M}$ of M. Throughout this section we make the following assumptions

(I.3.1)
- (a) $|\mathscr{L}|$ is nowhere dense in M;
- (b) each leaf of $\tilde{\mathscr{L}}$ is a closed subset of $\tilde{M}$;
- (c) if p and q are points of $\tilde{M} - \tilde{X}$ then there is a path $\omega : I \to \tilde{M}$, with $\omega(0) = p$, and $\omega(1) = q$, which is transverse to $\tilde{\mathscr{L}}$ and which crosses each leaf at most once.

In [5] it was proved that if $\Lambda = \mathbb{R}$, then under these assumptions there is a dual tree. Miniking this argument, we shall construct a dual Λ-tree for arbitrary Λ. What we actually do is construct a Λ-pre-tree T and then appeal to the general results which allow us to complete the Λ-pre-tree to a Λ-tree.

The points of T are the complementary regions of $\tilde{\mathscr{L}}$ in $\tilde{M}$, i.e., the connected components of $\tilde{M} - |\tilde{\mathscr{L}}|$. To define $d : T \times T \to \Lambda^{\geq 0}$ let R_0 and R_1 be complementary regions. Choose an arc $\omega : [0, 1] \to \tilde{M}$ transverse to $\tilde{\mathscr{L}}$ with $\omega(i) \in R_i$ for $i = 0, 1$ that meets each leaf of $\tilde{\mathscr{L}}$ in at most one point (here, $[0, 1]$ denotes the segment in the real numbers). We set $d(R_0, R_1) = \int_\omega \tilde{\mu} = \tilde{\mu}(\omega)$. The argument in [5, Lemma 5] generalizes mutatis-mutandis to show:

LEMMA I.3.2. *$d(R_0, R_1)$, as defined above, is independent of the choice of ω.* □

This then defines T and a function $d : T \times T \to \Lambda^{\geq 0}$. One sees easily that (T, d) is a Λ-*metric* space (i.e. d is symmetric, $d(x, y) = 0$ iff $x = y$, and $d(x, y) + d(y, x) \geq d(x, y)$).

LEMMA I.3.3. *T is a Λ-pre-tree.*

Proof. Since we know that T is a Λ-metric space we need only show that the three axioms in (I.1.3) hold. Let $x, y \in T$ correspond to complementary regions R_0 and R_1. Let ω be an arc in $\tilde{M}$ from R_0 to R_1 satisfying (I.3.1)(c). We claim that (x, y) is equal to the set $w \in T$ corresponding to complementary region that ω meets. Clearly $w \in T$ corresponds to a complementary region that ω meets then $d(x, w) + d(w, y) = d(x, y)$. Conversely, suppose that $d(x, w) + d(w, y) = d(x, y)$. Let ω_1 and ω_2 be a path as in (I.3.1)(c) from x to w and from w to y. We can assume that $\omega_1(1) = \omega_2(0)$.

Then $\omega = \omega_1 * \omega_2$ is a path from R_0 to R_1 and $\int_\omega \mu = d(x, y)$. Arguing as in [5, Lemma 5] we see that ω satisfies (I.3.1)(c).

Associating to each $w \in T$ corresponding to a complementary region that ω meets $d(x, w)$ gives an isometric embedding of $[x, y]$ into $[0, d(x, y)] \subset \Lambda$.

The argument in the proof of Lemma 5 in [5] shows that $[x, y] \cap [x, z] = [x, w]$ for some $w \in T$.

Lastly, Axiom (iii) is clear from the description of $[x, y]$. □

COROLLARY I.3.4. *Let $(\mathscr{L}, \mu) \subset M$ be a measured lamination. Let $\tilde{\mathscr{L}} \subset \tilde{M}$ be the induced lamination in the universal covering of M. If $\tilde{\mathscr{L}}$ satisfies Axioms* II.3.1, *then there is a dual Λ-tree $\Lambda T_{\mathscr{L}}$. The action of $\pi_1(M)$ on $\tilde{\mathscr{L}}$ and $\tilde{M} - \tilde{L}$ induces an action of $\pi_1(M)$ on $\Lambda T_{\mathscr{L}}$.* □

Having finished the construction of the Λ-tree dual to $(\tilde{\mathscr{L}}, \tilde{\mu}) \subset M$, let us describe the correspondences:

$$(\text{I.3.7}) \quad \begin{cases} \text{(a) A point of the pre-tree } T_{\mathscr{L}} \subset \Lambda T_{\mathscr{L}} \text{ is a component of } \tilde{M} - |\tilde{\mathscr{L}}|. \text{ The set of } \textit{directions} \text{ in } \Lambda T_{\mathscr{L}} \text{ at the point corresponding to } R \text{ is in natural one-to-one correspondence with the boundary leaves of } R. \\ \text{(b) Every point of } \Lambda T_{\mathscr{L}} - T_{\mathscr{L}} \text{ is an edge point of } \Lambda T_{\mathscr{L}}. \text{ Each edge of } \Lambda T_{\mathscr{L}}, \text{ whether or not it is represented by a point of } \Lambda T_{\mathscr{L}} - T_{\mathscr{L}}, \text{ corresponds to a unique leaf of } \tilde{\mathscr{L}}. \end{cases}$$

(I.3.8). It follows from (I.3.7) that the stabilizer of a point in $T_{\mathscr{L}}$ is the stabilizer of a complementary component of $\tilde{\mathscr{L}}$ in $\tilde{M}$, or equivalently is the image in $\pi_1(M)$ of the fundamental group of a complementary component of $\mathscr{L}$ in M. The stabilizer of any point of $\Lambda T_{\mathscr{L}} - T_{\mathscr{L}}$ is the stabilizer of a leaf of $\tilde{\mathscr{L}}$, or equivalently is the image in $\pi_1(M)$ of the fundamental group of some leaf of $\mathscr{L}$.

Let $(\mathscr{L}, \mu) \subset M$ be a measured lamination. A loop ω in M is said to be *transverse* to $\mathscr{L}$ if either ω lies in a leaf of $\mathscr{L}$ or ω is covered by intervals transverse to $\mathscr{L}$ in the sense of Section I.2. In either case we can define $\int_\mu \omega$. In the first case the integral is zero.

COROLLARY I.3.9. *Let $(\mathscr{L}, \mu) \subset M$ be a measured lamination. Suppose $\tilde{\mathscr{L}} \subset \tilde{M}$ satisfies Axioms* I.3.1 *and that $\pi_1(M) \times \Lambda T \to \Lambda T$ is the dual action. Then for each $\gamma \in \pi_1(M)$ there is a loop $\omega_\gamma \subset M$ representing γ and transverse to $\mathscr{L}$ such that the geometric intersection of ω_γ with $(\mathscr{L}, \mu)$ i.e. $\int_{\omega_\gamma} \mu$, is equal to the translation length $\tau(\gamma)$. Furthermore, $\int_{\omega'_\gamma} \mu \geq \int_{\omega_\gamma} \mu$ for any loop ω'_γ freely homotopic to γ and transverse to $\mathscr{L}$.*

Proof. Let ℓ' be a loop in the free homotopy class of γ transverse to $\mathscr{L}$. Let $x \in \ell'_\gamma \cap (M - |\mathscr{L}|)$. Cut ℓ'_γ open at x and lift it to an arc in $\tilde{M}$. The endpoints of this arc are of the form $\tilde{x}$ and $\gamma\tilde{x}$ and lie in complementary regions R and γR. We have

$$\int_{\ell'_\gamma} \mu \geq d(R, \gamma R) \geq \tau(\gamma).$$

Now we construct a loop ℓ_γ to minimize the integral. Suppose first that γ either has an axis or a fixed point meeting T. Let R_0 be a complementary region to $\tilde{\mathscr{L}}$ in $\tilde{M}$ corresponding to a point in the axis or fixed set of γ. Let ω be a path in M from R_0 to γR_0 crossing each leaf of $\tilde{\mathscr{L}}$ at most once. We can suppose that ω projects to a loop ℓ in M. This loop satisfies $\int_\ell \xi = \tau(\gamma)$ and is in the free homotopy class of γ.

The last case to consider is when γ is an inversion, i.e. γ fixes no point of T but has translation length 0. Such an element leaves invariant an edge of ΛT. Thus, the element is represented by a loop in a leaf of $\mathscr{L}$. Clearly $\int_\ell \mu = 0$. □

DEFINITION I.3.10. Let $(\mathscr{L}, \mu) \subset M$ be a measured lamination satisfying Axioms I.3.1. We define *geometric intersection with* $(\mathscr{L}, \mu)$, $i_{(\mathscr{L}, M)} : \pi_1(M) \to \Lambda^{\geq 0}$ as follows; $i_{(\mathscr{L},\mu)}(\gamma) = \min_{\ell_\gamma} \mu$ where ℓ_γ ranges over all transverse loops in the free homotopy class of γ. Clearly, by (I.3.9) $i_{(\mathscr{L},\mu)} : \pi_1(M) \to \Lambda^{\geq 0}$ is the same as the translation length function of the dual tree.

Section I.4

In the last section we gave general axioms (I.3.1) for a measured lamination to have a dual tree. In this section we gave an alternative set of axioms in the special case of a surface.

The work of this section is to establish that the axioms given here are simply the conditions expressed in (I.3.1) thus guaranteeing that the measured lamination has a dual tree. It is the axioms (I.4.1) with which we work in the next chapter.

Let M be a complete hyperbolic surface of finite area and let $\mathscr{L} \subset M$ be a codimension-1 lamination. The universal covering $\tilde{M}$ of M can be identified with the hyperbolic disc $\mathbb{D}^2$. Hence there is an ideal boundary at infinity identified with S^1. We denote it by $\partial\tilde{M}$. Let $\tilde{\mathscr{L}} \subset \tilde{M}$ be the lamination covering $\mathscr{L}$. Let $\ell \subset \tilde{M}$ be a non-compact leaf of $\tilde{\mathscr{L}}$ with the leaf topology, ℓ is homeomorphic to the real line. Let $A \subset \ell$ be a ray. We identify A with $[0, \infty) \subset \mathbb{R}$. Then the embedding $\ell \subset \mathbb{D}^2$ induces a map $\varphi_A = [0, \infty[\to \mathbb{D}^2$. We say that A has *endpoint* $a \in S^1$ if in the compactification of $\mathbb{D}^2$ by $\mathbb{D}^2 \cup S^1$, $\lim_{t\to\infty} \varphi_A(t) = a$. Notice that this notion is independent of the identification of A with $[0, \infty)$.

Now we formulate our alternative conditions. Let M be a complete, hyperbolic surface of finite area. Let $(\mathscr{L}, \mu)$ be a codimension-1, measured lamination in M. Let $\tilde{\mathscr{L}} \subset \tilde{M} = \mathbb{D}^2$ be its lift. We suppose:

(I.4.1)

(a) Each leaf of $\tilde{\mathscr{L}}$ is a proper and non-compact submanifold.

(b) Each $x \in |\tilde{\mathscr{L}}|$ has arbitrarily small flow box neighborhoods meeting $\tilde{M} - |\tilde{\mathscr{L}}|$ and meeting each leaf of $\tilde{\mathscr{L}}$ in a connected set.

(c) Each leaf ℓ of $\tilde{\mathscr{L}}$ possesses two distinct endpoints in $\partial\tilde{M} = S^1$ and so defines a point $e(\ell)$ in $(S^1 \times S^1\text{-diagonal})/(\text{flip of factors}) = \mathscr{L}$.

(d) The map from the leaf space of $\tilde{\mathscr{L}}$ to $\mathscr{L}$ is continuous in the sense that if $x_i \in |\tilde{\mathscr{L}}|$ converges to $x_\infty \in |\tilde{\mathscr{L}}|$ and x_i lies on the leaf ℓ_i of $\tilde{\mathscr{L}}$ for all $i \leq \infty$, then the points $e(\ell_i)$ converge to $e(\ell_\infty)$ in $\mathscr{G}$.

Notice that we don't assume that e is one-to-one. The main result of this section is the following.

THEOREM I.4.2. *Suppose that M is a complete hyperbolic surface of finite area. Suppose that $(\mathscr{L}, \mu) \subset M$ satisfies* (I.4.1). *Then it satisfies Axioms* I.3.1.

Proof. Axioms I.3.1(a) and (b) are clear. We establish Axiom I.3.1(c). That is to say if R_1 and R_2 are distinct complementary regions of $\tilde{\mathscr{L}}$, then there is an arc $\omega : [0, 1] \to \mathbb{D}^2$ with endpoints in R_1 and R_2, transverse to $\tilde{\mathscr{L}}$, and crossing each leaf of $\tilde{\mathscr{L}}$ at most once. Since each leaf ℓ of $\tilde{\mathscr{L}}$ is proper and non-compact, it divides $\tilde{M}$ into two simply connected pieces. We say that a leaf ℓ of $\tilde{\mathscr{L}}$ separates R_1 from R_2 if R_1 and R_2 lie in distinct components of $\tilde{M} - \ell$. Clearly, if ℓ separates R_1 and R_2 then any arc from R_1 to R_2 which is transverse to $\tilde{\mathscr{L}}$ must cross ℓ an odd number of times.

