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A maximum principle for biharmonic functions in Lipschitz and C^1 domains

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§1. Introduction

Our aim in this paper is to prove a maximum principle for functions biharmonic in a domain D whose boundary is Lipschitz or C^1. This result will be valid for Lipschitz domains $D \subset \mathbb{R}^n$, $n = 2$ or 3, but fails if $D \subset \mathbb{R}^n$, $n \geq 4$. The dimension dependent aspect of this theorem is a new phenomenon in the theory of elliptic partial differential equations and stands in sharp contrast to well-known results both for solutions to second order equations and for solutions of higher order equations in domains whose boundary is smooth. We then extend the method employed for three dimensional Lipschitz domains to show that the maximum principle holds for functions biharmonic in a C^1 domain $D \subset \mathbb{R}^n$, for any n. We begin with some background and the explicit statements of the results.

A function is *biharmonic* in D if it satisfies $\Delta^2 u = 0$ in D. A bounded domain D is Lipschitz if there exist finitely many doubly truncated, right circular cylinders Z_i (with ρZ denoting a dilation by ρ of Z) such that (i) $\partial D = \bigcup_i (\frac{1}{2} Z_i \cap \partial D)$, (ii) there is a change of coordinates and a Lipschitz function $\varphi_i : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $2Z_i \cap \partial D = \{(x, y) \in \partial D : y = \varphi_i(x)\}$, and (iii) $2Z_i \cap D$ is starlike with respect to some $P_i \in Z_i \cap D$.

Let $d\sigma$ denote the surface measure of ∂D. A function f defined on ∂D is said to belong to $L_1^p(\partial D)$ if for every cylinder Z with associated Lipschitz function φ there are $L^p(Z \cap \partial D, d\sigma)$ functions $g_1, \ldots, g_{n-1}$ such that

$$\int_{\mathbb{R}^{n-1}} h(x) g_j(x, \varphi(x))\, dx = -\int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_j} h(x) f(x, \varphi(x))\, dx$$

whenever $h \in C_0^\infty(R^{n-1} \cap Z)$. Thus to every function $f \in L_1^p(\partial D)$ is associated a unique vector $\nabla_T f$, called the tangential derivative of f, which in local coordinates

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may be realized as

$$(g_1(x, \varphi(x)), \ldots, g_{n-1}(x, \varphi(x)), 0) - \langle (g_1(x, \varphi(x)), \ldots, g_{n-1}(x, \varphi(x)), 0), N(x, \varphi(x)) \rangle N(x, \varphi(x))$$

where $N(x, \varphi(x)) \equiv N(Q)$ is the unit exterior normal to $Q = (x, \varphi(x))$ on ∂D.

If D is Lipschitz then to every point $Q \in \partial D$ there is a nontangential "cone" $\Gamma(Q) = \{X \in D : \text{dist}(X, \partial D) \leq (1+\alpha)|X - Q|\}$ for $\alpha > 0$ contained in D (see [J–K]). If v is a function in D, we may define $Nv(Q)$, the nontangential maximal function of v at $Q \in \partial D$, by $Nv(Q) = \sup_{X \in \Gamma(Q)} (v(X))$. The Dirichlet problem for the operator Δ^2 may be formulated as follows. Given $f \in L^p_1(\partial D)$ and $g \in L^p(\partial D)$, we seek a unique function u satisfying

$$\begin{aligned} &\text{(i)} \quad \Delta^2 u = 0 \quad \text{in } D \\ &\text{(ii)} \quad \lim_{X \to Q, X \in \Gamma(Q)} u(X) = f(Q) \quad \text{a.e. } d\sigma(Q) \\ &\text{(iii)} \quad \lim_{X \to Q, X \in \Gamma(Q)} \langle \nabla u(X), N(Q) \rangle = g(Q) \quad \text{a.e. } d\sigma(Q) \\ &\text{(iv)} \quad \|N(\nabla u)\|_{L^p(d\sigma)} < +\infty \end{aligned} \tag{1.1}$$

such that, for C depending only on the dimension and the Lipschitz character of D,

$$\|N(\nabla u)\|_{L^p(d\sigma)} \leq C\{\|f\|_{L^p_1(d\sigma)} + \|g\|_{L^p(d\sigma)}\}.$$

