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Homotopy and isotopy in dimension three

JOEL HASS[1] AND PETER SCOTT[2]

Let M be a closed P^2-irreducible 3-manifold. It is a long standing problem to decide if homotopic homeomorphisms of M must be isotopic. The answer is now known to be affirmative if M is Haken, [Wa1], see also [L], or if M is a Seifert fiber space [Ho–R] [Bon] [B–R] [A] [R] [Sc1] [B–O], and for a few other special manifolds [B–R]. Thus is now seems reasonable to conjecture that the answer is always affirmative. However, if one considers reducible manifolds, there is a counter example [F–W]. In this paper, we further enlarge the class of 3-manifolds for which the above conjecture can be proved. If a closed P^2-irreducible 3-manifold is non-orientable, it must be Haken, so we consider only orientable 3-manifolds in the rest of this paper.

Let M be an orientable 3-manifold, let F be a closed orientable surface not S^2 and let $f: F \to M$ be an immersion which injects $\pi_1(F)$. Let M_F denote the cover of M such that $\pi_1(M_F)$ equals $f_*(\pi_1(F))$ and let $\tilde{M}$ denote the universal cover of M. We will suppose that the lift of f into M_F is an embedding. (Note that this is automatic if f is least area in the smooth or PL sense.) Thus the pre-image in $\tilde{M}$ of $f(F)$ consists of an embedded plane Π which covers F in M_F and the translates of Π by $\pi_1(M)$. We will say that f has the *k-plane property* if, given k distinct translates of Π, some pair is disjoint. In this paper we will consider the case when k equals 3. A map with the 3-plane property has no transverse triple points. We will say that f has the *1-line-intersection property* if two distinct translates of Π are disjoint or intersect transversely in a single line. The main result of this paper is

THEOREM 1.1. *Let M be a closed orientable irreducible 3-manifold which is neither Haken nor a Seifert fiber space. If there is a closed orientable surface F, not S^2, and an immersion $f: F \to M$ which injects $\pi_1(F)$ and has the 3-plane and 1-line-intersection properties, then homotopic homeomorphisms of M are isotopic.*

In [H–S], we show that if M satisfies the hypotheses of this theorem, and M is homotopy equivalent to an irreducible 3-manifold N, then M and N are homeomor-

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phic. In fact we prove a more general result in which the 3-plane hypothesis is replaced by the assumption that f has the 4-plane property.

Note that as in [H–S], the hypotheses of Theorem 1.1 do not require that M have any finite cover which is Haken. But if M is double covered by a Haken manifold M_1 which admits an embedding of F injecting $\pi_1(F)$, then the immersion $f: F \to M$ obtained by projecting into M will have the 3-plane property. For the planes in $\tilde{M}$ above $f(F)$ will split into two families, each family consisting of mutually disjoint planes. Two out of any three planes in $\tilde{M}$ must belong to the same family and thus not intersect. If the embedding of F in M_1 also has the 1-line-intersection property then M will satisfy the hypotheses of the theorem unless it is Haken or a Seifert fiber space. This will happen, for example, if M_1 is hyperbolic and F is totally geodesic.

There is a general construction of manifolds which satisfy the hypotheses of Theorem 1.1 which we have discussed with Aitchison and Rubinstein and which will appear in future work of theirs. It is also discussed in [Sk]. One starts with a closed orientable surface F of genus at least two, chooses an even number of disjoint essential simple closed curves on F and chooses an identification of these curves in pairs. Then one thickens the resulting 2-complex to an orientable 3-manifold so that the two sheets of F cross where the curves are identified and one adds 2-handles and 3-handles to obtain a closed 3-manifold M which clearly has an immersion of F without triple points. If one chooses the 2-handles to be attached in a fairly complicated way, then one can show that M is irreducible, that the map of F into M injects $\pi_1(F)$ and that F has the 3-plane and the 1-line intersection properties. Presumably most of the manifolds obtained in this way will be non-Haken, though we cannot prove this. It follows from [HRS] that any non-Haken irreducible 3-manifold which admits such an immersion of F is obtained in this way.

The result of Theorem 1.1 and its proof are very closely related to those of [Sc1]. In [Sc1], Scott showed that homotopic homeomorphisms were isotopic for certain Seifert fiber spaces. Any irreducible Seifert fiber space M with infinite fundamental group admits an immersion of the torus T into M which injects $\pi_1(T)$, and such a map always has the 1-line-intersection property [FHS]. Scott's result in [Sc1] applied to exactly those non-Haken Seifert fiber spaces for which $f: T \to M$ could be found with the 3-plane property. Thus Theorem 1.1 is a natural extension of the main result of [Sc1]. The outline of the argument in this paper follows closely that of [Sc1], but our arguments differ in some important details. Note that, in Theorem 1.1, F cannot be the torus. For, as f has no triple points, this would imply that M was a Seifert fiber space. The reason for our assumption, in Theorem 1.1, that M is not a Seifert fiber space is that our arguments do not work well when F is the torus. In [Sc1], the map f was very special because it had only one double curve, and this allowed the arguments to work.

The idea of our proof of Theorem 1.1 is as follows. Let X denote the 2-complex $f(F)$ in M, and let h be a homeomorphism of M which is homotopic to the identity. Note that, as M is not Haken, f cannot be an embedding. As in [Sc1], we consider $hX \cap X$. This intersection cannot be empty. The key step is to isotop h until $hX \cap X$ contains no triple points. We do this in §1. In [Sc1], this almost completed the argument, but here we need to work a great deal harder. Let Δ denote the union of all the double curves of X. Our aim is to isotop h to the identity in stages, working first with $h \mid \Delta$, then with $h \mid X$ and finally with h itself. However this raises some tricky points which did not need to be considered in [Sc1]. We discuss these fully in §2.

There is an alternative approach which we discuss fully in §3. The first step is still to isotop h to be the identity on Δ, but the next step is to show that Δ is *free* in a sense which we define. Then we show that if h fixes a free link in M, then h is isotopic to the identity. This last result uses Thurston's hyperbolization theorem, whereas the arguments in §2 are more elementary and direct. However the arguments in §3 are likely to lead to other applications. We would like to acknowledge a helpful conversation with Francis Bonahon on this section of the paper.

§1. Removing triple points

We will use notation as close to that of [Sc1] as possible, as several results we need are proved in [Sc1]. Our main result is

THEOREM 1.1. *Let M be a closed orientable irreducible 3-manifold which is neither Haken nor a Seifert fiber space. If there is a closed orientable surface F, not S^2, and an immersion $f: F \to M$ which injects $\pi_1(F)$ and has the 3-plane and 1-line-intersection properties, then homotopic homeomorphisms of M are isotopic.*

We will arrange that f does not factor through a covering map of some surface F' by F, by replacing f if necessary. If f factors through a covering of an orientable surface F' by F and an immersion $f' : F' \to M$, we simply replace f by f', which will still have the 3-plane and 1-line-intersection properties. Thus we can assume that f does not factor through a covering of an orientable surface by F. If f factors through a covering of a non-orientable surface F'' by F, the degree of this covering must be two. In this case, we can perturb f so that it no longer factors through such a covering and still has the required properties. To do this, consider the cover M'' of M corresponding to $\pi_1(F'')$. The lift f'' of f to this cover must double cover an embedded copy of F''. We now replace f'' by an embedding with image equal to the boundary of a regular neighborhood of F'', and replace f by the composite of this embedding with the covering projection from M'' to M.