We claim that the set of leaves separating R_1 from R_2 is an open and closed sublamination of $\mathscr{L}$. To prove this notice that each of R_1 and R_2 has a single boundary component separating the two regions. We call these leaves ℓ_1 and ℓ_2. If $\ell_1 = \ell_2$ then the result is clear. Suppose $\ell_1 \neq \ell_2$. These leaves are the frontier of a sub-disk B in D_2 with $B \cap \partial D^2$ consisting of two closed intervals (possibly reduced to points). For a leaf ℓ of $\tilde{\mathscr{L}}$ to separate R_1 from R_2 it must be contained in B. Furthermore, it must have an endpoint in each of I_+ and I_-. It is easy to see that this is an open and closed condition on the leaves of $\tilde{\mathscr{L}}$ because of (I.4.1)(c) and (d).

Let us denote by $\tilde{\mathscr{L}}(R_1, R_2)$ the sublamination of $\tilde{\mathscr{L}}$ consisting of leaves separating R_1 from R_2. The leaves of $\tilde{\mathscr{L}}(R_1, R_2)$ have a natural total ordering which is described by the progression from R_1 to R_2: $\ell < \ell'$ if and only if ℓ separates R_1 and ℓ'. The first element in this ordering is a boundary component of R_1, and the last is a boundary component of R_2. Furthermore, one sees that each complementary region – other than R_1 and R_2 – which has a boundary component separating R_1 and R_2 has exactly two and these two are neighbors with respect to the ordering in the sense that there is no leaf of $\tilde{\mathscr{L}}(R_1, R_2)$ between these two leaves.

Consider the subset of leaves $\ell \subset \tilde{\mathscr{L}}(R_1, R_2)$ with the property that there is an arc $\omega : [0, 1) \to \mathbb{D}^2$ transverse to $\tilde{\mathscr{L}}$ with endpoint in R_1 which crosses each leaf of $\tilde{\mathscr{L}}$ at most once, and which crosses ℓ.

We claim that this set of leaves has the following properties:

(I.4.3)
- (a) It is closed under forming least upper bounds.
- (b) It is non-empty.
- (c) If it contains one boundary component of a region $R \neq R_1$ or R_2, then it contains the other boundary component of R which separates R_1 from R_2.

Let us show that (I.4.3)(a), (b), (c) imply that $\ell_2 \in \tilde{\mathscr{L}}(R_1, R_2)$. By (a) and (b) the lub of all leaves of $\tilde{\mathscr{L}}(R_1, R_2)$ exists and is an element of $\tilde{\mathscr{L}}(R_1, R_2)$. We shall show that this lub is ℓ_2. Clearly, since our paths are open paths the lub must be isolated in $\tilde{\mathscr{L}}$ on the side containing R_2. Thus, the lub, ℓ, is the boundary component of a region $R \neq R_1$. Unless $R = R_2$, the other boundary component ℓ' of R which separates R_1 and R_2 is contained in $\tilde{\mathscr{L}}(R_1, R_2)$ and satisfies $\ell < \ell'$. This is impossible. Hence, $R = R_2$. This implies that $\ell = \ell_2$.

This completes the proof that (I.3.1)(c) holds. □

Chapter II. Construction of measured laminations by transversability

The material in this section is the analogue for Λ-trees of Section I of [8].

DEFINITION II.1. Let N be an n-manifold. Let $\mathscr{L} \subset M$ be a codimension-1 lamination and let T be a Λ-tree. A *transverse map* $\varphi : (N - |\mathscr{L}|) \to T$ is a locally constant function with the property that for any point $x \in |\mathscr{L}|$ there is a flow box $U^{n-1} \times (0, 1)$ containing x in which $\mathscr{L}$ is horizontal (i.e. so that $|\mathscr{L}| = U \times X$, where X is a closed subset of $(0, 1)$ containing no neighborhood of either end of $(0, 1)$) and in which φ takes the form

$$U \times ((0, 1) - X) \xrightarrow{\text{proj}} ((0, 1) - X) \xrightarrow{\lambda} T$$

where λ is a locally constant, weakly monotone function inducing an order-preserving bijection from the components $(0, 1) - X$ to the points of a segment in T. We call such a neighborhood $U \times (0, 1)$ a *flow box for* φ.

Notice that a flow box for φ is also a flow box for the lamination $\mathscr{L}$ and the map $\lambda : ((0, 1) - X) \to T$ defines a transverse Λ-measure for $\mathscr{L}$ in this flow box. It is easy to see that these transverse Λ-measures are compatible, so that λ actually defines a global transverse Λ-measure for $\mathscr{L}$. This codimension-1, Λ-measured lamination is the *induced measured lamination.* The map $\varphi : (N - |\mathscr{L}|) \to T$ assigns to each complementary region of $\mathscr{L}$ in N a point of T. It also assigns to each leaf of $\mathscr{L}$ an edge of T. To define the map from the leaves of $\mathscr{L}$ to the edges of T, let $x \in |\mathscr{L}|$, and let $U \times (0, 1)$ be a flow box for $\mathscr{L}$ near x with x corresponding to (u, t). The point t determines a Dedekind cut of $(0, 1) - X$ which is mapped by λ to a Dedekind cut of a segment of T corresponds to an edge of T. It is clear that the resulting edge of T depends only on the leaf of $\mathscr{L}$ in which x lies. Two leaves of $\mathscr{L}$ passing through the same flow box will determine the same edge of T if and only if there are no complementary regions of $\mathscr{L}$ between them in that flow box.

Remark II.2. Suppose that $(\mathscr{L}, \mu) \subset M$ satisfies Axioms (I.3.1). Let T be the tree dual to $(\tilde{\mathscr{L}}, \tilde{\mu}) \subset \tilde{M}$ as constructed in Section I.3. The natural map $\rho : (\tilde{M} - |\tilde{\mathscr{L}}|) \to T$ is transverse. Furthermore, if $\varphi : (\tilde{M} - |\tilde{\mathscr{L}}|) \to T'$ is a transverse mapping, them we have a factorization $\varphi = \psi \circ \rho$ where ψ maps T to T'. By the definition of transverse, ψ is actually a morphism of trees, though there is no reason to think in general that it is an isomorphism.

The goal of this section is to prove that equivariant transverse map always exist. Namely, we have:

THEOREM II.3. *Let Γ be the fundamental group of a n-manifold M. Let $\Gamma \times T \to T$ be an action of Γ on a Λ-tree T. Then there exists a codimension-1 lamination $\mathscr{L} \subset M$ and a Γ-equivariant transverse map $\tilde{\varphi} : (\tilde{M} - \tilde{\mathscr{L}}) \to T$. If there is a neighborhood of infinity $Z \to M$ such that for each component Z_0 of Z,* Im $(\pi_1(Z_0) \to \pi_1(M))$ *fixes a point in T, then we can take $\mathscr{L}$ to have compact support.*

The rest of this section is devoted to proving this theorem. We begin by choosing a triangulation τ of M. We denote by $\tilde{\tau}$ the induced triangulation of $\tilde{M}$. We shall construct $\tilde{\varphi}$ by induction over the skeleta $\tilde{\tau}_i$ of $\tilde{\tau}$.

(a) 0-*skeleton*: We choose arbitrarily a Γ-equivariant map $\tilde{\varphi}_0$ of $\tilde{\tau}_0$ to T. We define $\mathscr{L} \cap \tilde{\tau}_0 = \phi$.

(b) 1-*skeleton*: To extend $\tilde{\varphi}_0$ to a map $\tilde{\varphi}_1$ defined on $\tilde{\tau}_1$ we need the following result.

LEMMA II.4. *Let $[a, b]$ be a segment in a countable ordered abelian group Λ. Let $[0, 1]_{\mathbb{R}} \subset \mathbb{R}$ be the unit interval. Then there is a Λ-measure μ on $[0, 1]_{\mathbb{R}}$ with the following two properties*:

- (i) *The support F of μ is nowhere dense and contained in the interior of* $[0, 1]_{\mathbb{R}}$.
- (ii) *The function $t \to a + \int_0^t d\mu$ defines a monotone bijection from the components of* $([0, 1]_{\mathbb{R}} - F)$ *to* $[a, b] \subset \Lambda$. *Finally, if μ_1 and μ_2 are Λ-measures satisfying* (i) *and* (ii) *then there is an orientation-preserving homeomorphism* $h : [0, 1]_{\mathbb{R}} \to [0, 1]_{\mathbb{R}}$ *such that* $h * \mu_1 = \mu_2$.

Proof. For any countable segment $[a, b]$ in a totally ordered set S, there is an order-preserving injection $f : S \mapsto (0, 1)_{\mathbb{R}}$ with the property that for some compact nowhere dense set $F \subset (0, 1)_{\mathbb{R}} - f([a, b])$, each connected component of $(0, 1)_{\mathbb{R}} - F$ contains a unique point of $f(S)$. The proof of this fact is elementary.

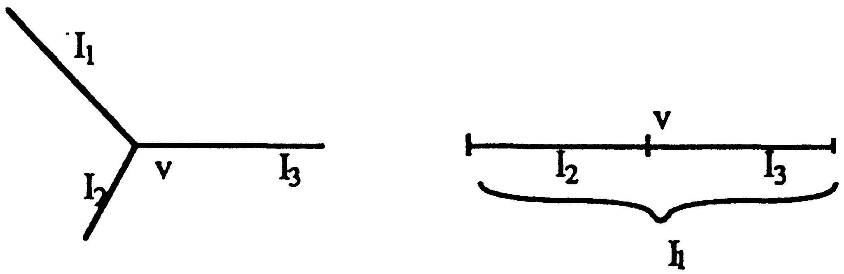
If $[a, b]$ is a segment in a totally ordered abelian group Λ, we can further define a measure μ on $(0, 1)_{\mathbb{R}}$ supported on F, as follows: for any two points α and β in

$(0, 1)_{\mathbb{R}} - F$, set $\mu(]\alpha, \beta[) = f^{-1}(\mu) - f^{-1}(\lambda)$ where λ and μ are the points of $f([a, b])$ belonging to the same connected component of $(0, 1)_{\mathbb{R}} - f([a, b])$ as α and β respectively.

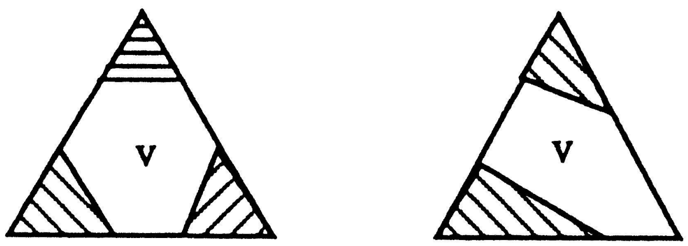
This proves the existence of a measure μ with properties (i) and (ii). The uniqueness of such a measure up to an orientation preserving homeomorphism is left to the reader. □

We are now ready to extend the map $\tilde{\varphi}_0$ to the 1-skeleton. Let e be an edge of τ with endpoints $\tilde{a}$ and $\tilde{b}$. Let $a = \tilde{\varphi}_0(\tilde{a})$ and $b = \tilde{\varphi}_0(\tilde{b})$. There is in T a unique segment with endpoints a and b. This segment is isomorphic to an interval of Λ. By using Lemma II.1.4 above, we can "pullback" this interval to e. More precisely, there is a nowhere dense closed set $F \subset \operatorname{int} e$, support of a measure μ, such that $x \to a + \int_{\tilde{a}}^{x} d_\mu$ furnishes a monotone bijection from the components of $e - F$ to $[a, b] \subset \tau$. The closed set F will be the *trace of* $\tilde{\mathscr{L}}$ on e and the above map the extension of $\tilde{\varphi}_0$ to e. We make such a choice for each 1-simplex e of $\tilde{\tau}$. Clearly, we can get an extension of $\tilde{\varphi}_1$ to the whole 1-skeleton. Making the choices equivariant (as we can always do) makes $\tilde{\varphi}_1$ equivariant.

(c) *2-skeleton.* Let Δ be a 2-simplex of $\tilde{\tau}$. Its boundary is the union of 3 edges e_1, e_2 and e_3, each endowed with a Λ-measure μ_i with support F_i. The mal $\tilde{\varphi}_1$ allows us to identify the components of $e_i - F_i$ with the intervals $I_i \subset T$. From the axioms for a tree, one sees that the intervals I_1, I_2, I_3 (after possibly a permutation of the labels) fit together in one of the two ways pictured below:



The vertex v corresponds to 3 components $c_i \subset e_i - F_{i'}$, which $\tilde{\varphi}_1$ sends to v. By the uniqueness of the μ_i, there exists a homeomorphism h_i between components of $e_i - C_i$ identified by $\tilde{\varphi}_1$. We use the h_i to identify the traces of the F_i on these components and to extend $\bigcup F_i \subset \partial\Delta$ to one of the laminations in Δ pictured below



We let this lamination be $\tilde{\mathscr{L}} \mid \Delta$. There is a unique map $\tilde{\varphi}_2 : (\Delta - \tilde{\mathscr{L}} \mid \Delta) \to T$ which is locally constant and extends $\tilde{\varphi}_1$ on the boundary. Working Γ-equivariantly, we define a lamination $\tilde{\mathscr{L}}$ restricted to $\tilde{\tau}_2$ and an equivariant transverse map $\tilde{\varphi}_2$.