In Dahlberg–Kenig–Verchota [D–K–V], the Dirichlet problem with data in (L^2_1, L^2) was solved in every Lipschitz domain $D \in \mathbb{R}^n$, for any n, and in analogy with the corresponding theory for the Laplacian, was shown to fail to be solvable with L^p data for $p < 2$. (As usual in this theory, once the L^2 result is known, a real variable argument gives an automatic improvement. Namely, there exists an $\varepsilon = \varepsilon(D)$ such that the Dirichlet problem, together with the appropriate L^p estimates on $N(\nabla u)$, is solvable with data in (L^p_1, L^p) for $2 - \varepsilon < p < 2 + \varepsilon$.) These results were extended in [V2] to show that the Dirichlet problem with data in (L^p_1, L^p), for all $1 < p < \infty$ is solvable if the domain $D \subset \mathbb{R}^n$ (any n) has C^1 boundary. For Lipschitz domains D, the solvability of the Dirichlet problem in (L^p_1, L^p), $p > 2$, was shown in Pipher–Verchota [PV1] to hold for $D \subset \mathbb{R}^n$, $n = 2$ or 3, and to fail for $p = p(n)$ sufficiently large where $D \subset \mathbb{R}^n$, $n \geq 4$. This failure of solvability for large p is in fact a failure of the following maximum principle and the example will be recalled in §2. Our (weak) version of the maximum principle is just the case $p = \infty$ of the Dirichlet problem:

(M.P.) If $|\nabla u| \in L^\infty(\partial D, d\sigma)$ and $\|N(\nabla u)\|_{L^2(d\sigma)} < \infty$, then the (unique) solution to the Dirichlet problem with data in (L_1^2, L^2) satisfies $\|\nabla u\|_{L^\infty(D)} \leq C\|\nabla u\|_{L^\infty(\partial D)}$ for some C depending only on the Lipschitz character of D.

A more classical version of the maximum principle may be phrased as follows.

(M.P.)* If $\Delta^2 u = 0$ in D, a bounded Lipschitz domain in $\mathbb{R}^n$, and $|\nabla u|$ is continuous in $\bar{D}$, then there is a constant C that depends only on the Lipschitz character of D such that

$$\sup_{X \in D} |\nabla u(X)| \leq C \max_{Q \in \partial D} |\nabla u(Q)|.$$

A consequence of standard elliptic theory ([A], [A–D–N]) and specialization to the biLaplacian of results on higher order operators due to Agmon and Miranda is that the maximum principle (M.P.) holds for C^4 domains in $\mathbb{R}^n$, for all n. Moreover, Miranda [M] has proven related maximum principle results (with sharp constants) relating $\max_D u$ to $\max_{\partial D} |\nabla u|$ for very general domains in $\mathbb{R}^2$, including Lipschitz domains. Our results show that Lipschitz domains are the sharp class of domains for which (M.P.) may fail.

THEOREM 1.2. *If D is a Lipschitz domain in $\mathbb{R}^n$, then* (M.P.) *is valid. Moreover, there exist Lipschitz domains in $\mathbb{R}^n$, $n \geq 4$, for which* (M.P.) *fails.*

THEOREM 1.3. *If D is a C^1-domain in $\mathbb{R}^n$, then* (M.P.) *is valid.*

In addition we obtain solutions to the Dirichlet problem (for $D \subset \mathbb{R}^3$) with data in *BMO* and appropriate Carleson measure estimates on the solution in virtue of the validity of (M.P.) and estimates on the Green's function. These estimates also give Hölder continuity of $|\nabla u|$ in dimension $n = 3$ and of the solution u itself in dimension $n = 4$ and $n = 5$. These will be discussed in §4, and indeed we shall rely on such estimates to prove (M.P.) for general Lipschitz domains, once it has been verified for special (starlike) Lipschitz domains.

§2. The maximum principle on starlike domains

The failure of the maximum principle on Lipschitz domains $D \subset \mathbb{R}^n$, $n \geq 4$, follows from the existence of a biharmonic function $u(x)$ in the exterior of a cone

$\Gamma(0)$, with vertex at the origin satisfying

(a) $u(x) = |x|^{\alpha}\varphi(x/|x|)$ where φ is a smooth function on the sphere,
(b) $u \equiv 0$ and $\partial u/\partial N \equiv 0$ on the lateral sides of the cone $\Gamma(0)$,
(c) If the aperture of the cone is small enough (i.e., the Lipschitz constant is large) then α may be chosen to be less than 1.