Let X denote the 2-complex $f(F)$ in M, and let h be a homeomorphism of M which is homotopic to the identity. Let Y denote $h(X)$. Let Δ denote the union of the double curves of X. A point of $\Delta \cap Y$ or of $h(\Delta) \cap X$ will be called a triple point of $X \cap Y$. We can always isotop h so that X and Y are in general position i.e. Δ and $h(\Delta)$ are disjoint, Δ meets Y transversely and $h(\Delta)$ meets X transversely at any triple points and X meets Y transversely in the usual sense at all other points. In this section, we prove

THEOREM 1.2. *Under the hypotheses of Theorem* 1.1, *let h be a homeomorphism of M which is homotopic to the identity. Let X and Y be as above. Then h is isotopic to a homeomorphism h_1 such that X and $h_1(X)$ are in general position and intersect without triple points and without nullhomotopic double curves.*

Proof. We first isotop h so that X and $Y = h(X)$ are in general position. Thus $X \cap Y$ has only finitely many triple points. As in [Sc1], the basic idea is to give a sequence of isotopies of X or Y in M, each of which reduces the number of triple points of $X \cap Y$. Any isotopy of Y can, of course, be extended to an isotopy of h, and any isotopy of X can be replaced by an isotopy of Y, and hence of h, which has the same effect on $X \cap Y$. However, we will need a measure of the complexity of $X \cap Y$ which is more subtle than just the number of triple points. Having defined this complexity, we will describe five types of isotopy each of which reduces our complexity. Finally, we will show that if $X \cap Y$ has least possible complexity, then it has no triple points. Taken together, these results will complete the proof of Theorem 1.2.

The points of $X \cap Y$ will be called XY-points, the points of Δ will be called XX-points and the points of $h(\Delta)$ will be called YY-points. A point of $\Delta \cap Y$ will be called an XX triple point and a point of $h(\Delta) \cap X$ will be called a YY triple point. A point in a covering space of M which projects to a XY-point in M will also be called a XY-point, and we use similar definitions for the other types of point.

In order to define the complexity of h, we need to consider the covering space M_F of M whose fundamental group is $f_*(\pi_1(F))$. Recall that f lifts to an embedding of F in M_F whose image we still denote by F. Let X_F denote the full pre-image in M_F of X, let Δ_F denote the full pre-image in M_F of Δ, and let h_F denote the lift of h to M_F obtained by lifting the homotopy of h to the identity. Note that the surfaces in X_F need not all be embedded. The points of $F \cap h_F(X_F)$ are XY-points, and the points of $F \cap \Delta_F$ are XX-points. We define the complexity of h to be the triple (s, t, d), where s is the number of XX triple points on F in M_F, t is the total number of triple points on F, and d is the number of null-homotopic curves of XY-points on F. These complexities are to be lexicographically ordered.

For the rest of our proof of Theorem 1.2, it will be more convenient to consider the universal covering space $\tilde{M}$ of M in place of M_F, as all surfaces and double curves in $\tilde{M}$ will be automatically embedded. Recall that the pre-image of $f(F)$ in $\tilde{M}$ consists of a family of embedded planes with the 3-plane property. We call these planes X-planes. The corresponding planes in the pre-image of Y will be called Y-planes. Two X-planes intersect in a XX-line or are disjoint, two Y-planes intersect in a YY-line or are disjoint, and the intersection of a X-plane and a Y-plane is a 1-manifold each component of which is called a XY-curve. In all diagrams, XX-lines and YY-lines will be drawn solid and XY-curves will be dotted.

Now we restrict attention to a single X-plane Π and the double curves which lie on Π. These consist of a family of disjoint XX-lines and a fairly arbitrary collection of XY-curves. The intersection of any two of these curves is a triple point. Similar comments apply to a Y-plane with the roles of X and Y reversed.

An XY-circle C in Π will be called *innermost* if the disc D in Π which is bounded by C has no double points in its interior. We let p denote the projection map $\tilde{M} \to M$.

LEMMA 1.3. *If Π is an X-plane or Y-plane which contains an innermost XY-circle, there is an isotopy of h which reduces its complexity.*

Proof. This result is the same as Lemma 2.2 of [Sc1], and the proof in [Sc1] applies unchanged to the present situation. The isotopy defined in [Sc1] reduces d by at least one and cannot increase s or t.

For our next results, we need some more definitions. A *2-gon* in an X-plane or a Y-plane Π is a 2-disc D in Π such that ∂D is the union of two arcs, each of which is a sub-arc of a double curve in Π. If both arcs lie on XY-curves, we will say that D is *of type XY*. Otherwise, we say D is of mixed type. A 2-gon D is *innermost* if its interior contains no double points.

LEMMA 1.4. *If Π is an X-plane or Y-plane which contains an innermost 2-gon D of mixed type, there is an isotopy of h which reduces its complexity.*

Proof. This result is the same as Lemma 2.3 of [Sc1]. The proof in [Sc1] needs the following slight modification.

As in [Sc1], assume that Π is an X-plane, and let ∂D consist of two arcs λ and μ, where λ is a sub-arc of an XX-curve L and μ is a sub-arc of an XY-curve. The isotopy required is defined as in [Sc1]. The proof that D projects injectively to M is also as in [Sc1] except for the following point. Let g be an element of $\pi_1(M)$ such that gD meets D. If $gL = L$, then it follows that $g\Pi = \Pi$. In [Sc1], this is proved by using the fact that $\pi_1(M)$ has center. In the present case, we note that the alternative is that g interchanges the two X-planes through L. Thus, in a neighborhood of L, g must act as a screw motion, so that g^2 must preserve Π and reverse its orientation, contradicting our hypothesis that F is orientable.

The isotopy defined in [Scl] reduces t by at least two. If Π is an X-plane, then it also reduces s by at least two. If Π is a Y-plane, then s is unchanged. In either case the complexity of h is reduced.

LEMMA 1.5. *If Π is an X-plane or Y-plane which contains an innermost 2-gon of type XY, there is an isotopy of h which reduces its complexity.*

Proof. This result is the same as Lemma 2.4 of [Scl]. The isotopy is constructed as in [Scl], and the proof in [Scl] only needs modifying as in Lemma 1.4. Namely, we need to point out that if g preserves an XX-line L, then it must preserve each of the two X-planes which contain L.

As for Lemma 1.4, the isotopy defined in [Scl] reduces t by at least two and reduces s by at least two or zero according as Π is a Y-plane or X-plane respectively.

Our next move can only be used in a situation where the three preceding lemmas are not applicable.

LEMMA 1.6. *Suppose that no X-plane or Y-plane contains an innermost circle or an innermost 2-gon, but that some plane Π contains a 2-gon of type XY. Then there is an isotopy of h which reduces its complexity.*

Proof. This result is the same as Lemma 2.5 of [Scl]. However, the proof needs substantial changes. We start by reproducing the first three paragraphs of the proof of Lemma 2.5 of [Scl]. We will first suppose that Π is an X-plane.

We have a 2-disc D in Π whose boundary consists of two arcs μ_1 and μ_2, each being a sub-arc of an XY-curve. We can assume that no sub-disc of D has the same property, by replacing D if necessary. As we are assuming that D is not innermost, the interior of D must contain some XX-points or XY-points. As D contains no sub-disc which is a 2-gon of type XY, we see that there cannot be an XY-arc in D with both ends on μ_1, or both on μ_2, and that any two XY-curves in D meet in at most one point. Also there cannot be an XY-arc joining μ_1 to μ_2 as this would contradict the 3-plane property for the Y-planes. It follows that any XY-curve in D, other than μ_1 and μ_2, must be a circle in the interior of D, and that any two such circles must be disjoint.

Suppose that D contains an XY-circle C. As no plane contains an innermost circle, some XX-line must meet C. As all XX-lines in Π are disjoint, this implies that D contains an innermost 2-gon of mixed type. As no plane contains an innermost 2-gon, we deduce that the interior of D contains no XY-points. Hence

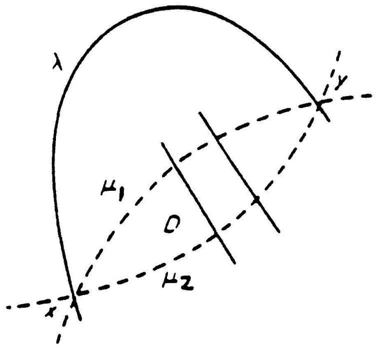

Figure 1.7

the interior of D must contain some XX-points. Let λ be an arc of XX-points in D. If $\partial\lambda$ lies on μ_1, then λ together with a sub-arc of μ_1 determines a 2-gon of mixed type. This 2-gon may contain other XX-arcs, but it follows that D contains an innermost 2-gon of mixed type. As we are assuming that Π has no innermost 2-gons, we deduce that all the XX-arcs in D join μ_1 to μ_2.