(d) *The higher skeleta.* Suppose by induction we have defined $\tilde{\mathscr{L}}$ restricted to $\tilde{\tau}_{i-1}$ and an equivariant transverse map $\tilde{\varphi}_{i-1}$ on $\tilde{\tau}_{i-1} - \tilde{\mathscr{L}} \mid \tilde{\tau}_{i-1}$, for some $i > 3$. Let Δ^i be an i-simplex. We have $(\tilde{\varphi}_{i-1} \mid \partial\Delta^i - \tilde{\mathscr{L}} \mid \subset \Delta^i)$. Before extending $\tilde{\varphi}_{i-1}$ over Δ^i, we must study $\tilde{\mathscr{L}} \mid \partial\Delta^i$. First recall the following definition contained in [7].

DEFINITION. A lamination $\mathscr{L} \subset M$ is a *family of parallel compact leaves* if there is a neighborhod U of $\mathscr{L}$, homeomorphic to $S \times I$, where S is a compact codimension $\mathscr{L}$ submanifold of M such that $\mathscr{L} \cap U \cong S \times F$ for a closed set $F \subset I$.

LEMMA II.5. *$\tilde{\mathscr{L}} \mid \partial\Delta^i$ decomposes as a finite disjoint union of sublaminations; each of them is a family of parallel compact leaves.*

Proof. First we show that each leaf of $\tilde{\mathscr{L}} \mid \partial\Delta^i$ is compact. Choose a finite set of transversality flow boxes for $\tilde{\varphi}_i \mid (\partial\Delta^i - \tilde{\mathscr{L}} \mid \partial\Delta^i)$, $U_\ell, \ldots, U_K$. Because of the transversality property, each leaf of $\tilde{\mathscr{L}} \mid \partial\Delta^i$ meets each U_j at most once. Hence, each leaf of $\tilde{\mathscr{L}} \mid \partial\Delta^i$ is compact. If a leaf ℓ had non-trivial holonomy, then a leaf near to ℓ would cross one of the flow boxes U_i at least twice. Since all leaves are compact and all holonomy groups are trivial, the argument in [7] shows that each leaf has a saturated neighborhood which is a family of parallel compact leaves. As $\partial\Delta^i$ is compact, there can be only a finite number of such families. □

We are now in a position to extend $\tilde{\varphi}_{i-1} \mid \partial\Delta^i$ to Δ^i. By the above result, $\tilde{\mathscr{L}} \cap \partial\Delta^i$ decomposes in a finite number of families of parallel compact leaves. A leaf of each family bounds in $\partial\Delta^i$ a codimension-0 submanifold. By pushing into Δ^i and using the product structure for a given family, we construct a lamination $\tilde{\mathscr{L}}'$ of Δ^i bounded by $\tilde{\mathscr{L}} \cap \partial\Delta^i$. We then define $\tilde{\varphi}_i \mid (\Delta^i - \tilde{\mathscr{L}})$ as the unique possible extension of $\tilde{\varphi} \mid (\partial\Delta^i - \tilde{\mathscr{L}})$. Clearly, we can choose these extensions to be Γ-equivariant. This completes the inductive definition of $\tilde{h}$ and $\tilde{\varphi}$.

If there is a neighborhood Z of infinity in M with the property that for each component Z_0 of Z, the group $\mathrm{Im}\,(\pi_1(Z_0) \to \pi_1(M))$ fixes a point, then we can define the map $\tilde{\varphi}$ to be locally constant on $\tilde{Z} \subset \tilde{M}$ sending each component of $\tilde{Z}$ to a point left invariant by the subgroup of $\pi_1(M)$ fixing that component. □

To end this section, we want to remark that the constructed lamination has only a C^1-regularity along the leaves, this is because the glueing maps h_i in the above construction at the level of the 2-skeleton are only homeomorphisms.

Chapter III. The case of surfaces continued

The main result of this chapter is the following.

THEOREM III.1.1. *Let M be a complete hyperbolic surface of finite area, and let $\pi_1(M) \times T \to T$ be an action of $\pi_1(M)$ on a Λ-tree with the property that, for each cusp E of M, the group $\pi_1(E) \subset \pi_1(M)$ has a fixed point in T. Then there is a codimension-1, measured lamination $(\mathscr{L}, \mu) \subset M$ with compact support such that:*

(a) *$(\mathscr{L}, \mu)$ satisfies Axioms* (I.4.1);
(b) *there is a transverse equivariant map $\tilde{\varphi} : (\tilde{M} - |\tilde{\mathscr{L}}|) \to T$ such that the transverse measure for $\tilde{\mathscr{L}}$ by $\tilde{\varphi}$ is $\tilde{\mu}$.*

COROLLARY III.1.2. *Let M be a complete hyperbolic surface of finite area. Let $\pi_1(M) \times T \to T$ be an action on a Λ-tree with the property that, for each cusp E of M, $\pi_1(E) \supset \pi_1(M)$ fixes a point of T. Then there exists a Λ-measured lamination $(\mathscr{L}, \mu) \subset M$ of compact support and a transverse $\pi_1(M)$-equivariant map $\tilde{\varphi} : (\tilde{M} - \tilde{\mathscr{L}}) \to T$ such that:*
(i) *$(\tilde{\mathscr{L}}, \tilde{\mu})$ has a dual tree $T_{\mathscr{L}}$, and*
(ii) *the Λ-measure induced by $\tilde{\varphi}$ is $\tilde{\mu}$.*
The map $\tilde{\varphi}$ factors as the natural map $\tilde{\varphi}_{\mathscr{L}} : (\tilde{M} - |\tilde{\mathscr{L}}|) \to T_{\mathscr{L}}$ by a $\pi_1(M)$-equivariant morphism of trees $\psi : T_{\mathscr{L}} \to T$.

Proof that III.1.1 ⇒ III.1.2. This is immediate given Theorem I.4.2 and Remark II.2. □

Section III.1

In this section, we find a measured lamination $(\mathscr{L}, \mu) \subset M$ and a transverse equivariant map $\tilde{\varphi} : (\tilde{M} - |\tilde{\mathscr{L}}|) - T$ such that Axioms I.4.1(a) and (b) hold and such that Axioms I.4.1(c) and (d) hold for all leaves of $\tilde{\mathscr{L}}$ lying over non-compact leaves of $\mathscr{L}$. In the next section, we shall deal with the leaves lying over compact leaves of $\mathscr{L}$.

By Theorem II.3, there is a lamination $\mathscr{L} \subset M$ of compact support and an $\pi_1(M)$-equivariant, transverse map $\tilde{\varphi} : (\tilde{M} - |\tilde{\mathscr{L}}|) \to T$. We shall modify $\mathscr{L}$ and $\tilde{\varphi}$ until Axioms (I.4.1) are satisfied. First, however, let us show that some of what we need to establish is automatically true.

Cover M by a finite set of flow boxes for $\mathscr{L}$ which lift to standard flow boxes for the map $\tilde{\varphi}$ as in Definition II.1. We label the lifted flow boxes in $\tilde{M}$ by $\tilde{V}_1, \tilde{V}_2, \ldots$

CLAIM III.1.3. *$\mathscr{L}$ satisfies (III.5.1(b)) and each leaf of $\tilde{\mathscr{L}}$ is a proper submanifold of $\tilde{M}$.*

Proof. Since $|\tilde{\mathscr{L}}|$ is nowhere dense in $\tilde{M}$ by construction, if a leaf of $\tilde{\mathscr{L}}$ crosses $\tilde{V}_i$ twice, then a complementary region of $\tilde{\mathscr{L}}$ in $\tilde{M}$ also crosses $\tilde{V}_i$ twice. This contradicts the fact that $\tilde{V}_i$ is a flow box for $\tilde{\varphi}$. Hence, each leaf of $\tilde{\mathscr{L}}$ crosses $\tilde{V}_i$ at most once. The claim follows immediately. □

We must modify $\tilde{\mathscr{L}}$ and $\tilde{\varphi}$ in order to arrive at a situation which satisfies the conclusion of Theorem III.3.1. The first modification is to remove the compact leaves of $\tilde{\mathscr{L}}$.

Step A. *Elimination of the compact leaves of $\tilde{\mathscr{L}}$*

Suppose that I is a closed interval in a leaf of $\tilde{\mathscr{L}}$. We say that I follows *the path of flow boxes* $\tilde{V}_{i_1}, \ldots, \tilde{V}_{i_t}$ if there is a partition of I into subintervals $I_1, I_2, \ldots, I_t$ (in order) with $I_j \subset \tilde{V}_{i_j}$ for $j = 1, \ldots, t$. (Notice that I can follow many different paths of flow boxes; also notice that there is a completely analogous notion of an entire half-leaf following an infinite path of flow boxes.)

Any closed leaf follows a finite path $\tilde{V}_{i_1}, \ldots, \tilde{V}_{i_t}$ which is periodic in the sense that $\tilde{V}_{i_1} = \tilde{V}_{i_t}$.

There are analogous notions of paths of flow boxes in M. These paths are obtained from projections paths for the lifted leaf in $\tilde{M}$ down into M. The compact leaves of $\tilde{\mathscr{L}}$ project in M onto the compact leaves of $\mathscr{L}$ homotopic to zero. To eliminate them, we need the following:

LEMMA III.1.4. *The leaves of $\mathscr{L}$ homotopic to zero decompose into a finite number of equivalence classes under the relation "following the same path of flow boxes". Each equivalence class is a sublamination.*

Proof. Suppose that we have a fixed finite path of flow boxes in $\tilde{M}$. The set of leaves of $\tilde{\mathscr{L}}$ containing intervals following this path is an open and closed subset. If the path of flow boxes is periodic, then each leaf containing an interval following the path must close to form a compact leaf following the periodic path (this is because each leaf crosses $\tilde{V}_{i_j}$ at most once).

This proves that those compact leaves of $\tilde{\mathscr{L}}$ with a given periodic path of flow boxes form an open and closed sublamination of $\tilde{\mathscr{L}}$. They project into an open and closed sublamination of $\mathscr{L}$ of leaves which are trivial in M and follow a given path of flow boxes in M. The proof will be complete when we show that the image sublaminations in $\mathscr{L}$ form a finite family of sublaminations.

Let V be one of the flow boxes in M, say $V = U \times (0, 1)$. We claim that each leaf ℓ of $\mathscr{L}$ in the image of the compact leaves of $\tilde{\mathscr{L}}$ meets V at most twice. Let Δ be an embedded disc in M, with boundary ℓ; each component of $\Delta \cap V$ is of the form $U \times (0, a)$, $U \times (b, 1)$, of $U \times (a, b)$ with $0 < a, b < 1$. But if $\Delta \cap V$ has a component of the third type, then there is a lifting of Δ to $\tilde{\Delta} \subset \tilde{M}$ which meets a lift $\tilde{V}$ of V also in $U \times (a, b)$. This would mean that there is a lift $\tilde{\ell}$ of $\tilde{\mathscr{L}}$ meeting $\tilde{V}$ twice. This contradicts Claim III.1.3.

From this it follows easily that the sublaminations of trivial leaves with a given path of flow boxes in M form a finite family. □

Now we are ready to eliminate the leaves of $\mathscr{L}$ which are homotopically trivial. We have decomposed these leaves into a finite collection of open and closed sublaminations, say $C_1, \dots, C_j$, in $\mathscr{L}$. Each C_i consists of a nested family of circles with an outermost member which bounds a disk containing all the rest. Let D_i be this maximal disk associated to the sublamination C_i. Of course, ∂D_i is a leaf of C_i which is isolated in $\mathscr{L}$ on the outside of D_i. Let D_i be a slightly larger disk whose boundary is contained in $M - |\mathscr{L}|$, and which meets $\mathscr{L}$ exactly in C_i.

We begin by eliminating C_1. We define $\mathscr{L}_1 = \mathscr{L} - C_1$; we then construct a new transverse map $\tilde{\varphi}_1$ on $\tilde{M} - |\tilde{\mathscr{L}}_1|$ to be equal to $\tilde{\varphi}$ on $(\tilde{M} - |\tilde{\mathscr{L}}|) \cap (\tilde{M} - \tilde{\bar{D}}_1)$ and to be locally constant on $\tilde{\bar{D}}_1$. This has the effect of removing C_1 and all other families contained in D_1.

Continuing in this manner, we construct a new lamination $\mathscr{L}' \subset \mathscr{L}$ consisting of $\mathscr{L}$ minus its trivial leaves and a transverse equivariant map $\tilde{\varphi} : \tilde{M} - |\tilde{\mathscr{L}}'| \to T$.

At this point, we have Axioms I.4.1(a) and (b) satisfied. Let us take up the study of the endpoints of half-leaves of $\tilde{\mathscr{L}}$.

Step B. *Existence of endpoints for all half-leaves of* $\tilde{\mathscr{L}}$

The main result of this study is:

PROPOSITION III.1.5. *Let* $\tilde{\ell}_0$ *be a half-leaf of* $\tilde{\mathscr{L}}$, *then* $\tilde{\ell}_0$ *has an end in* S'.

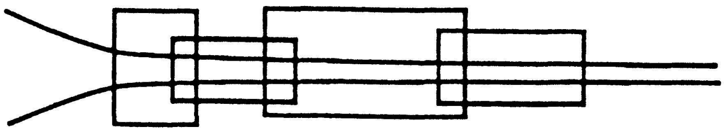
The proof of this proposition is an adaptation of [3, p. 124] to our case. Let us start with a classical result.