The existence of such a u was established in [MPN] in dimensions $n \geq 5$ and shown in [PV1] in dimension $n = 4$ where the limiting value $\alpha = 1/2$ was computed.

A bounded domain D may be constructed so that ∂D is smooth except at the origin, and if $B(0, r)$ denotes the ball of radius r centered at the origin, we may require that

$$B(0, \tfrac{1}{2})\backslash\Gamma(0) \subset D \subset B(0, 1)\backslash\Gamma(0), \qquad \text{with } \partial D \cap \partial\Gamma(0) = \partial\Gamma(0) \cap B(0, \tfrac{1}{2}).$$

By the smoothness of ∂D away from the origin and interior estimates, the solution u given above will satisfy $|\nabla u| \in L^{\infty}(\partial D)$, yet near the origin, $|\nabla u(x)| = O(|x|^{\alpha - 1})$ which fails to be bounded when $\alpha < 1$.

Consider now a bounded starlike domain Lipschitz domain $D \subset \mathbb{R}^3$. Then the fundamental solution of the biLaplacian is $\Gamma(X, Y) = |X - Y|$. The Green's function $G(X, Y)$ for Δ^2 satisfies $\Delta_Y^2 G(X, Y) = \delta(X - Y)$ and both $G(X, \cdot)|_{\partial D}$ and $(\partial G/\partial N_Q)(X, \cdot)$ vanish on ∂D. An integration by parts gives the following representation of a biharmonic function u in D for which $N(\nabla u) \in L^2(d\sigma)$:

$$u(X) = \int_{\partial D} u(Q) \frac{\partial}{\partial N_Q} \Delta_Q G(X, Q)\, d\sigma(Q) + \int_{\partial D} \frac{\partial u}{\partial N_Q} \Delta_Q G(X, Q)\, d\sigma(Q). \tag{2.1}$$

To establish Theorem (1.2) for starlike domains, we first recall the regularity problem for Δ^2 and the known results on its solvability. A Whitney array ([V3]) in $WA^p(\partial D)$ is a sequence of functions $\langle f_0, f_1, f_2, f_3\rangle = \dot{f}$ with $f_j \in L_1^p(\partial D)$ such that $\dot{f}$ belongs to the completion, in $L_1^p(\partial D)$ norm, of the sequence space $\{\langle F|_{\partial D}, D_1 F|_{\partial D}, D_2 F|_{\partial D}, D_3 F|_{\partial D}\rangle : F \in C_0^{\infty}(\mathbb{R}^n)\}$. In a starlike bounded Lipschitz domain D, this is equivalent to the statement that a compatibility condition holds:

Let $\partial D = \{\rho(\theta)\theta : \theta \in S^{n-1}\}$ for $\rho : S^{n-1} \to \mathbb{R}_+$ and assume $\operatorname{dist}(\partial D, 0) = 1$. Given $f|_{\partial D}$ and a boundary point $Q = \rho(X)X/|X|, \tilde{f}$ denotes the homogeneous degree zero extension of f defined by $\tilde{f}(X) = f(\rho(X)X/|X|)$. The vectors $e_j, j = 1, 2,$ or 3 denote the standard basis for $\mathbb{R}^3$. Then $\dot{f} = \{f_\alpha \in L_1^p(\partial D) : |\alpha| \leq 1\}$ belongs to

$WA^p(\partial D)$ iff

$$D_j\tilde{f}_{(0,0,0)}(Q) = f_{e_j}(Q) - \frac{N_Q^j}{\langle Q, N_Q \rangle} \sum_{k=1}^{3} Q_k f_{e_k}(Q),$$

$1 \leq j \leq 3$ holds a.e. on ∂D.

The regularity problem for D is, roughly speaking, the assertion that a biharmonic function whose second "tangential" derivatives on ∂D are in $L^p(\partial D)$ possesses an L^p estimate on $N(\nabla\nabla u)$. In particular, the next theorem was shown for $p \in (2-\varepsilon, 2+\varepsilon)$ in Verchota [V3] and for $1 < p < 2 - \varepsilon$ in $\mathbb{R}^3$ in Pipher–Verchota [P–V1].