Let Π_i denote the Y-plane containing μ_i, for $i = 1, 2$, let L denote the line $\Pi_1 \cap \Pi_2$ and let λ denote the segment of L with the vertices of D as end points. Then $\lambda \cup \mu_i$ bounds a 2-disc D_i in Π_i, and $D \cup D_1 \cup D_2$ forms a 2-sphere in $\tilde{M}$ which must bound a 3-ball B. See Figure 1.7.

In [Sc1], Scott was able to show that B projects injectively into M. Then he defined an isotopy of Y in M which isotoped $p(D_1 \cup D_2)$ across $p(B)$ and just past $p(D)$. The present situation may be more complicated as B may not inject into M. However, we will show how to define a similar isotopy of Y in M.

Suppose now that gB meets B. Exactly as in [Sc1], it follows that $gL = L$. Thus, as in our proof of Lemma 1.4, g must preserve each of Π_1 and Π_2. If g also preserves Π, then either g is the identity or $gx = y$, or $gy = x$. But the last two cases imply that gB is on the opposite side of Π from B which contradicts the fact that Π projects to a 2-sided surface in M. Thus, if g preserves Π, then g must be the identity. Hence if g is a non-trivial element of $\pi_1(M)$ such that gB meets B, then g does not preserve Π. Also g^{-1} does not preserve Π. Now g acts on L as a non-trivial translation, so that g^2 acts on L as a non-trivial translation. It follows that g^2 also cannot preserve Π, so that $g\Pi$ and $g^{-1}\Pi$ are distinct. Now the 3-plane property applied to Π, $g\Pi$, and $g^{-1}\Pi$ shows that $g\Pi$ and $g^{-1}\Pi$ are disjoint. Hence Π and $g^2\Pi$ are disjoint.

Let α denote a generator of the stabiliser of L. We deduce from the above that if gB meets B then g must be one of 1, α or α^{-1}. If B does not meet αB, we can define the required isotopy of Y in M exactly as in [Sc1]. If B does meet αB, the picture must be as in Figure 1.8. We choose an isotopy of $D_1 \cup D_2$ across B such that the isotopy restricted to $(D_1 \cup D_2) \cap \alpha B$ is obtained from the isotopy restricted to $(D_1 \cup D_2) \cap \alpha^{-1}B$ by applying α. This can be done precisely because αB and $\alpha^{-1}B$

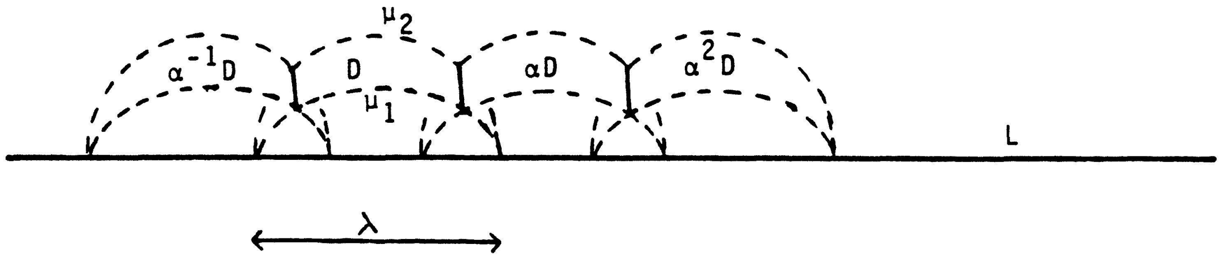

Figure 1.8

are disjoint. This isotopy extends to an isotopy of the union of all the $\alpha^n(D_1 \cup D_2)$ which is equivariant under the action of the group generated by α. Finally, this extends to an isotopy of all the translates of $D_1 \cup D_2$ by $\pi_1(M)$ which is $\pi_1(M)$-equivariant. This isotopy extends equivariantly to the union of all the Y-planes by the identity. Such an isotopy must descend to an isotopy of Y in M, as required. As in the preceding two lemmas, this isotopy reduces t by at least two and does not increase s as Π is an X-plane. If Π is a Y-plane, we can apply the same arguments reversing the roles of X and Y to obtain an isotopy of X which reduces s and t by at least two. This completes the proof of Lemma 1.6.

Our final move can only be used in a situation where none of the preceding lemmas is applicable. Unlike the previous moves, this move may increase the total number t of triple points of $X \cap Y$, but it does reduce the number s of XX triple points.

LEMMA 1.9. *Suppose that no X-plane or Y-plane contains an innermost circle or an innermost 2-gon, and that no plane contains a 2-gon of type XY. If an X-plane Π contains a 2-gon, there is an isotopy of h which reduces its complexity.*

Proof. This is the same statement as Lemma 2.6 of [Sc1]. However, as with the previous lemma, the proof needs substantial changes to apply to our situation. We start by reproducing the first paragraph of the proof of Lemma 2.6 of [Sc1].

Let D be a 2-gon in the X-plane Π with one edge λ being a sub-arc of an XX-curve and the other edge μ being a sub-arc of a XY-curve. We can assume that no sub-disc of D is a 2-gon, by replacing D by a sub-disc, if necessary. Thus the interior of D contains no XX-points. As D cannot be an innermost 2-gon, its interior must contain some XY-points. No XY-circle can lie in D as Π contains no innermost circles and no 2-gons of type XY. It follows that the XY-points of D lie on arcs. None of these arcs can have both end points on λ, as this would yield a sub-disc of D which is a 2-gon. Also none of these arcs can have both end points on μ, as Π contains no 2-gons of type XY. Hence each XY-arc in D joins λ to μ. Finally these arcs are disjoint from each other by the 3-plane property for the Y-planes. Thus we have a configuration as shown in Figure 1.10.

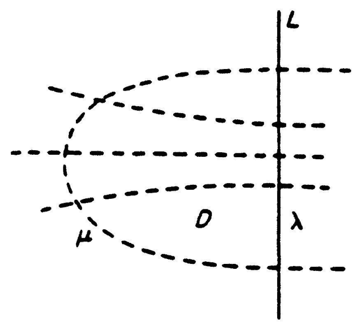

Figure 1.10

In [Sc1], Scott showed that D projected injectively into M. As for the previous lemma, this need not be true in the present situation, but we can still define the required isotopy of X in M in a similar way to that in [Sc1].

Suppose now that gD meets D. If $g\Pi$ does not equal Π, then $gD \cap D$ must consist of XX-points in $g\Pi \cap \Pi$. As the only XX-points in D lie on λ, and the only XX-points in gD lie on $g\lambda$, we deduce that gL must meet L. But this implies that $gL = L$ as L projects to a simple closed curve in M, and this implies that g preserves Π, as in the proof of Lemma 1.4. This contradicts our assumption that $g\Pi$ does not equal Π. Thus $g\Pi = \Pi$. Hence $g(\partial D)$ must meet ∂D. Hence one of $g\lambda$ and $g\mu$ must meet one of λ and μ. If $g\mu$ meets μ, then $g\mu$ must contain one of the XY-arcs which crosses D, so that $g\mu$ must meet λ. It follows that, in all cases, $gD \cap D$ contains points of λ or of $g\lambda$. As the only XX-points of D lie on λ and the only XX-points of gD lie on $g\lambda$, we conclude that $g\lambda$ meets λ. Hence gL meets L and so $gL = L$.

Let Π' denote the Y-plane which contains μ. If g is a nontrivial element of $\pi_1(M)$ such that gD meets D, the above shows that $g\Pi = \Pi$ and $gL = L$, so that $g\mu$ must cross μ. Hence $g\Pi'$ must cross Π', and also $g^{-1}\Pi'$ must cross Π'. By the 3-plane property for Y-planes, $g\Pi'$ and $g^{-1}\Pi'$ must be disjoint.