LEMMA III.1.6. *Let* ℓ_0 *be a half-leaf of a lamination on a complete hyperbolic surface of finite area* M. *The set* C *of cluster points of* ℓ_0 *is a minimal lamination* (*i.e. each leaf of* C *is dense in* C).

Proof. It is well-known (and easy to show) that C is a sublamination of $\mathscr{L}$. We establish that it is minimal. If C contains a compact leaf ℓ, then ℓ_0 must spiral into

ℓ. Clearly, then $C = \ell$. Otherwise, let λ be a noncompact leaf of C and denote by $\bar{\lambda}$ the closure of λ in C, then $\bar{\lambda}$ is a sublamination. If $\Lambda_0 \subset \bar{\lambda}$, then $C \subset \bar{\lambda}$ and hence λ is dense in C. If $\ell_0 \not\subset \bar{\lambda}$, then ℓ_0 is contained in a connected component of $M - |\bar{\lambda}|$. Let us study these complementary components.

DEFINITION. A *spike* for K is an end of a complementary component R of K which is bounded by two rays in leaves of K which follows the same infinite path of flow boxes.



CLAIM. *Let K be a lamination in M. Each complementary region R of K in M consists of a compact part together with a finite number of spikes.*

For a proof, see [7]. □

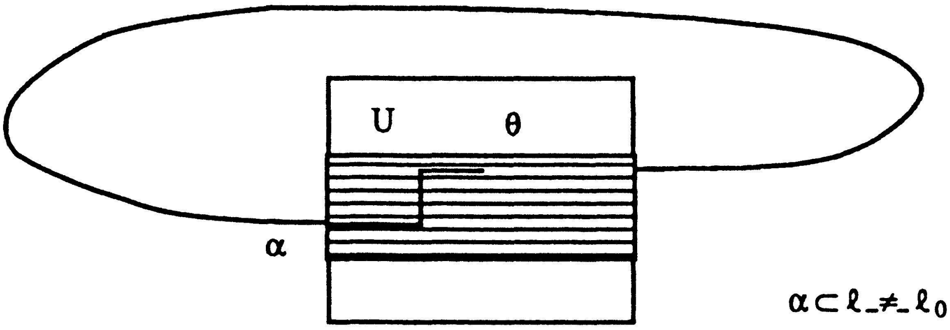
Let $X = M - |\bar{\lambda}|$. If ℓ_0 is contained in a compact part of X, then the closure of ℓ_0 is contained in the complement of the spikes of X. This is impossible since the boundaries of the spikes are contained in $|\bar{\lambda}| \subset |C|$. Thus, ℓ_0 must enter at least one of the spikes of X. Once it enters a spike it must run out to infinity in that spike, following the same infinite path of flow boxes that the sides of the spike do. One sees easily then that the set of cluster points of ℓ_0 is the same as the set of cluster points of either half leaf of ∂X which bounds the spike. Since $\partial X \subset |\bar{\lambda}|$, it follows that $C \subset \bar{\lambda}$. This proves that $C = \bar{\lambda}$, or equivalently that λ is dense in C. This completes the proof of the lemma. □

Proof of Proposition III.1.5. If $\tilde{\ell}_0 \subset \tilde{\mathscr{L}}$ projects down to a compact leaf of $\mathscr{L} \subset M$, then it has an endpoint in ∂D which is the attracting fixed point of the hyperbolic transformation associated with the oriented simple closed curve (homotopically nontrivial) on M.

If $\tilde{\ell}_0$ projects to a non-compact half leaf ℓ_0 of $\mathscr{L}$, then we apply Lemma III.1.6. Let C be the limit set of ℓ_0. If C is a simple closed curve, then ℓ_0 spirals into C and has the same endpoint as the hyperbolic transformation associated to C. If C is an exceptional minimal set, there is a transversality flow box U for $\mathscr{L}$ which meets C.

Choose a leaf $\ell \neq \ell_0$ in C, which is not isolated on either side in C. By using that ℓ is dense in C, one constructs easily a simple closed curve $\gamma = \alpha \cup \beta$ where β is an arc transverse to $\mathscr{L}$, contained in a flow box U, and α is an arc $\subset \ell$ which intersects

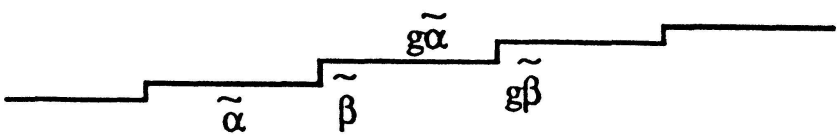
β with the same orientation at both endpoints, as pictured below:



Since ℓ is not isolated on one side, $C \cap \beta \neq \phi$, so that ℓ_0 intersects β infinitely many times. Consequently $\tilde{\ell}_0$ will meet the full preimage $\tilde{\gamma}$ of γ infinitely often.

FACT. *Let $\tilde{\gamma}_0$ be a component of $\tilde{\gamma}$; then, the intersection $\tilde{\gamma}_0 \cap \tilde{\ell}_0$ is either empty or a single point.*

Proof. We note g the hyperbolic translation corresponding to $\tilde{\gamma}_0$. It acts on $\tilde{\gamma}_0$ by translation with a fundamental domain a union of a lift $\tilde{\alpha}$ of α and a lift $\tilde{\beta}$ of β (cf. picture).



Assume that $\tilde{\gamma}_0 \cap \tilde{\ell}_0$ has more than a single component. Then $\tilde{\ell}_0$ crosses successively two translates of $\tilde{\beta}$, say $g^r\tilde{\beta}$ and $g^s\tilde{\beta}$, $r \leq s$.

By considering the leaves of $\mathscr{L}$ containing the translates $g^t\tilde{\alpha}$, $r \leq t \leq s$, we find one which intersects some $g^t\tilde{\beta}$ twice.

This contradicts the fact that U is a transversality flow box for $\mathscr{L}$. □

To complete the proof of Proposition III.1.5, we argue as in [3, p. 124]. Choose a sequence $\tilde{\gamma}_i$ of components of $\tilde{\gamma}$, each meeting $\tilde{\ell}_0$ in one point such that, if $i < j$, then $\tilde{\gamma}_j \cap \tilde{\ell}_0$ is farther out in the end of $\tilde{\ell}_0$ than is $\tilde{\gamma}_i \cap \tilde{\ell}_0$. The intersection of the half spaces bounded by $\tilde{\gamma}_i$ defines a point in $\partial \mathbb{D}$ which is the end of $\tilde{\ell}_0$. □

Remark III.1.7. If the cluster points of ℓ_0 form an exceptional minimal set, then the half spaces $h_1 > h_2 > \cdots$ determine one endpoint of $\tilde{\ell}_0$. Since $\tilde{\ell}_0$ crosses the boundary of each h_i at most once, the other endpoint of $\tilde{\ell}_0$ lies outside $\bar{h}_1$. Hence, the endpoints of $\tilde{\ell}_0$ are distinct in this case.

Step C. Continuity of the endpoints of half-leaves of $\tilde{\mathscr{L}}$ lying above non-compact leaves of $\mathscr{L}$

PROPOSITION III.1.8. *Let $\tilde{\ell}_0 \subset \tilde{\mathscr{L}}$ be a half-leaf which projects to a non-compact leaf of $\mathscr{L}$. All half-leaves of $\tilde{\mathscr{L}}$ which pass sufficiently close to the initial point of $\tilde{\ell}_0$ and with the same orientation as $\tilde{\ell}_0$ have ends in $\partial\mathbb{D}$ which are close to the end of $\tilde{\ell}_0$.*

Proof. If the image of $\tilde{\ell}_0$ in M spirals around a closed loop C, then the same is true for all sufficiently close half-leaves. Thus all half-leaves which are sufficiently close to $\tilde{\ell}_0$ have the same end in $\partial\mathbb{D}$.

Now let us consider the case when the cluster points of ℓ_0 form an exceptional minimal set. In M, construct a simple close curve γ which meets ℓ_0 infinitely often (as in the proof of Proposition III.1.5). Let $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots$ be the sequence of lifts of γ, in order, that $\tilde{\ell}_0$ meets. We know that the half spaces that they bound, $h_1, h_2, \ldots,$ form a decreasing family. The closures of these half-spaces in $\mathbb{D}$ have a single point in common – the endpoint of $\tilde{\ell}_0$. For any neighborhood N of this endpoint, there is an i such that $\bar{h}_i \cap \partial D \subset N$. Any half-leaf $\tilde{\ell}_\alpha$ that passes sufficiently close to the initial point of $\tilde{\ell}_0$ (with the same orientation as $\tilde{\ell}_0$) will meet $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_i$. Since $\tilde{\ell}_\alpha$ cannot cross $\tilde{\gamma}_i$ twice, its endpoint must be contained in $\bar{h}_i \cap S^1$ and hence in N. □

This completes the proof that Axioms II.4.1(c) and (d) hold for leaves of $\tilde{\mathscr{L}}$ lying about leaves of $\mathscr{L}$ whose closures contain exceptional minimal sets. We have also established the continuity of the endpoint map for oriented leaves at any leaf of $\tilde{\mathscr{L}}$ lying above a non-compact leaf of $\mathscr{L}$.

Section III.2

In this section, we complete the proof of Theorem III.1.1 by studying the endpoint phenomena near compact leaves of $\mathscr{L}$. We keep the notation and assumptions of the last section. In particular, $\mathscr{L} \subset M$ satisfies Axioms I.4.1(a) and (b).

LEMMA III.2.1. *If $\tilde{\ell}_0$ is a leaf of $\tilde{\mathscr{L}}$ which projects to a compact leaf $\ell_0 \subset \mathscr{L}$, then any leaf $\tilde{\ell}_\alpha$ with distinct endpoints which is sufficiently close to $\tilde{\ell}_0$ has endpoints close to the endpoints of $\tilde{\ell}_0$.*

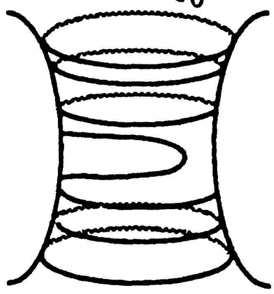
Proof. Any leaf in $\mathscr{L}$ nearby to ℓ_0 contains some end which spirals around ℓ_0. The spiralling end has the same endpoint as $\tilde{\ell}_0$ moving in the same direction. If the nearby leaf $\tilde{\ell}_0$ has distinct endpoints, by equivariance under the cyclic group generated by ℓ_0, we see that the other endpoint of $\tilde{\ell}_\alpha$ converges to the other endpoint of $\tilde{\ell}_0$ as $\tilde{\ell}_\alpha$ gets closer and closer to $\tilde{\ell}_0$. □

Lemma III.2.1 and Proposition III.1.8 imply the following.

COROLLARY III.2.2. *If every leaf of $\tilde{\mathscr{L}} \subset \tilde{M}$ has distinct endpoints, then the map from the leaf space of $\tilde{\mathscr{L}}$ to $\mathscr{I}$ is continuous.* □

Our next task is to modify $\tilde{\mathscr{L}}$ until each leaf has distinct endpoints. First let us give an example to show an example where leaves with the same endpoints actually arise.

Example. Let ℓ_0 and ℓ_1 be parallel simple closed curves on M and form a *Reeb component* between them;



All leaves in this Reeb component have both endpoints equal in $\partial\mathbb{D}$.

Elimination of Reeb-type components

We define an *annulus* for $\mathscr{L}$ to be an annulus $A \subset M$ bounded by two closed leaves of $\mathscr{L}$. An annulus for $\mathscr{L}$ is *maximal* if it is contained in no larger annulus for $\mathscr{L}$. Let us show first that any annulus for $\mathscr{L}$ is contained in a maximal annulus. Let $(A_i)_{i\in N}$ be an increasing sequence of annuli bounded by the leaves ℓ_i^+ and ℓ_i^- of $\mathscr{L}$. In the universal cover of M choose an increasing sequence $\tilde{A}_i$ of lifts of the A_i. The $\tilde{A}_i$ are infinite strips bounded by leaves $\tilde{\ell}_i^+$ and $\tilde{\ell}_i^-$ of $\tilde{\mathscr{L}}$; each $\tilde{A}_i$ is invariant under a fixed element $\gamma \in \pi_1(M)$ and the endpoints of ℓ_i^+ and ℓ_i^- are the fixed points of γ on $\partial\tilde{M}$. Let x_i^+ be a sequence of points on $\tilde{\ell}_i$ converging to a point $x_\infty^+ \in \tilde{H}$. If the leaf $\tilde{\ell}_\infty^+$ of $\tilde{\mathscr{L}}$ passing through $\tilde{\ell}_\infty^+$ projects down to a non-compact leaf of $\mathscr{L}$, then by the continuity of the endpoint map (Proposition III.1.8), endpoints of $\tilde{\ell}^+_\infty$ coincide with those of the $\tilde{\ell}^+_i$. By the assumption that the A_i form an increasing sequence, the leaf $\tilde{\ell}^+_\infty$ is invariant under γ, contradicting the fact that $\tilde{\ell}^+_\infty$ projects to a non-compact leaf.