THEOREM (9.3 of [P–V1]). *Let $D \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary. Then, given $\dot{f} \in WA^p(\partial D)$ there exists a unique function u in D such that*

$$\begin{aligned}
&\text{(i)} \quad \Delta^2 u = 0 \quad \text{in } D \\
&\text{(ii)} \quad \lim_{X \to Q, X \in \Gamma(Q)} u(x) = f_0(Q) \quad \text{a.e.} \\
&\text{(iii)} \quad \lim_{X \to Q, X \in \Gamma(Q)} \nabla u(X) = \langle f_1, f_2, f_3 \rangle(Q) \quad \text{a.e.} \\
&\text{(iv)} \quad \|N(\nabla\nabla u)\|_{L^p} < \infty
\end{aligned} \tag{2.2}$$

and in addition, if $\vec{f} = \langle f_1, f_2, f_3 \rangle$,

$$\text{(v)} \quad \|N(\nabla\nabla u)\|_{L^p(\partial D)} \leq C \|\nabla_T \vec{f}\|_{L^p(\partial D)}.$$

The duality between the regularity problem with data in L^p and the Dirichlet problem with data in $L^{p'}(1/p + 1/p' = 1)$ (see [V3]), established the solvability of the Dirichlet problem for $2 < p < \infty$ as a consequence of the above theorem. Theorem (9.3) above was proven by establishing the correct "$p = 1$" analog and interpolating between this endpoint result and the known $p = 2$ result ([D–K–V]). The endpoint result which gave this interpolation is formulated, as is typical in this theory (see Dahlberg–Kenig [D–K] for example), in terms of a Hardy space H^1_{at} estimate. We recall some definitions from [D–K] and [V3].

If Λ denotes the graph of a Lipschitz function, a compactly supported function f is an $H^1_{1,\text{at}} - L^q$ atom if $|\nabla_T f| \in L^q(\Lambda, d\sigma)$ and f and hence each $(\partial/\partial T_j)f$ (which automatically has mean value zero) is a $(1, q)$ atom, i.e., each $(\partial/\partial T_j)f$ is supported on a ball $B \subseteq \Lambda$ and satisfies the estimate

$$\left\| \frac{\partial}{\partial T_j} f \right\|_{L^q(\Lambda)} \leq C\sigma(B)^{1/q - 1}.$$

Then ([D–K]) $f \in H^1_{1,\mathrm{at}}$ if there are $H^1_{1,\mathrm{at}}(\Lambda) - L^q$ atoms a_k so that

$$\frac{\partial}{\partial T_j} f = \sum_{k=1}^{\infty} \lambda_k \frac{\partial}{\partial T_j} a_k, \qquad \sum |\lambda_k| < \infty \tag{$*$}$$

and $H^1_{1,\mathrm{at}}(\Lambda)$ is a Banach space modulo constants if $\|f\|_{H^1_{1,\mathrm{at}}} = \inf\{\Sigma\,|\lambda_k| : \lambda_k \text{ as in } (*)\}$. More generally, f is an $H^1_{1,\mathrm{at}}(\partial D) - L^q$ atom, for D a bounded Lipschitz domain, if f is supported in a cylinder Z and $Z \cap \partial D = \Lambda$, the graph of a lipschitz function and f is in fact an $H^1_{1,\mathrm{at}}(\Lambda) - L^q$ atom. A Whitney array $\dot{f}$ belongs to $H^1_{2,\mathrm{at}}(\partial D)$ if $\dot{f} = \langle f_0, f_1, \ldots, f_n \rangle$ and the f_j are in $H^1_{1,\mathrm{at}}(\partial D)$.

THEOREM (9.6 of [P–V1]). *Let $D \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary. Given $\dot{f} \in H^1_{2,\mathrm{at}}(\partial D)$ there exists a unique function u in D such that*

$$\begin{aligned}
&\text{(i)} \quad \Delta^2 u = 0 \quad \text{in } D \\
&\text{(ii)} \quad \lim_{X \to Q, X \in \Gamma(Q)} u(X) = f_0(Q) \quad \text{a.e.} \\
&\text{(iii)} \quad \lim_{X \to Q, X \in \Gamma(Q)} \nabla u(Q) = \vec{f} = \langle f_1, f_2, f_3 \rangle \quad \text{a.e.} \\
&\text{(iv)} \quad \|N(\nabla\nabla u)\|_{L^1(\partial D)} < \infty
\end{aligned} \tag{2.3}$$

and in addition,

$$\text{(v)} \quad \|N(\nabla\nabla u)\|_{L^1(\partial D)} \le C \sum_{k=0}^{2} \|f_k\|_{H^1_{\mathrm{at}}}.$$