Let α denote a generator of the stabiliser of L. The above arguments show that if gD meets D then g must be one of 1, α or α^{-1}. See Figure 1.11. Now, as in the

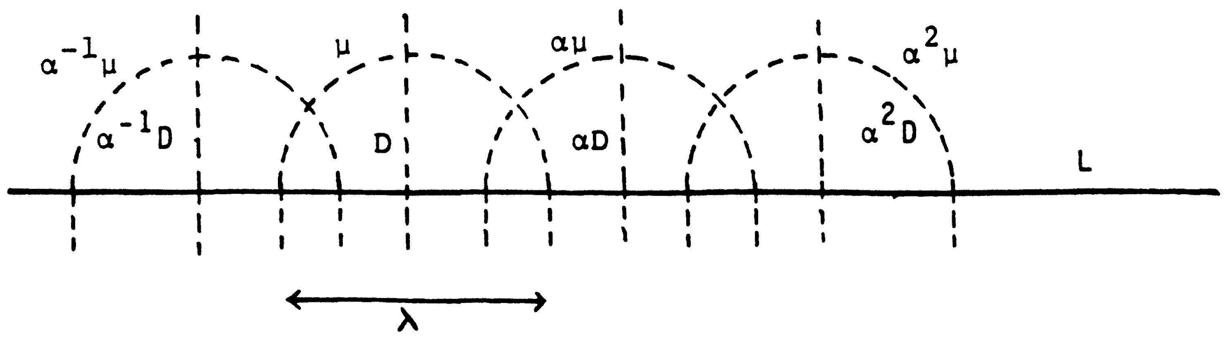

Figure 1.11

preceding lemma, we can define the required isotopy of X in M, by defining an equivariant isotopy of the X-planes in $\tilde{M}$, which isotops λ across D and just past μ.

As in [Sc1], this isotopy increases the total number t of triple points of $X \cap Y$, but it reduces the number s of XX triple points by two, thus reducing the complexity of h.

Now we are in a position to conclude the proof of Theorem 1.2, which asserts that h can be isotoped to arrange that $X \cap Y$ has no triple points. We have just seen five ways of isotoping h to reduce its complexity. The next lemma tells us that if none of these isotopies can be carried out, then $X \cap Y$ has no triple points. It also says that there are no nullhomotopic double curves.

LEMMA 1.12. *Suppose that the homeomorphism h minimises the complexity* (s, t, d) *among all homeomorphisms isotopic to h. Then the complexity of h is* $(0, 0, 0)$.

Proof. This is essentially the same statement as Lemma 2.7 of [Sc1]. However, the proof needs substantial changes to apply to our situation. We start by reproducing the first paragraph of the proof of Lemma 2.7 of [Sc1].

Lemma 1.3 shows that no X-plane or Y-plane contains an innermost XY-circle. Lemmas 1.4 and 1.5 show that no plane contains an innermost 2-gon. Lemma 1.6 then shows that no plane can contain a 2-gon of type XY. Finally, Lemma 1.9 shows that no X-plane contains a 2-gon of any type. It follows that no XY-curve can be a circle. Thus d must equal zero.

Let Π denote an X-plane. We have just seen that all the XY-curves in Π are lines and that any two of the XY-lines and XX-lines in Π meet in at most one point.

Recall the covering M_F of M and the lift of f to M_F whose image is denoted by F. The complete pre-image of X is denoted X_F and the lift of h is denoted h_F. Let Π denote the X-plane in $\tilde{M}$ which is the pre-image of F. We consider the image in F of the XX-lines and XY-lines in Π. The XX-lines project to a family of disjoint essential simple closed curves on F, and each XY-line projects to some essential closed curve. Recall that the XX-curves on F are the intersection curves of F with the other components of X_F. The XY-curves on F are the intersection curves of F with $h_F(X_F)$. If two of these curves C_1 and C_2 intersect, then they are the image of a pair of lines in Π which intersect. As these lines intersect in only one point, we deduce that C_1 and C_2 intersect essentially i.e. they cannot be homotoped to the disjoint.

Suppose that $X \cap Y$ has some XX triple points. Then there is a XY-curve C on F which crosses a XX-curve. It follows from the above that C cannot be homotopic to any XX-curve. Let F' denote the component of X_F such that $F \cap h_F(F')$ contains C. Note that F' may not be embedded or compact. The 1-line-intersection property for the X-planes implies that F' is disjoint from F, or F' equals F, or F' is an annulus and meets F in one simple closed XX-curve. The last case is impossible as C would then have to be homotopic to a XX-curve. Thus F' is disjoint from F or F' equals F, but $h_F(F')$ crosses F. In either case, it follows that $h_F(F')$ intersects one (and possibly both) of the closures of the components of $M_F - F$ in a compact surface.

Suppose that F' is an embedded surface in M_F. It follows, as in Lemma 4.1 of [FHS], that there must be a compact product region W in M_F between F and $h_F(F')$. Let Ω and Ω' denote the surfaces in which W meets F and $h_F(F')$. Taking the pre-image of W in $\tilde{M}$ yields a Y-plane Π' in $\tilde{M}$ above $h_F(F')$ and a product region $\tilde{W}$ between Π and Π' which projects to W in M_F. Let $\tilde{\Omega}$ and $\tilde{\Omega}'$ denote the surfaces in which $\tilde{W}$ meets Π and Π'. Of course, $\tilde{W}$ will probably not be compact.

There is an XY-line L in Π which projects to C and which crosses $\partial\tilde{\Omega}$. As L projects to C, each component of $L \cap \tilde{\Omega}$ projects to a compact subinterval of C. It follows that $L \cap \tilde{\Omega}$ consists of compact arcs. In fact, $L \cap \tilde{\Omega}$ must be a single arc λ, as otherwise there would be a 2-gon in Π between L and a component of $\partial\tilde{\Omega}$. Let Π'' denote the Y-plane which contains L, and consider the intersection of Π'' with $\tilde{\Omega}'$. This is a 1-manifold with exactly two boundary points, the endpoints of λ, as $\partial\tilde{\Omega}'$ equals $\partial\tilde{\Omega}$. As $\Pi'' \cap \Pi'$ is a single line, it follows that $\Pi'' \cap \tilde{\Omega}'$ is a single arc λ'. But this implies that Π'' contains a 2-gon bounded by λ and λ', which is a contradiction, as X-planes contain no 2-gons.

If F' is not embedded in M_F, we can make a very similar argument to the above, as follows. Let M_1 denote the covering of M_F such that F' lifts to M_1 by a homotopy equivalence. The lift will be an embedded surface F'_1. Let F_1 denote the pre-image of F in M_1. Then F_1 cuts M_1 into two pieces. Now the projection map of F'_1 to F' is proper. It follows that $h_1(F'_1)$ intersects at least one of the closures of the components of $M_1 - F_1$ in a compact surface. As before, it follows that there is a compact product region W in M_1 between F_1 and $h_1(F'_1)$, and we can make all the arguments in the above paragraph to deduce the required contradiction.

We have just shown that $X \cap Y$ cannot have XX triple points, i.e. that s must be zero. We will show below that this implies that no Y-plane contains a 2-gon of any type. Assuming this we can apply all the above arguments with the roles of X and Y reversed, showing that t must be zero.

Here is the proof of the above claim that if there are no XX-triple points then no Y-plane contains a 2-gon of any type. We know that no Y-plane can contain a 2-gon of type XY from the arguments at the start of the proof of Lemma 1.12. Suppose that Π is a Y-plane which contains a 2-gon. Then Π contains a 2-gon D which has no 2-gon properly inside it. This must be a 2-gon of mixed type which cannot be innermost. The proof of Lemma 1.9 applies to show that D must be as shown in Figure 1.10, i.e. the only double curves inside D are disjoint XY-arcs which join the YY-edge of D to the XY-edge. But this implies, in particular, that there are XX-triple points on the XY-edge of D. This contradiction completes the proof of the claim and hence the proof of Lemma 1.12.

§2. The main result

Using the notation of §1, we can now assume that our homeomorphism h of M is such that X and Y intersect in general position without triple points and without nullhomotopic double curves. Our aim is to isotop h further until h is the identity on X. It will then be trivial to isotop h to be the identity on a regular neighborhood N of X. Now the closure of each component of $M - N$ is a handlebody by Lemma 1.4 of [HRS], and any homeomorphism of a handlebody U which is the identity of ∂U is isotopic to the identity fixing ∂U. Thus h is isotopic to the identity as required.