If the leaf $\tilde{\ell}^+_\infty$ projects to a compact leaf ℓ^+_∞, then, by Lemma III.2.1, all the nearby leaves ℓ^+_i are parallel to ℓ^+_∞. The same reasoning applied to the ℓ^-_i, shows that the union of the A_i is the interior of an annulus bounded by the two leaves ℓ^+_∞ and ℓ^-_∞. Hence, any annulus is contained in a maximal one. Furthermore, maximal annuli are disjoint and non-parallel; they are in finite number.

DEFINITION. We say that an annulus A for $\mathscr{L}$ is *taut* if when we consider a component $\tilde{A}$ of the pre-image of A in the universal covering of M, there is a spanning arc α transverse to $\tilde{\mathscr{L}}$ such that $\tilde{\varphi} \mid (\alpha - |\tilde{\mathscr{L}}| \cap \alpha)$ is a strictly monotone function from the components of $(\alpha - |\tilde{\mathscr{L}}| \cap \alpha)$ onto (a possibly infinite) segment of T.

One sees easily that if A is taut, then $\mathscr{L} \mid A$ consists of leaves parallel to ∂A and leaves which spiral between them, always spiralling in the same direction.

We are going now to define $\mathscr{L}$ and $\tilde{\varphi}$ until all the maximal annuli for $\mathscr{L}$ are taut, proving:

LEMMA III.2.3. *Given an action $\pi_1(M) \times T \to T$, there is a measured lamination $\mathscr{L} \subset M$ without trivial leaves and a transverse map $\tilde{\varphi} : \tilde{M} \to \tilde{\mathscr{L}} \to T$ with only taut annuli. Furthermore, each leaf of $\tilde{\mathscr{L}}$ has distinct ends in the Poincaré disk, and the induced map from the leaves of $\tilde{\mathscr{L}}$ to $\mathscr{I}$ is continuous.*

Clearly Theorem III.1.1 is immediate from Lemma III.2.3.

Proof. We begin with the lamination $\mathscr{L}$ obtained in the previous section. The proof is by induction on the number of maximal non-taut annuli. There are several cases of non-taut maximal annuli to consider. In each case we shall succeed either in removing the annulus entirely or replacing it by only taut ones. Since we shall never create new non-taut annuli, this will complete the proof.

Let $A \subset M$ be a non-taut maximal annulus for $\mathscr{L}$. Let ℓ_0 and ℓ_1 be its boundary components. Let $\tilde{A}$ be a component of the pre-image of A in the universal covering of $\tilde{M}$ with boundary $\tilde{\ell}_0$ and $\tilde{\ell}_1$. Let e_0 and e_1 be the edges of T given by $\tilde{\varphi}(\tilde{\ell}_0)$ and $\tilde{\varphi}(\tilde{\ell}_1)$. Let s_0 and s_1 be the sides of e_i in T determined by the normals to $\tilde{\ell}_0$ and $\tilde{\ell}_1$ pointing *out* of $\tilde{A}$.

Case (1). $e_0 = e_1$ and s_0 and s_1 are opposite sides of $e_0 = e_1$ in T.

In this case, we change the lamination $\mathscr{L}$ by removing the interior of A and gluing ℓ_0 to ℓ_1. In the universal covering this removes the interior of each component of the pre-image of A and glues the sides together. The map $\tilde{\varphi}$ induces a map on the new manifold. One sees easily that, under the conditions of Case 1, this new map is transverse. Choosing a product structure in a neighborhood of the original annulus defines an identification of this new manifold with the original one M. Thus, under this identification, we have constructed a new lamination $\mathcal{L}' = \mathcal{L} \mid (M - A) \cup \gamma$ where γ is the image in M' of the gluing circle. We have a new transverse equivariant map $\tilde{\varphi}' : (M' - |\tilde{\mathcal{L}}|) \to T$. This operation has the effect of removing the annulus A for $\mathcal{L}$ and replacing it by a single circle. Hence, it reduces the number of non-taut maximal annuli by one.

Case (2). $e_0 = e_1$ and $s_0 = s_1$.

In this case we shall do a surgery and then remove all trivial leaves. Choose an arc α_0 through a point $x_0 \in \ell_0$ which lifts to an arc $\tilde{\alpha}_0$ in $\tilde{M}$ which is a vertical arc in a transversality flow box for $\tilde{\varphi}$. Of course α_0 is transverse to $\mathcal{L}$. There is a subinterval $J_0 \subset \alpha_0$, meeting A exactly in one endpoint, such that the Poincaré map P around ℓ_0 is defined on $J_0 \cap |\mathcal{L}|$ and sends it to $\alpha_0 \cap |\mathcal{L}|$.

We can take the other endpoint of J_0 to miss $|\mathcal{L}|$.

If $J_0 \cap |\mathcal{L}| = J_0 \cap \ell_0$, then ℓ_0 is isolated on the side $M - A$. Otherwise, at the expense of reversing the direction of ℓ_0, all leaves crossing J_0 spiral towards ℓ_0 as we go once around ℓ_0.

Using this we construct a simple closed curve γ_0 in $M - A$ which is parallel to ℓ_0 and transverse to $\mathcal{L}$. Take an arc α which spirals around ℓ_0 once parallel to the leaves of $\mathcal{L}$ beginning at a point $x \in J_0 - J_0 \cap |\mathcal{L}|$; complete α to a simple closed curve γ, by adjoining the appropriate subinterval of J_0.

Let C_0 be the annulus in M bounded by ℓ_0 and γ_0.

Let $\tilde{A} \subset \tilde{M}$ be a component of the pre-image of A to the universal covering of M. Let $\tilde{J}_0$ be a lift of J_0 ending at $\tilde{\ell}_0 \subset \tilde{A}$.

Since $s_0 = s_1$, if J_0 is short enough, there will be an arc $\tilde{J}_1$ with endpoint in $\tilde{\ell}_1 \subset \tilde{A}$ and a homeomorphism $\tilde{h} : \tilde{J}_0 \to \tilde{J}_1$ such that the following diagram commutes

$$\begin{array}{ccc} \tilde{J}_0 - \tilde{J}_0 \cup |\tilde{\mathcal{L}}| & \xrightarrow{\ \tilde{h}\ } & \tilde{J}_1 - \tilde{J}_1 \cap |\tilde{\mathcal{L}}| \\ & {\scriptstyle\tilde{\varphi}} \searrow \quad \swarrow {\scriptstyle\tilde{\varphi}} & \\ & T & \end{array}$$

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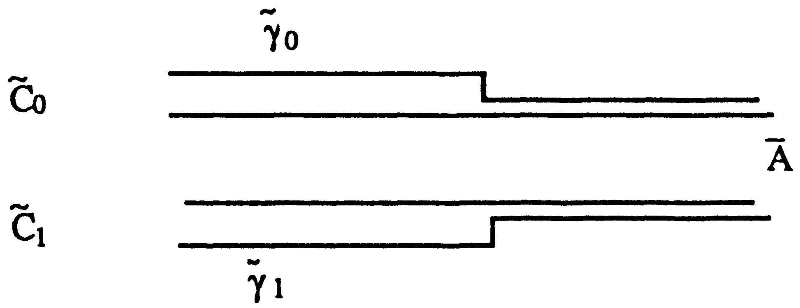
Let $\tilde{x} \in \tilde{J}_0$ cover x. Then, we can construct a loop γ_1 parallel to ℓ_1 in M using the point $\tilde{h}(\tilde{x}) \in \tilde{J}_1$ analogous to γ_0. Let C_1 be the collar bounded by ℓ_1 and γ_1.

Let $\tilde{C}_0$ and $\tilde{C}_1$ be the components of C_0 and C_1 which meet the component $\tilde{A}$ of A in $\tilde{M}$.



We can extend $\tilde{H} : \tilde{J}_0 \to \tilde{J}_1$ to an equivariant homeomorphism $\tilde{h} : \tilde{C}_0 \to \tilde{C}_1$ which sends $\tilde{C}_0 \cap |\tilde{\mathscr{L}}|$ to $\tilde{C}_1 \cap |\tilde{\mathscr{L}}|$ and which makes the diagram below commute

$$\begin{array}{ccc} \tilde{C}_0 - \tilde{C}_0 \cup |\tilde{\mathcal{L}}| & \xrightarrow{\tilde{h}} & \tilde{C}_1 - \tilde{C}_1 \cap |\tilde{\mathcal{L}}| \\ & {\scriptstyle\tilde{\varphi}} \searrow \quad \swarrow {\scriptstyle\tilde{\varphi}} & \\ & T & \end{array}$$

Now we shall change $\tilde{\mathscr{L}}$ and $\tilde{\varphi}$. The change will take place in the pre-image of $C_0 \cup A \cup C_1$. Fix a component $\tilde{C}_0 \cup \tilde{A} \cup \tilde{C}_0$. We have a homeomorphism $\tilde{h} : \tilde{\gamma}_0 \to \tilde{\gamma}_1$ commuting with the map to T and with the cyclic subgroup of $\pi_1(M)$ leaving this component invariant. We remove int $(\tilde{C}_0 \cup \tilde{A} \cup \tilde{C}_1)$ and glue $\tilde{\gamma}_0$ to $\tilde{\gamma}_1$ by $\tilde{h}$.

Making this change equivariantly over all components of the pre-image of $C_0 \cup A \cup C_1$ in $\tilde{M}$ constructs a new transverse map $\tilde{\varphi}' : \tilde{M} \to T$. Let $\tilde{\mathscr{L}}'$ denote the induced, measured lamination. First, notice that $\tilde{\varphi}' : (\tilde{\gamma}_0 - \tilde{\gamma}_0 \cap |\tilde{\mathscr{L}}|) \to T$ induces a bijection between the components and the points of a segment in T. Thus, any leaf of $\tilde{\mathscr{L}}'$ crosses $\tilde{\gamma}_0$ at most once.

CLAIM. *$\tilde{\mathscr{L}}'$ has no trivial leaves and each annulus of $\mathscr{L}' \subset M$ is either an annulus of $\mathscr{L} \subset M$ or is taut.*

Proof. Let $\tilde{\ell}'$ be a leaf of $\tilde{\mathscr{L}}'$. Either it is also a leaf of $\tilde{\mathscr{L}}$ or it crosses some lift of γ_0 exactly once. Since $\tilde{\mathscr{L}}$ has no trivial leaves neither does $\tilde{\mathscr{L}}'$.

Let $A' \subset M$ be an annulus for $\tilde{\mathscr{L}}'$, bounded by ℓ'_1 and ℓ'_2. If both ℓ'_1 and ℓ'_2 are leaves of $\mathscr{L}$, then A' is an annulus for $\mathscr{L}$. Suppose that $\ell'_1 \notin \mathscr{L}$. Let $\tilde{A}' \subset \tilde{M}$ be a component of the pre-image of A, bounded, say by $\tilde{\ell}'_1$ and $\tilde{\ell}'_2$. Since $\tilde{\ell}'_1 \notin \tilde{\mathscr{L}}$, there is a component $\tilde{\gamma}_0$ of the pre-image of γ_0 so that $\tilde{\ell}'_1$ crosses $\tilde{\gamma}_0$ once. We divide into 2 sub-cases.

Case a: ℓ_1' is not parallel to γ_0. In this case, the ends of $\tilde{\ell}_1'$ are distinct from the ends of γ_0. Since $\tilde{\ell}_1'$ and $\tilde{\ell}_2'$ have the same ends, $\tilde{\ell}_2'$ must also cross $\tilde{\gamma}_0$.

The arc $\tilde{\gamma}_0 \cap \tilde{A}'$ projects to a spanning arc for A'. Since $\tilde{\varphi} \mid (\tilde{\gamma}_0 - \tilde{\gamma}_0 \cap |\tilde{\mathscr{L}}|)$ is injective on components, this proves that A' is taut.

Case b: ℓ_1' is parallel to γ_0. In this case, ℓ_2' is also parallel to γ_0. Thus, since the original annulus A was maximal for $\mathscr{L}$, ℓ_2' cannot be a leaf of $\mathscr{L}$. Thus, $\tilde{\ell}_2'$ also crosses some lift $\tilde{\gamma}_0$ of γ_0. Since $\tilde{\ell}_2'$ and $\tilde{\ell}_1'$ have the same endpoints, they cross the same lift $\tilde{\gamma}_0$ of γ_0. Once again, the arc $\tilde{\gamma}_0 \cap \tilde{A}'$ projects to a spanning arc for A', showing that A' is taut. □

Case (3). $e_0 \neq e_1$; e_0 is not on the s_1-side of e_1 and e_1 is not on the s_0-side of e_0.

In this case, we replace this maximal annulus by a taut one with the same boundary. There is (a possibly infinite) segment I of T joining the edge e_0 to e_1. This segment is, as an oriented segment, invariant under the cyclic group generated by the loop ℓ_0. Let α be a spanning arc for A. Let $\tilde{\alpha}$ be a lift of α to $\tilde{A}$. We define $\tilde{\mathscr{L}}' \mid \tilde{\alpha}$ and $\tilde{\varphi}' \mid (\tilde{\alpha} - \tilde{\alpha} \cap \tilde{\mathscr{L}}')$ so that the latter is a strictly monotone map of $\tilde{\alpha} - \tilde{\alpha} \cap \tilde{\mathscr{L}}'$ into I. Since I is invariant under the cyclic subgroup of $\pi_1(M)$ which leaves $\tilde{A}$ invariant, it is an easy matter to extend $\tilde{\mathscr{L}}'$ and $\tilde{\varphi}'$ to all of $\tilde{A}$. This, when extended by equivariance to the other components of the pre-image of A, replaces $\mathscr{L}$ by a lamination $\mathscr{L}'$ which is taut in A and which agrees with $\mathscr{L}$ outside A.