At one point in our proof it will be convenient to quote results on least area surfaces. In order to do this, we pick a Riemannian metric on M and choose $f: F \to M$ to be least area in its homotopy class. Lemma 2.4 of [HS], shows that f will still have the 3-plane property and the 1-line intersection property. For the rest of this section we will assume that f is least area.

We start with a result which extends our earlier observation that F cannot be the torus.

LEMMA 2.1. *With the hypotheses of Theorem* 1.1, $\pi_1(M)$ *cannot contain a subgroup isomorphic to* $\mathbb{Z} \times \mathbb{Z}$, *nor can it contain an infinite cyclic normal subgroup.*

Proof. Suppose that $\pi_1(M)$ does contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$. As M is not Haken, the version of the Torus Theorem in [Sc2] shows that $\pi_1(M)$ has an infinite cyclic normal subgroup. We will show that this is impossible.

Recall that we have an incompressible immersion $f: F \to M$ which satisfies the 3-plane condition. In particular, f has no triple points. Thus if F is the torus, then M must be a Seifert fiber space which contradicts our hypothesis. If F is not the torus, then $\pi_1(F)$ has no infinite cyclic normal subgroup. It follows that M is finitely covered by a manifold M' whose fundamental group has an infinite cyclic normal subgroup with quotient group isomorphic to $\pi_1(F)$. In particular, M is finitely covered by a Haken manifold. Again this implies that M is a Seifert fiber space by [Sc2]. This contradiction completes the proof of Lemma 2.1.

Now we show how to isotop h to be the identity on the double curves of X. Recall that Δ denotes the union of these curves.

LEMMA 2.2. *We can isotop* h *to a homeomorphism* h' *such that* $h' \mid \Delta$ *is the identity on* Δ. *Further,* h' *can be chosen so that* h' *is homotopic to the identity rel* Δ.

i.e. h' is homotopic to the identity by a homotopy which fixes all points of Δ at all times.

Remark. Of course, X and $h'(X)$ are not in general position and the points of C are quadruple points of $X \cap h'(X)$, but we can arrange that $X - C$ and $h'(X - C)$ are in general position and without triple points or nullhomotopic double curves by choosing the isotopy to be supported in a small neighborhood of C.

Proof. Let C denote a component of Δ. We will first show that we can isotop h to h' such that $h'(C)$ equals C. Then we can isotop h to h'' which is the identity on C unless h' reverses the orientation of C. But this would imply that the element of $\pi_1(M)$ represented by C is conjugate to its inverse, which in turn would imply that $\pi_1(M)$ contains a copy of the fundamental group of the Klein bottle and hence contains a copy of $\mathbb{Z} \times \mathbb{Z}$ which is impossible by Lemma 2.1.

Consider a component l of the pre image in $\tilde{M}$ of C. Thus l is the intersection of two X-planes P_1 and P_2 in $\tilde{M}$. Let $\tilde{h}$ be a lift of h to $\tilde{M}$ which is homotopic to the identity. This choice of lift determines corresponding lifts of h for each covering of M. Then $\tilde{h}(l)$ is the intersection of $\tilde{h}(P_1)$ and $\tilde{h}(P_2)$. We will show that $\tilde{h}(P_2)$ meets P_1 in a line m which is parallel to l in P_1, i.e. l and m are disjoint and have the same stabiliser. Also m will be parallel to $\tilde{h}(l)$ in $\tilde{h}(P_2)$. Let G denote the stabiliser of P_1 and let M_G denote the quotient of $\tilde{M}$ by G. The image of P_1 in M_G is an embedded surface F_1 homeomorphic to F which separates the two ends of M_G. The image of P_2 in M_G is an annulus A_2 which runs from one end of M_G to the other and meets F_1 in a circle C_1 which projects to C in M. Let h_G denote the homeomorphism of M_G determined by our choice of $\tilde{h}$. Thus h_G is homotopic to the identity. Further, h_G moves points of M_G a uniformly bounded distance. It follows that $h_G(A_2)$ still runs from one end of M_G to the other and so must meet F_1 in a closed curve D_1 homotopic to C_1 in F_1. As C_1 and D_1 must be disjoint, we see that $\tilde{h}(P_2)$ meets P_1 in a line m which is parallel to l in P_1. Also m will be parallel to $\tilde{h}(l)$ in $\tilde{h}(P_2)$.

Let S_1 denote the infinite strip in P_1 bounded by l and m, and let S_2 denote the infinite strip in $\tilde{h}(P_2)$ bounded by m and n. Let S denote $S_1 \cup S_2$. We will use the projection of S into M to define the required isotopy of $h(C)$ to C. Note that S may contain several XX, YY or XY-lines, but all these lines must be parallel to m and so split S into sub-strips. Each of these sub-strips projects to an embedded annulus in M_G and this annulus projects into X or Y by an embedding or by a map which embeds apart from identifying the two boundary components. This defines an isotopy of $h(C)$ to C by isotoping along these annuli in turn and hence yields an isotopy of h to h' such that $h'(C)$ equals C. Note that when we lift this isotopy between h and h' to an isotopy between $\tilde{h}$ and $\tilde{h}'$, we find that $\tilde{h}'(l)$ equals l. Now we isotop $h' \mid C$ to the identity, and extend to an isotopy from h' to h''. This lifts to an isotopy from $\tilde{h}'$ to $\tilde{h}''$. We choose our isotopy of h' in such a way that $\tilde{h}'' \mid l$ is the identity. As h'' is isotopic to h which is homotopic to the identity, we have a homotopy H'' of h'' to the identity which lifts to a homotopy $\tilde{H}''$ of $\tilde{h}''$ to the identity and $\tilde{h}'' \mid l$ is the identity. We claim that h'' is homotopic to the identity rel C. For the restriction of H'' to C yields a homotopy $H_C : C \times I \to M$ whose lift to $\tilde{M}$ is a homotopy of the inclusion of l in $\tilde{M}$ to itself. It follows that we can homotop H_C rel $C \times \partial I$ to be the identity at all times, and extend to obtain a homotopy from H'' to H, where H is a homotopy from h'' to the identity which fixes C at all times. Thus h'' is homotopic to the identity rel C, as required.

Finally, we consider the situation when Δ has more than one component. Let the components be denoted $C_1, \ldots, C_r$. Suppose that we have isotoped h to h_{i-1} such that $h_{i-1} \mid C_j$ is the identity, when $j \leq i-1$, and that h_{i-1} is homotopic to the identity rel $C_1 \cup \cdots \cup C_{i-1}$. We apply the foregoing argument to C_i. If the track of the isotopy we find does not meet $C_1 \cup \cdots \cup C_{i-1}$, we can isotop h_{i-1} rel $C_1 \cup \cdots \cup C_{i-1}$ to h_i such that $h_i \mid C_i$ is the identity. The argument that h_i is homotopic to the identity rel C_i also shows that h_i is homotopic to the identity rel $C_1 \cup \cdots \cup C_i$. If the track of the isotopy does meet some C_j, $j \leq i-1$, then the track must contain C_j. In this case we simply perturb the track slightly so as to avoid C_j. Now we can construct h_i as required.

The next step is to isotop h so as to be the identity on X. This turns out to be a very delicate problem. The point is that although we have just shown that we can isotop h to be the identity on Δ, there are non-isotopic ways to do this. i.e. one can isotop h to h_1 and to h_2 each the identity on Δ and each homotopic to the identity rel Δ, but h_1 and h_2 need not be isotopic rel Δ. A simple example of this phenomenon can be described as follows. Suppose that M contains a fibered solid torus V which contains two double curves of X embedded in V as fibers. Let h_1 be the identity on M and let h_2 be obtained from h_1 by an isotopy which rotates V through a full turn about the center fiber of V. Then h_2 is homotopic to the identity rel Δ but is not isotopic to the identity rel Δ. Obviously much more complicated examples are possible. It follows from the above that although we can isotop h to be the identity on Δ and arrange that h is homotopic to the identity rel Δ, we may need to alter h on Δ in order to isotop h to the identity on X.

Another technical problem is that when we have isotoped h so as to be the identity on the image in M of a subsurface S of F, we will need to know that h is homotopic to the identity rel $f(S) \cup \Delta$ in order to carry out further isotopies. The following result shows that once we can isotop h to be the identity on a "non-trivial" subsurface of X then this is automatic.