Case (4). $e_0 \neq e_1$; e_1 is not on the s_0-side of e_0, but e_0 is on the s_1-side of e_1.

Consider the leaves of $\tilde{\mathscr{L}}$ in $\tilde{A}$ which map to the edge e_1 of T. Clearly, $\tilde{\ell}_1$ is such a leaf. Since a neighborhood of $\tilde{\ell}_1$ in $\tilde{A}$ maps into the side of e_1 opposite to s_1 and hence opposite from the side of e_1 containing e_0, there must be other leaves $\tilde{\ell}$ in $\tilde{A}$ which map to e_1. We claim that each of these leaves projects to a closed leaf in $\mathscr{L} \cap A$. If one such leaf $\tilde{\ell}$ projects to a non-closed leaf of $\mathscr{L} \cap A$, then ℓ must accumulate to points $x \in \mathscr{L} \cap A$ not contained in ℓ. Since ℓ is contained in A, the various elements of $\pi_1(M)$ which measure the differences of the various branches of ℓ near x all lie in the same cyclic group and this group fixes the edge e_1. In the universal cover, since $\tilde{\varphi}_{\#}(\tilde{\ell}) = e_1$ and $g_i e_1 = e_1$ for all i, we see that $\tilde{\varphi}_{\#}(g_i\tilde{\ell}) = e_1$ for all i. Hence $\tilde{\varphi}_{\#}(\tilde{\ell}') = e_1$. This contradicts the transversality of $\tilde{\varphi}$. This proves that all leaves of $\tilde{A}$ mapping to the edge e_1 project to simple closed curves in A.

We consider this set of leaves projected into A. They form a finite subset of parallel simple closed curves. There is one such $\ell \subset A$ which is closer to ℓ_0. We know that $\ell \subset \operatorname{int} A$. We use ℓ to divide A into two annuli A_- from ℓ_0 to ℓ, and A_+ from ℓ to ℓ_1. Clearly, the argument above shows that as we pass from A_- to A_+, we pass from the side of e_1 containing e_0 to the opposite side. Thus, we can apply the argument in Case (3) to replace $\tilde{\mathscr{L}} \mid \tilde{A}_-$ and $\tilde{\varphi}$ be a taut lamination and map. Then, we apply the argument in Case (2) to do surgery on $\mathscr{L} \mid A_+$. As before, we see all new annuli created are taut.

Case (5). $\varepsilon_0 \neq e_1$; e_0 is not on the s_1-side of e_1, but e_1 is on the s_0-side of e_0. This case is identical with Case (4) with the roles of e_0 and e_1 reversed.

Case (6). $e_0 \neq e_1$; e_0 is on the s_1-side of e_1 and e_1 is on the s_0-side of e_0.
This case is the "union" of Cases (4) and (5). We divide the annulus A into three annuli A_-, A_0, A_+, A_- going from ℓ_0 to ℓ'_0, A_0 from ℓ'_0 to ℓ'_1 and A_+ from ℓ'_1 to ℓ_1, where the single curves ℓ'_0 and ℓ'_1 are constructed as above with respective images ℓ_0 and ℓ_1 under $\tilde{\varphi}$. We replace then A_0 by a taut annulus and do surgery to remove A_- and A_+. This proves the first half of Lemma III.2.3.

To finish the proof, we need just to show the following

SUBLEMMA III.2.4. *If $\mathscr{L} \subset M$ is a lamination and $\tilde{\varphi} : \tilde{M} - \tilde{\mathscr{L}} \to T$ a transverse equivariant map, then if $\mathscr{L}$ has no trivial leaves and no non-taut annuli, each leaf ℓ of $\mathscr{L}$ has two distinct endpoints in $\partial \mathbb{D}$.*

Proof. We have already seen (Proposition III.1.5) that each half-leaf of $\tilde{\ell}$ has a well-defined endpoint in $\partial \mathbb{D}$ and (Remark III.1.7) that these endpoints are distinct if either end of ℓ adheres to an exceptional minimal set inside $\mathscr{L}$. Clearly, if ℓ is compact it has two distinct endpoints. Thus, if the endpoints of ℓ are the same, then both ends of ℓ must spiral around simple closed curves, ℓ_0 and ℓ_1. Let $\tilde{\ell}_0$ and $\tilde{\ell}_1$ be the lifts of ℓ_0 and ℓ_1 with the property and $\tilde{\ell}_0$ and one end of $\tilde{\ell}$ are asymptotic whereas $\tilde{\ell}_1$ and the other end of $\tilde{\ell}$ are asymptotic. Since $\tilde{\ell}_0$ and $\tilde{\ell}_1$ have an endpoint in common and represent closed loops in M, they must have both endpoints in common. By transversality, we see that $\tilde{\ell}_0 \neq \tilde{\ell}_1$. In particular, the region between $\tilde{\ell}_0$ and $\tilde{\ell}_1$ projects to an annulus A in M. Since a spanning arc for A must cross the image of $\tilde{\ell}$ at least twice, A is not taut. This contradicts our assumption that all the annuli for $\mathscr{L}$ are taut. Hence, all the leaves of $\tilde{\mathscr{L}}$ have distinct endpoints. □

This finishes the proof of Lemma III.2.3. □

Chapter IV. Geometric actions of surfaces groups

Section IV.1. *Characterization of geometric actions of surface groups*

The point of this section is to prove the following theorem characterizing those actions of a surface group on a Λ-tree which are geometric.

THEOREM IV.1.1. *Let M be a complete hyperbolic surface of finite area. Let Γ be the fundamental group of M. Suppose $\Gamma \times T \to T$ is a minimal action on a Λ-tree with the property that, for each cusp E of M, the group $\pi_1(E) \subset \Gamma$ has a fixed point in T. Then, the actionn is dual to a Λ-valued measured lamination $(\mathscr{L}, \mu) \subset M$ if and only if the following two conditions are satisfied*:

(a) *The stabilizer in Γ of any edge is cyclic.*

(b) *If γ_0 and γ_1 are elements of Γ whose axes in $\tilde{M}$ cross and if B_0 and B_1 are partial axes for γ_0 and γ_1, respectively, in T, then there is no non-degenerate segment $[x, y]$ with $B_0 \cap [x, y] = \{x\}$ and $B_1 \cap [x; y] = \{y\}$.*

N.B. Whether or not the axes of γ_0 and γ_1 cross in $\mathbb{D}^2$ is independent of the hyperbolic structure on M.

Proof. Suppose $\Gamma \times T \to T$ is dual to a measured lamination $(\mathscr{L}, \mu)$ in M. Then, the stabilizer of an edge is the stabilizer of the corresponding leaf in $\tilde{\mathscr{L}}$ (see (I.3.8)). These are clearly cyclic groups.

We show that Condition (b) holds as well. For any element $\gamma \in \pi_1(M)$, the axis of γ in T (assuming it has one) consists of a sequence of complementary regions for $\tilde{\mathscr{L}}$ in $\tilde{M}$. The sequence is invariant under γ. Choose a transverse arc ω from one of these regions R_0 to $\gamma \cdot R_0$ crossing each leaf of $\mathscr{L}$ at most once. We can suppose that ω projects to a closed loop in M. Thus, $\bigcup_{n \in \mathbb{Z}} \gamma^n \omega = \tilde{A}_\gamma$ is a properly embedded copy of $\mathbb{R}$ in $\mathbb{D}^2$ invariant under γ. The regions that $\tilde{A}_\gamma$ passes through all correspond to points of T on an axis for γ. Now suppose that γ' is another element of $\pi_1(M)$ which acts without fixed point on T. We construct a properly embedded line $\tilde{A}_{\gamma'}$ as above. If the axes of γ and γ' in $\mathbb{D}^2$ intersect, then $\tilde{A}_\gamma \cap \tilde{A}_{\gamma'} \neq \varnothing$. Since we can deform A_γ slightly, we can assume $\tilde{A}_\gamma \cap \tilde{A}_{\gamma'} \subset \tilde{M} - \tilde{\mathscr{L}}$. A complementary region containing a point of $\tilde{A}_\gamma \cap A_{\gamma'}$ is common to the axes for γ and γ' in T.

We turn now to the sufficiency of the two conditions. Suppose $\Gamma \times T \to T$ is a minimal action satisfying Conditions (a) and (b). By Lemma III.2.3, there is a measured lamination $\mathscr{L} \subset M$ with compact support and a transverse equivariant map $\tilde{\varphi} : (\tilde{M} \subset \tilde{L}) \to T$ such that

$$\text{(IV.1.2)} \quad \begin{cases} \text{(i) All leaves of } \tilde{\mathscr{L}} \subset \tilde{M} \text{ are proper and non-compact;} \\ \text{(ii) All the annuli of } \mathscr{L} \subset M \text{ are taut.} \end{cases}$$

We claim that $\psi : T_{\mathscr{L}} \to T$ is an isomorphism. Proving this, of course, will establish that the action on T is dual to $\mathscr{L}$. By Corollary I.1.2, either ψ is an isomorphism or ψ has folding. Our goal, then, is to show that ψ has no folding. Before beginning the proof of this, we list two facts about actions on Λ-trees. Let T_0 be a Λ-tree and let $\Gamma \times T_0 \to T_0$ be an action.

Fact 1. Let $\gamma \in \Gamma$ and $x \in T_0$. Suppose the translation length of γ is positive. Let τ_0 be the direction of the segment $[x, \gamma_x]$ at x and τ_1 its direction at γ_x. Then, x is contained in a partial axis for γ if and only if $\gamma_x \tau_0 \neq \tau_1$. If $\gamma_x \tau_0 = \tau_1$, then there is a partial axis B_γ for γ missing x, and a non-degenerate segment $[x, z]$ leaving x in the direction τ_0 such that $[x, z] \cap B_\gamma = \{z\}$.

Fact 2. Let γ_1 and γ_2 have partial axes B_1 and B_2 in T_0 with $B_1 \cap B_2 \neq \varnothing$, such that the positive directions of B_1 and B_2 induced by γ_1 and γ_2 disagree on $B_1 \cap B_2 = [x, y]$. If $\tau(\gamma_1)$ and $\tau(\gamma_2)$ are both greater than the length of $B_1 \cap B_2$, then $\gamma_2 \gamma_1^{-1}$ has a partial axis meeting $B_1 \cup B_2$ in exactly $[\gamma_1 x, y] \cup [y, \gamma_2 x]$.

Now we return to the proof that ψ has no folding. For simplicity, we denote by the same letters a component of $\tilde{M} - \tilde{\mathscr{L}}$ and its image in $T_{\mathscr{L}}$; likewise a leaf of $\tilde{\mathscr{L}}$ and the corresponding edge in $T_{\mathscr{L}}$.

If R is a component of $\tilde{M} - \tilde{\mathscr{L}}$, then R is the union of a compact piece and spikes. Each spike is bounded by two half-leaves following the same infinite path of flow boxes and having the same endpoint in $\partial \mathbb{D}^2$. Conversely, any end of a non-compact leaf in ∂R is part of the boundary of a spike.

LEMMA IV.1.3. *Let R be a complementary region of $\tilde{\mathscr{L}} \subset \tilde{M}$. Suppose that leaves $\tilde{\ell}_1$ and $\tilde{\ell}_2$ of $\tilde{\mathscr{L}}$ bound a spike S in R. Then, the directions $\tilde{\varphi}_\#(\tilde{\ell}_1)$ and $\tilde{\varphi}_\#(\tilde{\ell}_2)$ at $\tilde{\varphi}(R)$ in T are distinct. Furthermore, there is an element $\gamma \in \Gamma$ with a partial axis B'_γ in $T_{\mathscr{L}}$ such that*:

(a) *B'_γ passes through R,*

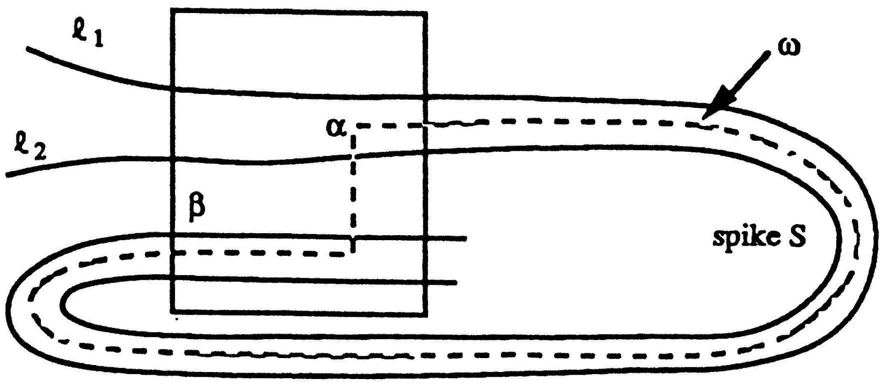
(b) *its directions at R are given by $\tilde{\ell}_1$ and $\tilde{\ell}_2$, and*

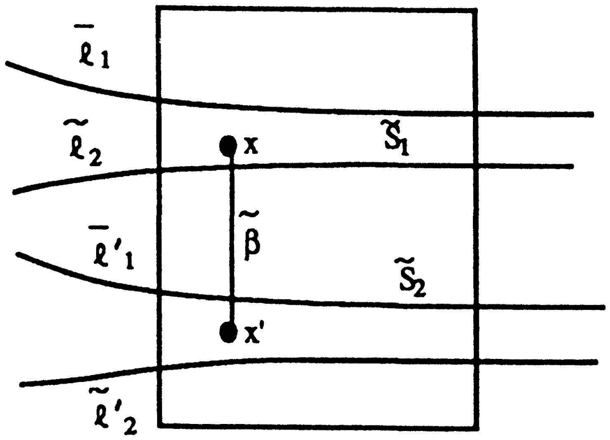
(c) *$\psi \mid B'_\gamma : B'_\gamma \to T$ is an injection.*

Proof. Clearly, the existence of a partial axis B'_γ satisfying (a), (b) and (c) implies that $\tilde{\varphi}_\#(\tilde{\ell}_1) \neq \tilde{\varphi}_\#(\tilde{\ell}_2)$. There is a transversality flow box $U \subset M$ for φ which the spike of R bounded by $\tilde{\ell}_1$ and $\tilde{\ell}_2$ crosses infinitely often. Choose two crossings which go in the same direction.

Let ω be the loop formed from an arc in the spike α and a vertical arc β in U as pictured on following page. Let $\tilde{U}$ in $\tilde{M}$ be a component of the pre-image of U. We have $\tilde{\beta} \subset \tilde{U}$ with endpoints in lifts of the spike S.

Thus, there is an element $\gamma \in \Gamma$ whose conjugacy class is represented by the loop ω with $\gamma \cdot \tilde{S}_1 = \tilde{S}_2$. The image of $\tilde{\beta}$ in $T_{\mathscr{L}}$ is a fundamental segment for the action of γ on a partial axis $B'_\gamma \subset T_{\mathscr{L}}$. Since $\tilde{U}$ is a transversality flow box for $\tilde{\varphi}$, $\tilde{\varphi} \mid : (\tilde{\beta} - \tilde{\beta} \cap \tilde{\mathscr{L}}) \to T$ is an injection onto a segment u of T. Furthermore, if τ_0 denotes the direction of this segment at $x' = \tilde{\varphi}(x)$, then $\gamma \cdot \tau_0$ is the direction $\tilde{\varphi}_\#(\tilde{\ell}'_2)$ which is distinct from $\tau_1 = \tilde{\varphi}_\#(\tilde{\ell}'_1)$ again because $\tilde{U}$ is a transversality flow box for





$\tilde{\varphi}$. Thus, by Fact 1, the translates of u under the powers of γ form a partial axis B_γ for γ in T. Clearly $\psi : B'_\gamma \to B_\gamma$ is an isomorphism. □

COROLLARY IV.1.4. *ψ has no folding at a vertex $v \in T_{\mathscr{L}}$ of order* 2.

Proof. Let R be the complementary region of $\tilde{\mathscr{L}} \subset \tilde{M}$ corresponding to v. Then ∂R consists of two leaves $\tilde{\ell}_1$ and $\tilde{\ell}_2$ of $\mathscr{L}$.

Consider the map $R \to M$. It is either one-to-one or an infinite cyclic covering. If the map is one-to-one, then R has finite area and $\tilde{\ell}_1$ and $\tilde{\ell}_2$ bound a spike in R. By the above lemma, $\tilde{\varphi}_\#(\tilde{\ell}_1) \neq \tilde{\varphi}_\#(\tilde{\ell}_2)$. If $R \to M$ is an infinite cyclic covering, then $\tilde{\ell}_1$ and $\tilde{\ell}_2$ project to parallel simple closed curves and R projects to an annulus. Since all annuli in $\tilde{M}$ are taut for $\tilde{\varphi}$, $\tilde{\varphi}_\#(\tilde{\ell}_1) \neq \tilde{\varphi}_\#(\tilde{\ell}_2)$ in this case as well. □

Assume that ψ has a folding. By the above result, this folding must be at a vertex of order >2. Let R be the corresponding component of $\tilde{M} - \tilde{\mathscr{L}}$ and $\tilde{\ell}_1$, $\tilde{\ell}_2$ two leaves in ∂R which are identified. By Corollary IV.1.4, R has order at least 3.

The completion of the proof of Theorem IV.1.1 is based on the following lemma:

LEMMA IV.1.5. *Let $\tilde{\ell}$ be a leaf of $\tilde{\mathscr{L}}$ in ∂R which projects to a non-compact leaf in M; let $\hat{\ell}^+$ and $\hat{\ell}^-$ be the leaves of $\tilde{\mathscr{L}}$ tangent to the ends of $\tilde{\ell}$. Then, there is an element $\omega \in \Gamma$ such that:*

(1) *The axis B'_ω in $T_{\mathscr{L}}$ for ω passes through R from the $\hat{\ell}^+$ to the $\hat{\ell}^-$ direction.*

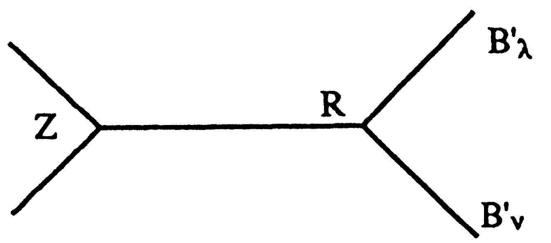
(2) *If $\tilde{\varphi}_\#(\hat{\ell}^+) \neq \tilde{\varphi}_\#(\hat{\ell}^-)$, then the axis B_ω for ω in T passes through $\tilde{\varphi}(R)$ from the $\tilde{\varphi}_\#(\hat{\ell}^-)$ to the $\tilde{\varphi}_\#(\hat{\ell}^+)$ direction.*

(3) *If $\tilde{\varphi}_\#(\hat{\ell}^+) = \tilde{\varphi}_\#(\hat{\ell}^-)$, then the axis B_ω for ω in T misses $\tilde{\varphi}(R)$ and is connected to $\tilde{\varphi}(R)$ by a segment $[\tilde{\varphi}(R), \omega]$ which leaves $\tilde{\varphi}(R)$ in the $\tilde{\varphi}_\#(\hat{\ell}^+)$ direction.*

Proof. By Lemma IV.1.3, there are elements λ and ν of Γ such that:

(IV.1.6)
- (i) There is a partial axis $B'_\lambda \subset T_\ell$ for λ that crosses R from $\hat{\ell}^+$ to $\tilde{\ell}$ when oriented in the direction that λ operates.
- (ii) There is a partial axis $B'_\nu \subset T_{\mathscr{L}}$ for ν that crosses R from $\tilde{\ell}$ to $\tilde{\ell}^-$ when oriented in the direction that ν operates.
- (iii) $\psi \mid B'_\lambda$ and $\psi \mid B'_\nu$ are injections.

In $T_{\mathscr{L}}$, the picture of the axes of λ and ν are:

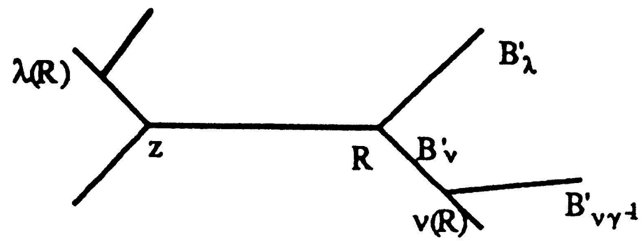


where $z \neq R$ since B'_ν enters R and B'_λ leaves R both in the $\tilde{\ell}_1$-direction. Let $\tau(\lambda)$ and $\tau(\nu)$ be the translation lengths of λ and ν in $T_{\mathscr{L}}$. We say that these lengths are *comparable* if there are integers $m, n > 0$ with $n(\tau(\lambda)) > \tau(\nu) > \tau(\lambda)/m$.

CLAIM IV.1.7. *There exist elements λ and ν satisfying* (IV.1.6) *with $\tau(\lambda)$ and $\tau(\nu)$ comparable.*

Proof. By symmetry, we can suppose $\tau(\lambda) \geq \tau(\nu)$. Let r be the length of the segment $B'_\lambda \cap B'_\nu$. If $r > \tau(\nu)$, then λ and $\nu\lambda^{-1}\nu^{-1}$ have all the properties required in (IV.1.6) for λ and ν. In addition $\tau(\nu\lambda^{-1}\nu^{-1}) = \tau(\lambda)$. Thus, replacing ν by $\nu\lambda^{-1}\nu^{-1}$ proves Claim IV.1.7 is this case.

Now suppose $\nu \leq \tau(\nu)$. Consider the element $\nu\lambda^{-1}$. By Fact 2, it has a partial axis as pictured below:

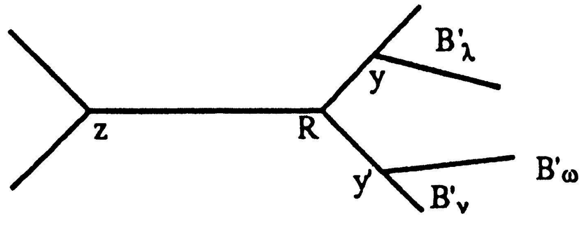


Clearly, $\tau(\nu\lambda^{-1}) = \tau(\nu) + \tau(\lambda) \leq 2\tau(\lambda)$, and $\tau(\nu\lambda^{-1}) \geq \tau(\lambda)$. Thus, replacing ν with $\nu\lambda^{-1}$ will prove the result provided that $\psi|_{B'_{\nu\lambda^{-1}}}$ is injective, or equivalently that the image of the segment $I = [\lambda R, R] \cup [R, \nu R]$ is a fundamental domain in T for the action of $\nu\lambda^{-1}$ on a partial axis. Clearly, ψ restricted to I is an injection. Furthermore, $\psi_{\#}$ of the direction at λR pointing toward R is transformed by $\nu\lambda^{-1}$ to $\psi_{\#}$ of a direction at νR pointing away from R. Hence, $\psi(I) \cap \psi(\nu\lambda^{-1}I)$ is equal to $|\nu R|$. Hence, $\psi|_{B'_{\nu\lambda^{-1}}}$ is an injection. □

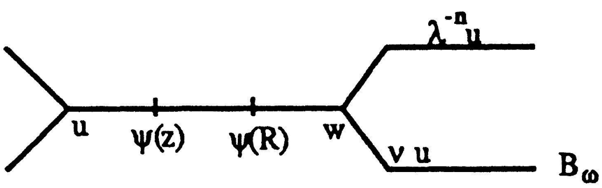
CLAIM IV.1.8. *Let λ and ν be elements as in* (IV.1.7). *There is an integer $n > 0$ such that the intersection of the axes $B_\lambda \cap B_\nu \subset T$ is a segment of length less than* $n \cdot \min(\tau(\lambda), \tau(\nu))$.

Proof. Since $\tau(\lambda)$ and $\tau(\nu)$ are comparable, the length of $B_\lambda \cap B_\nu$ is greater than $n \cdot \min(\tau(\lambda), \tau(\nu))$ for all n if and only if it is greater than $n \cdot (\tau(\lambda) + \tau(\nu))$ for all n. Let us suppose that this is the case. The elements λ and ν generate in Γ a free group of rank 2 $\langle\lambda, \nu\rangle$. If $B_\lambda \cap B_\nu$ has length greater than $n(\tau(\lambda) + \tau(\nu))$ for all n, then any finite set of words in the commutator subgroup $[\langle\lambda, \nu\rangle, \langle\lambda, \nu\rangle]$ fixes some non-degenerate segment in T. This contradicts Condition (a). □

We fix now elements λ and ν as in (IV.1.7) and an integer $n > 0$ such that the length of $B_\lambda \cap B_\nu$ is less than $n \cdot \min(\tau(\lambda), \tau(\nu))$. Of course, it is also true that the length of $B'_\lambda \cap B'_\nu$ is less than $n \cdot \min(\tau(\lambda), \tau(\nu))$. We wish to study the element $\omega = \nu^n\lambda^n$ and its axes in $T_{\mathcal{L}}$ and in T. First, let us consider the situation in $T_{\mathcal{L}}$. Let $y = \lambda^{-n}z$ and $y' = \nu^{-n}z$.



The axis B'_ω satisfies: $B'_\omega \cap (B'_\lambda \cup B'_\nu) = [y, R] \cup [R, y']$. In particular, B'_ω passes through R, entering from the $\hat{\ell}^+$-direction and leaving via the $\hat{\ell}^-$-direction. Now $\psi \mid B'_\nu$ and $\psi \mid B'_\lambda$ are injections, but of course $B_\nu \cap B_\lambda$ can be longer than $B'_\nu \cap B'_\lambda$. Suppose that the endpoints of $B_\lambda \cap B_\nu$ are u and w as pictured below.