LEMMA 2.3. *Let S be a compact incompressible subsurface of F. Suppose that the restriction of h to $f(S)$ is the identity. If no component of S is an annulus, then h is homotopic to the identity rel $f(S)$.*

Remarks. This result may fail if S is an annulus which f embeds in M.

The methods of Lemma 2.2 show that we can isotop h rel $f(S)$ to be the identity on $f(S) \cup \Delta$ in such a way that h is homotopic to the identity rel $f(S) \cup \Delta$.

Proof. First we consider the case when S is connected. Recall that h is homotopic to the identity. The homotopy of h to the identity yields a map $H : S^1 \times f(S) \to M$, such that $H \mid \{1\} \times f(S)$ is the identity. Let α denote a generator of $\pi_1(S^1)$ and let β denote $H_*(\alpha)$ in $\pi_1(M)$. If β is trivial then H is homotopic rel $\{1\} \times f(S)$ to the identity homotopy. This shows that h is homotopic to the identity rel $f(S)$. If β is non trivial, then $H_*(\pi_1(S^1 \times f(S)))$ is a subgroup of $\pi_1(M)$ with non trivial center. This subgroup cannot be of finite index, as this would imply that $\pi_1(M)$ itself has an infinite cyclic normal subgroup, by Lemma 4.2 of [Sc2], which is impossible by Lemma 2.1. Thus $H_*(\pi_1(S^1 \times f(S)))$ is of infinite index in $\pi_1(M)$ and hence is the fundamental group of a non-compact covering space of M. This covering will have a compact irreducible submanifold Σ with the same fundamental group [Sw]. Σ must be a Seifert fiber space [Wa2], as $\pi_1(\Sigma)$ has non-trivial center and Σ is orientable irreducible and with non-empty boundary. As $\pi_1(M)$ cannot contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$, it follows that $H_*(\pi_1(S^1 \times f(S)))$ is infinite cyclic. Thus $\pi_1(f(S))$ must also be infinite cyclic, so that S must be an annulus. It follows that if S is not an annulus, then h is homotopic to the identity rel $f(S)$.

If S is disconnected, we apply the above argument to each component of S in order to complete the proof of Lemma 2.3.

Now we come to the main part of the work in this section. We choose the closure R of a component of $F - f^{-1}(\Delta)$ such that R is not an annulus. Such a component exists as F is not a torus. Recall that M_F denotes the covering of M with $\pi_1(M_F) = f_*(\pi_1(F))$ and that f lifts to an embedding of F into M_F whose image we denote by F. Let h_F denote the lift of h to M_F which is the identity on Δ_F, the pre-image in M_F of Δ in M. We will consider how $h_F(R)$ meets F. Certainly $h_F(\partial R)$ lies in F as ∂R lies in Δ_F and h_F fixes Δ_F. Now $h_F(R) \cap F$ divides R into subsurfaces. As R is not an annulus, the closure S of some component of $h_F(R) - h_F(R) \cap F$ is also not an annulus. As S lies in the closure of one component of $M_F - F$, it follows that there is a subsurface T of F with ∂T equal to ∂S and that there is a product region W in M_F between S and T. The idea is to isotop S across W to T. If W projects injectively into M, it is clear that this can be achieved by an isotopy of h. But although S projects injectively into M, this need not be true of W nor even of T. However, we will eventually isotop h to be the identity on the image of a subsurface S' of S which consists of S with a collar neighborhood of ∂S removed.

We need to start by analysing how the surfaces of X_F and Y_F meet W. Note that if no surface of X_F or Y_F meets the interior of W then T and W must project injectively into M. Thus we can isotop h so that the induced isotopy of h_F carries S across W to T. Now h is homotopic to the identity. It follows that we can further isotop h to be the identity on $f(S)$, and that this further isotopy keeps $h(f(S))$ in $f(F)$.

LEMMA 2.4. *If W contains a component C of Δ_F, then C lies in some regular neighborhood of ∂S in W.*

Proof. First we suppose that C lies in ∂W. If C lies in S, it must be a component of ∂S, so that the result is trivial. Suppose that C lies in the interior of T. Every loop in T is homotopic in M_F into S and hence into R, as S lies in $h_F(R)$ and h_F is homotopic to the identity. As $\pi_1(M_F) = f_*(\pi_1(F))$, every loop in T is homotopic in F into R. As each component of ∂T is disjoint from ∂R or coincides with a component of ∂R, it follows that T consists of a subsurface of R and of annuli in F which meet R in one boundary component. Hence C is a boundary component of T or lies in one of these annuli and so must be parallel in T to a component of ∂T, so that again the result of Lemma 2.4 is clear.

Now we suppose that C lies in the interior of W. The point here is that h_F is a homeomorphism of M_F which fixes C and so C must lie on the same side of F and of $h_F(F)$. It follows that there must be a second component S' of $h_F(F) - h_F(F) \cap F$, such that the product region W' between S' and some subsurface T' of F contains C. As S' and S are disjoint, it follows that W contains W' or vice versa. We will suppose first that W' is contained in W. Then S' is homotopic in W into S, as W is a product region. Hence S' is homotopic in $h_F(F)$ into S. As S' and S are disjoint subsurfaces of $h_F(F)$, is follows that S' must be an annulus and that S' is homotopic into ∂S. Thus C lies in a regular neighborhood of ∂S as required. If W were contained in W', the above argument would show that S is an annulus, contradicting our choice of S. This completes the proof of Lemma 2.4.

Now we analyse how X_F and Y_F meet W. Of course, S is contained in Y_F and T is contained in X_F. Each sheet of X_F is really an immersed surface in M_F. The pre-image of ∂W in this surface divides it into subsurfaces which are immersed into W or into $M_F - W$. The subsurfaces which are mapped into W will be referred to as the components of $X_F \cap W$. Note that this is an abuse of standard terminology as components in this sense may meet in W.

LEMMA 2.5. *Each component of $X_F \cap W$ and of $Y_F \cap W$ is an embedded surface. Each component of $X_F \cap W$ other than T is either an annulus with one end on S and the other on T which lies in a regular neighborhood of ∂S in W or it is a surface with all its boundary in S. Each component of $Y_F \cap W$ other than S is a surface with its boundary in T.*

Proof. Let Σ be a component of $X_F \cap W$ other than T. Then Σ is a surface immersed in W whose boundary embeds in ∂W. If $\partial\Sigma$ meets T it must do so in a circle parallel in T into ∂T, by Lemma 2.4. But any component of X_F which meets F must be an embedded annulus meeting F in a single circle, because our immersion of F into M has the 3-plane property and the 1-line intersection property. Thus any component of $X_F \cap W$ which meets T is an embedded annulus with one end in S and the other on T, and any such annulus lies in a regular neighborhood of ∂S in W.

Now suppose that Σ does not meet T. As W deformation retracts to S, we can homotop Σ rel $\partial\Sigma$ into S. Thus Σ can be homotoped to have no crossings. This is the point at which we use the fact that we chose $f: F \to M$ to be least area in its homotopy class. Theorem 1 of [H] shows that Σ is least area in its homotopy class rel boundary in M_F and hence in W. Now we apply Theorem 6.3 of [FHS] modified as discussed in §7 of that paper so as to apply to surfaces with boundary. This shows that Σ must be embedded as it is properly homotopic rel $\partial\Sigma$ to a surface without crossings and it cannot factor through a proper covering as $\partial\Sigma$ is embedded in ∂W.

Finally let Σ be a component of $Y_F \cap W$ other than S. Then Σ cannot meet S so that $\partial\Sigma$ lies in T. As W deformation retracts to T, we can homotop Σ rel $\partial\Sigma$ into T. Again it follows that Σ must be embedded. This time Σ is not least area, but $h_F^{-1}(\Sigma)$ is least area and the above arguments show that it must be embedded. This completes the proof of Lemma 2.5.

LEMMA 2.6. *There is a subsurface S' of S consisting of S with a collar of ∂S removed, and an isotopy of h to h' rel Δ such that $h'_F(S')$ lies in T. There is a further isotopy of h' to h'' such that $h'' \mid S'$ is the identity.*

Proof. Suppose first that there is a regular neighborhood of ∂S in W which contains all components of $X_F \cap W$ and $Y_F \cap W$ other than S and T. Then $X_F \cap W$ cuts S into annuli parallel to ∂S and a subsurface which we denote by S''. Let W'' denote the component of $W - X_F \cap W$ which contains S''. Note that $W'' \cap S$ may not equal S''. If W'' contains any components of ∂S, we let W' denote W'' with a small regular neighborhood N of these components removed, and let T' denote $W' \cap T$ and let S' denote $W' \cap S''$. Otherwise, W' equals W''.

The surfaces S' and T' both project injectively into M. Further S' embeds in a component V of $M - X$ which contains T' in its boundary. Recall that the boundary components of S' in W' are either in X_F or in ∂N. It follows that the boundary components of S' in V are either in ∂V or in the image of ∂N. The second type of component can be isotoped in V to lie in ∂V. Thus we can suppose that S' is properly embedded in V. Now S' is homotopic in W' into T' so that S' is homotopic in V into T'. The Annulus Theorem provides an embedded annulus in V between each component of $\partial S'$ and corresponding components of $\partial T'$. This allows us to further isotop S' in V until $\partial S'$ lies in T'. Now there must be a product region in V between S' and T', so that we can isotop S' to coincide with T'.

This completes the proof of Lemma 2.6, under the assumption that there is a regular neighborhood of ∂S in W which contains all components of $X_F \cap W$ and $Y_F \cap W$ other than S and T. If this is not the case, we will need to make similar arguments for product regions inside W. We apply the above arguments to remove all of such a product region except possibly for subregions which lie in a regular neighborhood of ∂S in W.

Lemmas 2.4, 2.5 and 2.6 tell us that there is an incompressible subsurface S_1 of the interior of a component R of $F - f^{-1}(\Delta)$ such that S_1 is not an annulus and we can isotop h rel Δ to be the identity on S_1. Lemma 2.3 tells us that the new h is homotopic to the identity rel $f(S_1) \cup \Delta$. If the closure of $R - S_1$ has a component S which is not an annulus, we want to repeat the above argument.

LEMMA 2.7. *If the closure of $R - S_1$ has a component which is not an annulus, there is a subsurface S_2 of the interior of $S - S_1$ such that S_2 is not an annulus and we can isotop h rel $f(S_1) \cup \Delta$ to be the identity on $f(S_1 \cup S_2) \cup \Delta$.*

Proof. As $R - S_1$ has a component which is not an annulus, there is a component of $h_F(R) - h_F(R) \cap F$ whose closure S is not an annulus. The arguments of Lemmas 2.4, 2.5 and 2.6 apply to show that we can isotop h to be the identity on $f(S_2)$, where S_2 is some surface in the interior of S which is homeomorphic to S. We need to check that h remains fixed on S_1 during this isotopy. For this, it suffices to show that the interior of S_1 cannot meet the product region W in M_F between S and the subsurface T of F. If the interior of S_1 meets W, then it must be contained in T and hence is homotopic in M_F into S. This implies that $h(S_1)$ is homotopic in $h_F(F)$ into S. As $h(S_1)$ and S are disjoint subsurfaces of $h_F(F)$, it follows that S_1 is an annulus. This contradiction completes the proof of Lemma 2.7.

By repeatedly applying Lemma 2.7, we will obtain the following corollary.

COROLLARY 2.8. *There are disjoint subsurfaces $S_1, \ldots, S_k$ of R such that $R - \bigcup S_i$ consists only of annuli and we can isotop h so that $h \mid \Delta \cup (\bigcup S_i)$ is the identity and that h is homotopic to the identity rel $\Delta \cup (\bigcup S_i)$.*

Now we can prove the following:

LEMMA 2.9. *We can isotop* h to be the identity on $f(R)$.

Remark. Note that f may not embed R in M. Also note that our isotopy of h need not fix Δ.

Proof. First we consider the annuli of $R - \bigcup S_i$ which do not meet ∂R. Let A be such an annulus. We will consider the situation in M rather than in the cover M_F. The intersection $h_F(A) \cap A$ cuts A and $h_F(A)$ into sub-annuli. We can find innermost sub-annuli A' of A and A'' of $h_F(A)$ with the same boundary. Their union is an embedded torus in M. As M is not Haken, and A carries a non-trivial element of $\pi_1(M)$, this torus must bound a solid torus W in M. If the inclusion of A' in W induces an isomorphism of fundamental groups, we can isotop $h(A')$ across W so as to remove two components from $h_F(A) \cap A$, or, if this intersection equals ∂A, we can isotop $h(A)$ across W so as to fix A. This isotopy can be done keeping h fixed on $\bigcup S_i$. For S_i cannot meet W, as S_i cannot meet A or $h(A)$. But this isotopy may not fix Δ as W may contain a component of Δ. Conceivably, the inclusion of A' in W may not induce an isomorphism of fundamental groups, but this problem can occur for only one of the solid tori which can be obtained this way. For the union of two such solid tori and a neighborhood of a sub-annulus of A joining them would be a Seifert fiber space in M which was not a solid torus as it would have two singular fibers. We can find at least two such solid tori unless $h_F(A) \cap A$ equals ∂A. Thus we can always isotop $h(A)$ until we have $h_F(A) \cap A$ equal to ∂A. Now we consider the lift of A into M_F. As h is homotopic to the identity rel $\bigcup S_i$, we have $h_F(A) \cap A$ equals ∂A. It follows that the solid torus bounded by $A \cup h_F(A)$ in M_F projects injectively into M and hence that we can isotop h to be the identity on A, as required.

We repeat the above argument until h is the identity on $\bigcup S_i$ and on all components of $R - \bigcup S_i$ which do not meet ∂R. Denote this union by R_1.

If f embeds R in M, it is now trivial to isotop h to be the identity on R while keeping h fixed on R_1. We can do this by contracting the collars of ∂R along hR and expanding along R. Of course, such an isotopy does not fix Δ.

If f fails to embed R, let C and D be components of ∂R identified by f and let A and B denote the components of $R - R_1$ which contain C and D respectively. As before it is trivial to isotop h to be the identity on B, and we now need to show that we can isotop h rel $R_1 \cup B$ to be the identity on A. First we assume that, by repeating the above argument, we have arranged that h is the identity on $f(\partial R)$. Now we argue as in the case of an annulus in the interior of R. The new problem is that it is conceivable that a product region W in M between a sub-annulus of A

and of hA contains a component E of $f(\partial R)$. We will show that this cannot happen. If W contains E, then there are two annuli of $X \cap W$ which cross at E and so are cut by E into four half annuli. One of these half annuli is fixed by h because of the way that we chose h to fix $f(\partial R)$. But such an annulus cannot cross A or hA, yielding the required contradiction. This completes the proof of Lemma 2.9.

Finally we are in a position to complete the arguments of this section.

THEOREM 2.10. *There is an isotopy of h to be the identity on $X = f(F)$.*

Proof. Lemma 2.9 tells us that we can isotop h to be the identity on $f(R)$, where R is the closure of a component of $F - f^{-1}(\Delta)$ which is not an annulus. Now we proceed to consider those components which are adjacent to R. If we have an annulus adjacent to R which f embeds in M it is trivial to isotop h to be the identity on the union of R and the annulus. If we have a component R' which is not an annulus and is adjacent to R, we argue as for R. Lemmas 2.4–2.9 tell us how to isotop h to fix R'. We need to check that this isotopy can be done keeping h fixed on R. This is not obvious as our isotopy of R obtained from Lemmas 2.4–2.9 did not need to fix ∂R. The problem is that we might find a product region W between a subsurface of R' and of $h(R')$ which contains a component of ∂R in its interior. But the argument at the end of Lemma 2.9 shows that this cannot happen.