Then, ω has a partial axis B_ω in T with $B_\omega \cap (B_\lambda \cup B_\nu) = [\lambda^{-n}u, w] \cup [w, \nu^n u]$. This completes the proof of Lemma IV.1.5. □

Now we return to the proof of Theorem IV.1.1. Let $\tilde{\ell}_1$ and $\tilde{\ell}_2$ be leaves in ∂R which are identified under $\tilde{\varphi}_\#$. From Lemma IV.1.3, we deduce that $\tilde{\ell}_1$ and $\tilde{\ell}_2$ have no common endpoints. There are three cases.

Case 1. The images ℓ_1 and ℓ_2 of $\tilde{\ell}_1$ and $\tilde{\ell}_2$ in $\mathscr{L}$ are non-compact.
Let $\hat{\ell}_1^\pm$ be the leaves adjacent to $\tilde{\ell}_1$.

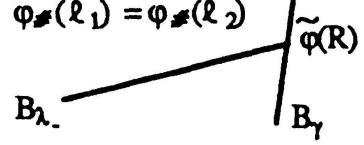
Subcase 1a. $\tilde{\varphi}_\#(\hat{\ell}_1^+) \neq \tilde{\varphi}_\#(\hat{\ell}_1^-)$. By Lemma IV.1.5 there is an element $\omega \in \Gamma$ whose axis B'_ω in $T_{\mathscr{L}}$, passes through R from direction $\hat{\ell}_1^+$ go $\hat{\ell}_1^-$, and whose axis B_ω in T, passes through $\tilde{\varphi}(R)$ from the $\tilde{\varphi}_\#(\hat{\ell}_1^+)$-direction to the $\tilde{\varphi}_\#(\hat{\ell}_1^-)$-direction.

By Lemma IV.1.5 there are elements λ_+, λ_-, γ whose axes in $T_{\mathscr{L}}$ pass through R from directions $\tilde{\ell}_1$ to $\hat{\ell}_1^+$, $\tilde{\ell}_1$ to $\hat{\ell}_1^-$ and $\tilde{\ell}_2$ to $\hat{\ell}'_2$ where $\hat{\ell}'_2$ is tangent to $\tilde{\ell}_2$ and such that ψ is an injection on each of these axes.

Since $\tilde{\varphi}_\#(\hat{\ell}_1^+) \neq \tilde{\varphi}_\#(\hat{\ell}_1^-)$, we know that $\tilde{\varphi}_\#(\hat{\ell}'_2)$ is dintinct from one of $\tilde{\varphi}_\#(\hat{\ell}_1^\pm)$. By symmetry we can assume that $\tilde{\varphi}_\#(\tilde{\ell}'_2) \neq \tilde{\varphi}_\#(\hat{\ell}_1^-)$.

Now consider $\mu = \lambda^{-1}\gamma$. Since $B'_\gamma \cap B'_{\lambda_-} = |R|$ in $T_{\mathscr{L}}$, the axis B_μ in $T_{\mathscr{L}}$ passes through R from the $\tilde{\ell}_2$-direction to the $\tilde{\ell}_1$-direction. Clearly, then $B'_\mu \cap B'_\omega = |R|$. It follows that the geometric axes for μ and ω in hyperbolic space also cross.

Let us look at the picture in T. B_γ passes through $\tilde{\varphi}(R)$ from the $\tilde{\varphi}_\#(\tilde{\ell}_2)$-direction to the $\tilde{\varphi}_\#(\hat{\ell}'_2)$-direction and B_{λ_-} passes through $\tilde{\varphi}(R)$ from the $\tilde{\varphi}_\#(\tilde{\ell}_1)$-direction to the $\tilde{\varphi}_\#(\hat{\ell}_1^-)$-direction. By Lemma IV.1.3, $\tilde{\varphi}_\#(\tilde{\ell}_1) \neq \tilde{\varphi}_\#(\hat{\ell}_1^\pm)$. Since $\tilde{\varphi}_\#(\tilde{\ell}_1) = \tilde{\varphi}_\#(\tilde{\ell}_2)$, and $\tilde{\varphi}_\#(\tilde{\ell}'_2) \neq \tilde{\varphi}_\#(\tilde{\ell}_1^-)$, we have



The element $\mu = \lambda_{-\gamma}^{-1}$ has an axis in T unless $\lambda_-^{-1}(\tilde{\varphi}(R)) = \gamma^{-1}(\tilde{\varphi}(R))$. Thus, replacing λ_- by λ_-^2 if necessary, we can assume that $\mu = \lambda_-^{-1}\gamma$ has an axis in T. Clearly, this axis does not pass through $\tilde{\varphi}(R)$, but rather is connected to $\tilde{\varphi}(R)$ by a non-degenerate segment leaving $\tilde{\varphi}(R)$ in the $\tilde{\varphi}_\#(\tilde{\ell}_1)$-direction. Either the axis for ω in T passes through $\tilde{\varphi}(R)$ from the $\tilde{\varphi}_\#(\hat{\ell}_1^+)$-direction to the $\tilde{\varphi}_\#(\hat{\ell}_1^-)$-direction or it misses $\tilde{\varphi}(R)$ and is connected to it by a segment which leaves $\tilde{\varphi}(R)$ in the $\tilde{\varphi}_\#(\hat{\ell}_1^+)$-direction. Since $\tilde{\varphi}_\#(\tilde{\ell}_1) \neq \tilde{\varphi}_\#(\hat{\ell}_1^\pm)$, B_μ and B_ω are connected by a non-degenerate segment in T. This contradicts Condition (b).

Subcase 1b. $\tilde{\varphi}_\#(\hat{\ell}_1^+) \neq \tilde{\varphi}_\#(\hat{\ell}_1^-)$.

Since $\hat{\ell}_1^+$ is part of the boundary of a spike, it does not project to a compact leaf in $\mathscr{L}$. Let $\tilde{\ell}'$ be the leave distinct from $\tilde{\ell}_1$ asymptotic to $\hat{\ell}_1^+$. By Lemma IV.1.5 there are elements ω and μ such that the axis B'_ω of ω in $T_{\mathscr{L}}$ passes through R from the $\tilde{\ell}_1^+$ to the $\tilde{\ell}_1^-$ direction and the axis B'_μ of μ passes through R from the $\tilde{\ell}'$ direction to the $\tilde{\ell}_1$ direction. Furthermore, the axis B_ω in T misses $\tilde{\varphi}(R)$ and is joined to it by a segment leaving $\tilde{\varphi}(R)$ in the $\varphi_\#(\ell_1^+)$ direction. Likewise, either B_μ passes through $\tilde{\varphi}(R)$ from the $\tilde{\varphi}_\#(\ell')$ direction to the $\tilde{\varphi}_\#(\tilde{\ell}_1)$ direction or B_μ misses $\tilde{\varphi}(R)$ and is joined to it by a segment leaving $\tilde{\varphi}(R)$ in the $\tilde{\varphi}_\#(\tilde{\ell}_1)$ direction. Since $\tilde{\varphi}_\#(\tilde{\ell}_1)$ and $\tilde{\varphi}_\#(\ell')$ are distinct from $\tilde{\varphi}_\#(\hat{\ell}_1^+)$ by Corollary IV.1.3, this contradicts condition (b).

Case 2. ℓ_1 is compact but ℓ_2 is non-compact.

$\tilde{\ell}_1$ corresponds to an element $\gamma \in \Gamma$ which stabilizes $\tilde{R}$. Hence, $\gamma(\tilde{\ell}_2) \in \partial R$. The map $\tilde{\varphi}$ is Γ-equivariant, hence the directions $\tilde{\ell}_2$ and $\gamma(\tilde{\ell}_2)$ are identified in T. As $\tilde{\ell}_2$ and $\gamma(\tilde{\ell}_2)$ project down on ℓ_2, non-compact, we can apply the first case. Thus, once again Case 2 leads to a contradiction.

Case 3. ℓ_1 and ℓ_2 are compact.

If the region R is not an annulus, then the stabilizers γ_1 of $\tilde{\ell}_1$ and γ_2 of $\tilde{\ell}_2$ do not lie in a cyclic subgroup of Γ. But $\langle\gamma_1, \gamma_2\rangle$ stabilizes the direction $\tilde{\varphi}_\#(\tilde{\ell}_1) = \tilde{\varphi}_\#(\tilde{\ell}_2)$. This contradicts Condition (a).

Hence, R must be an annulus. It is taut for the map $\tilde{\varphi}$. This means that $\tilde{\varphi}_\#(\tilde{\ell}_1) \neq \tilde{\varphi}_\#(\tilde{\ell}_2)$. This is a contradiction.

This completes the proof of Theorem IV.1.1. □

Section IV.2. The convergence theorem for surfaces

Let us recall the following definition from [6]. Let M be a surface of finite area, and P the set of cusps of M. We denote by $D_{\mathbb{R}}(M, P)$ the set of characters of discrete and faithful representations of $\Gamma = \pi_1(M)$ in $SL_2(\mathbb{R})$, which sends peripheral elements to parabolics. From an unbounded sequence (x_i) in $D_{\mathbb{R}}(M, P)$ one can extract a subsequence (x_{i_j}) and define an action $\varphi : \Gamma \times T \to T$ of Γ on a Λ-tree T with cyclic edge stabilizers such that (x_{i_j}) "converges" to the action $\varphi : \Gamma \times T \to T$.

The basic property of the action $\Gamma \times T \to T$ is that for any $\alpha, \beta \in \Gamma$ such that $|\mathrm{tr}\,(\beta(x_{i_j}))| \to \infty$, we have

$$\lim_{j \to \infty} \frac{|\mathrm{tr}\,(\alpha(x_{i_j}))|}{|\mathrm{tr}\,(\beta(x_{i_j}))|} = \frac{\tau_\varphi(\alpha)}{\tau_\varphi(\beta)} \in [0, \infty]$$

where τ_φ is the translation length function for T and where the right-hand side is the non-archimedean ratio on $\Lambda^{>0}$.

PROPOSITION IV.2.1. *Given an unbounded sequence* (x_{i_j}) *and an action* $\varphi : \Gamma \times T \to T$ *as in* IV.1, *for any elements* $g, h \in \Gamma$ *with* $A_g^{\mathrm{hyp}} \cap A_h^{\mathrm{hyp}} \neq \mathbb{H}^2$ *and with* g *and* h *having partial axes* B_g *and* B_g *in* T, *it cannot be the case that there is a non-degenerate segment* $[x, y] \subset T$ *with* $B_g \cap [x, y] = \{x\}$ *and* $B_h \cap [x, y] = \{y\}$.

Proof. We note by $\ell_i(\gamma)$ the translation length for the representation x_i of $\gamma \in \Gamma$. Since the axes for g and h meet for each of the hyperbolic structures x_i, it is easy to see that

$$\ell_i(gh) \leq \ell_i(g) + \ell_i(h)$$

so that, for some constant C, one has

$$\log |\mathrm{tr}\,(gh(x_{i_j}))| \leq \log |\mathrm{tr}\,(g(x_{i_j}))| + |\mathrm{tr}\,(h(x_{i_j}))| + C.$$

Since g and h have partial axes in the action φ, it follows that $\log |\mathrm{tr}\,(g(x_{i_j}))|$ and $\log |\mathrm{tr}\,(h(x_{i_j}))|$ go to infinity with j. Clearly

$$\lim_{j \to \infty} \frac{\log |\mathrm{tr}\,(gh(x_{i_j}))|}{\log |\mathrm{tr}\,(h(x_{i_j}))|} \leq \lim_{j \to \infty} \frac{\log |\mathrm{tr}\,(g(x_{i_j}))|}{\log |\mathrm{tr}\,(h(x_{i_j}))|} + 1.$$

Thus,

$$\tau_\varphi(gh) \leq \tau_\varphi(g) + \tau_\varphi(h). \qquad (*)$$

If there were a segment $[x, y]$ as in Proposition IV.2.1, then a fundamental domain for gh on a partial axis would be $[g^{-1}x, x] \cup [x, y] \cup [x, hy] \cup [hy, hx]$. Thus, $\tau(gh) = \tau(g) + \tau(h) + 2$ length $[x, y]$, contradicting $(*)$. □

We come now to our main result.

THEOREM IV.2.2. *Let $(x_i) \in D_{\mathbb{R}}(M, P)$ be a sequence which defines a growth function for the lengths of closed curves in M. Then, there is a codimension-*1 *nonarchimedeanly measured lamination $(\mathscr{L}, \mu) \subset M$ such that geometric intersection with $(\mathscr{L}, \mu)$ defines the same non-archimedean ratio on the set of homotopy classes of closed curves as the sequence (x_i) does.*

Proof. According to [6] the non-archimedean growth function associated to the sequence (x_i) is also associated to the translation length function of a minimal action $\Gamma \times T \to T$ of Γ on a Λ-tree with cyclic edge stabilizer. Since the cusps of M are represented by parabolic elements throughout the sequence, for each cusp E, $\pi_1(E)$ has a fixed point in T. According to IV.2.1 the action satisfies Condition (b) of Theorem IV.1.1. Thus, by Theorem IV.1.1., this action is dual to a measured lamination $(\mathscr{L}, \mu) \subset M$. By (I.3.10) the geometric intersection with $(\mathscr{L}, \mu)$ is the translation length function of the action. This completes the proof of Theorem IV.2.2. □

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