Suppose there is a component A of $F - f^{-1}(\Delta)$ which is an annulus adjacent to R, that A is not embedded by f, and that only one component of ∂A lies in ∂R. Then the other component of ∂A lies in a component R' which cannot be an annulus unless f embeds R'. (This is by the argument in the first paragraph of the proof of Lemma 2.9.) Thus we can isotop h to fix R' whether R' is an annulus or not. By repeating all the above arguments, we can arrange that h fixes all of $f(F)$ except those annuli of $F - f^{-1}(\Delta)$ which are not embedded by f. Let $A_1, \ldots, A_k$ be the annuli in F not fixed by h. Then $f(A_i)$ is an embedded torus T_i in M. Let C_i denote $f(\partial A_i)$. As M is not Haken and T_i carries a non trivial element of $\pi_1(M)$, T_i must bound a solid torus W_i. We consider the double curves $h_F T_i \cap (\bigcup T_j)$. By arguing as before, particularly as in the proof of Lemma 2.9, we can isotop h rel $X - \bigcup T_i$, so as to arrange that $h_F T_i \cap (\bigcup T_j)$ is empty when i and j are distinct and that $h_F T_i \cap T_i$ consists only of the circle C_i. It follows that, for each i and j, hW_i and W_j are disjoint or one is contained in the other. Now it follows that, for distinct i and j, hW_i and W_j are disjoint, and that either hW_i is contained in W_i or vice versa. Hence we can isotop h rel $X - \bigcup T_i$ to be the identity on each T_i. Thus we have isotoped h to fix X as required.

§3. Homeomorphisms fixing free links

Let M be a closed orientable irreducible non-Haken 3-manifold with infinite fundamental group. Let h be a homeomorphism of M which is homotopic to the identity and equals the identity on a link Δ in M. We would like to be able to deduce that h is isotopic to the identity. If M is a Seifert fiber space it is known that h is isotopic to the identity with no further assumptions [Sc1] [B–O], but it is unreasonable to expect to be able to do this in general, as one can always isotop h to fix a 3-ball in M and hence fix any circle in the 3-ball. So we restrict our attention to links which have irreducible complement in M. We call such a link essential. Note that if each component represents a non-trivial element of $\pi_1(M)$, it is automatically essential. We will also need one further condition on our link. Given a link L in a manifold M, let $\tilde{M}$ denote the universal cover of M and let $\tilde{L}$ denote the full pre-image of L in $\tilde{M}$. We say that L is *free* in M if the fundamental group of $\tilde{M} - \tilde{L}$ is free.

Now we show that the link Δ of §2 is free.

LEMMA 3.1. *Let $f : F \to M$ be an immersion with the 3-plane property and with double curve set Δ. Then Δ is free in M.*

Proof. In the universal cover $\tilde{M}$ of M the pre-image of $f(F)$ consists of embedded planes which intersect each other in the lines of $\tilde{\Delta}$. In [HRS], it is shown that the 3-dimensional regions into which these planes split $\tilde{M}$ are all simply connected, and that the 2-dimensional regions into which the planes are cut by $\tilde{\Delta}$ are also simply connected. Now Van Kampen's Theorem tells us that the fundamental group of $\tilde{M} - \tilde{\Delta}$ is free as required.

LEMMA 3.2. *Let M be a 3-manifold containing a free link L. Let $g : T \to M - L$ be a map of the 2-torus which injects $\pi_1(T)$. Then the image of $\pi_1(T)$ in $\pi_1(M)$ cannot be trivial.*

Proof. If the image of $\pi_1(T)$ in $\pi_1(M)$ were trivial, then g would lift to a map $\tilde{g} : \tilde{T} \to \tilde{M} - \tilde{L}$ which would inject $\pi_1(T)$. But this is impossible as $\pi_1(\tilde{M} - \tilde{L})$ is free.

Now we prove the main result of this section.

THEOREM 3.3. *Let M be a closed orientable irreducible non-Haken 3-manifold with infinite fundamental group, and let L be an essential free link in M. Let h be a homeomorphism of M which is homotopic to the identity and equals the identity on L. Then h is isotopic to the identity.*

Proof. Let N denote the closure of M minus a regular neighborhood W of L. As M is irreducible and L is essential, it follows that N is irreducible. Thus N is Haken. If ∂N were compressible in N, then N would be a solid torus. Thus M would be the union of N and the solid torus W. But this would contradict our assumption that M is irreducible and has infinite fundamental group as M would have to be a lens space or $S^1 \times S^2$. Thus we can assume that ∂N is incompressible in N. Now we consider the characteristic torus decomposition of N.

Suppose first that the characteristic submanifold of N is empty. Thurston's work [Th] implies that N admits a complete hyperbolic structure. Isotop h to be the identity on W and let h_1 denote the restriction of h to N. Mostow's Rigidity Theorem [Mo] implies that h_1 is homotopic to an isometry g_1 of N. As N is Haken and is not homeomorphic to $T \times I$, Waldhausen's result in [Wa1] implies that h_1 is isotopic to g_1. Now any isometry of N is periodic, so it follows that we can isotop h to a periodic homeomorphism g of M. A theorem of Conner and Raymond [C–R] asserts that if M is an orientable aspherical closed manifold with centerless fundamental group and if g is a periodic homeomorphism of M which is homotopic to the identity then g is the identity. We deduce in our case that if $\pi_1(M)$ is centerless, then h is isotopic to the identity as required. Otherwise, $\pi_1(M)$ has non-trivial center and so M is a Seifert fiber space by the recent work of Casson and Jungreis [C–J] and Gabai [G]. At this point, we could quote the general result that homotopic homeomorphisms of a Seifert fiber space are isotopic, or we can use the results of Meeks and Scott in [M–S] which tell us that a periodic homeomorphism of a Seifert fiber space with infinite fundamental group preserves some Seifert fibration. As h is homotopic to the identity, if follows that h embeds in a circle action on M and in particular is isotopic to the identity.

Next suppose that N is a Seifert fiber space. Then Lemma 3 of [E–M] tells us that, as M is irreducible, M must be a Seifert fiber space with a Seifert fibration extending a fibration of N. Again we could quote the general result that homotopic homeomorphisms of a Seifert fiber space are isotopic, or instead we can use the easier fact that any homeomorphism of N is isotopic to a fiber preserving one. Thus h is isotopic to a fiber preserving homeomorphism of M and now it is easy to show that h is isotopic to the identity.

Finally suppose that the characteristic submanifold of N is neither empty nor equal to N. Let Σ denote the frontier of the characteristic submanifold of N and let T be a component of Σ. As M is non-Haken, T must be compressible in M. Hence either T lies in a ball in M or T bounds a solid torus. But Lemma 3.2 tells us that the image of $\pi_1(T)$ in $\pi_1(M)$ cannot be trivial. We deduce that T bounds a solid torus V in M. Let T and T' be components of Σ which bound solid tori V and V' respectively in M, and suppose that V and V' are not disjoint. Then V contains V' or V' contains V or their union equals M. In the last case, we use the fact that there must be a closed curve C on ∂V which is essential in M as the image of $\pi_1(T)$ in $\pi_1(M)$ is non-trivial, by Lemma 3.2. Note that C must be essential in V and in V'. We consider the cover M_C of M whose fundamental group is the cyclic group carried by C. Thus C lifts to M_C, and we denote the lift by C also. The components of the pre-images in M_C of V and V' which contain C must be finite covers V_C and V'_C of V and V'. Again one of these solid tori is contained in the other or their union equals M_C. The first two cases are impossible as they would imply that V is contained in V' or vice versa. Thus M_C is the union of two solid tori and, in particular, is closed. But our hypotheses on M imply that M is aspherical and hence that M_C is an aspherical closed 3-manifold with infinite cyclic fundamental group which is impossible. We deduce that V must be contained in V' or vice versa. Now we let $\bar{V}$ denote the union of all the solid tori in M bounded by components of Σ. The preceding arguments imply that $\bar{V}$ is a disjoint union of solid tori. We consider the decomposition of M into $\bar{V}$ and its complement $\bar{N}$. We can isotop h so as to preserve the characteristic submanifold of N and hence preserve $\bar{V}$ and $\bar{N}$. By construction, $\bar{N}$ is a Seifert fiber space or is hyperbolic. Now the preceding arguments can be applied to show that our homeomorphism of M can be isotoped to the identity. This completes the proof of Theorem 3.3.

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