Fix a starlike domain $D \subset \mathbb{R}^3$ and $X \in D$ with $\operatorname{dist}(X) = 1$. Consider now the Whitney array $\dot{f}$ with $f_0(Q) = |X - Q|$ and $f_j(Q) = D_j(|X - \cdot\,|)(Q)$. Observe that $|\nabla_T f_j| \approx |X - Q|^{-1}$ and hence belongs to $L^{2+\varepsilon}$ for any $\varepsilon > 0$. Let $\gamma(X, Y)$ be the biharmonic solution to this $L^{2+\varepsilon}$ regularity problem guaranteed by [V3]. Then if $G(X, Y)$ is the Green's function for Δ^2 in D we have $G(X, Y) = \Gamma(X - Y) - \gamma(X, Y)$. Moreover (since $\operatorname{dist}(X) = 1$), there is a constant C such that

$$\|N(\nabla\nabla\gamma)\|_{L^{2+\varepsilon}(\partial D)} \le C \tag{2.4}$$

where C depends only on the Lipschitz character of D. Therefore "two tangential derivatives" of G are in $L^{2+\varepsilon}(\partial D)$. The next lemma is the main estimate needed for the maximum principle and asserts that this $L^{2+\varepsilon}$ estimate can be improved to an L^1 estimate. The next lemma will be shown on an arbitrary bounded Lipschitz domain D. However, to prove (M.P.) in general Lipschitz domains it will be convenient to specialize first to the situation of starlike domains. It will be important to keep in mind the situation where D is starlike and the X (named below) is at a distance much smaller than diam (D). In this case, the domain D' defined in this argument can *also* be assumed to be starlike.

LEMMA 2.5. *Let D be a bounded Lipschitz domain with Lipschitz constant M and let $X \in D$ with* dist $(X) = 1$. *Then there exists a constant C such that*

$$\int_{\partial D} |\Delta_Q G(X, Q)| \, d\sigma(Q) \leq C \tag{2.6}$$

where C is independent of diam (D).

Proof. Cover ∂D by cylinders Z_i and assume that there exists a Lipschitz function φ and a cylinder Z such that $Z \cap \partial D = \{(x, \varphi(x)) : (x, \varphi(x)) \in \partial D\}$, $X = (x_0, \varphi(x_0) + 1)$ and for some $r = r(M) > 1$, $\Delta' = \{(x, \varphi(x)) : |x - x_0| < r\} \subseteq Z \cap \partial D$. Let $X' = (x_0, \varphi(x_0) + \beta)$ and Γ' be the cone with vertex at X' such that $\Gamma' \cap \partial D = \Delta'$. Let D' be the (Lipschitz) domain defined by $D' = D \backslash \Gamma'$. Then $\partial D' \cap \partial D = \partial D \backslash \Delta'$ and by suitable choice of β, the Lipschitz constant of D' will be a bounded multiple of that of D.

The integral in (2.6) will be estimated in two parts. First,

$$\int_{\Delta'} |\Delta_Q G(X, Q)| \, d\sigma(Q) \leq C \left\{ \int_{\Delta'} |\Delta_Q G(X, Q)|^{2+\varepsilon} \, d\sigma(q) \right\}^{1/(2+\varepsilon)}$$

and since $\Delta_Q G(X, Q) = \Delta_Q \Gamma(X - Q) - \Delta_Q \gamma(X, Q)$ and both Γ and γ have two derivatives in $L^{2+\varepsilon}$, that part of the integral over Δ' is bounded by an absolute constant which depends only on dimension and the Lipschitz constant of D. It remains to estimate that part of the integral over $\partial D \backslash \Delta'$ and to this end we will show that

$$\int_{\partial D'} |N(\nabla_Q \nabla_Q G(X, Q))| \, d\sigma(Q) \leq C. \tag{2.7}$$

(In proving (M.P.), it will be important that the estimate in (2.7) is obtained for the nontangential maximum function on $\partial D'$.) The inequality (2.7) will follow from (2.3)(v) if we show that the f_j (i.e. the $D_j G(X, \cdot)(Q)$) are in fact $H^1_{1,\mathrm{at}} - L^{2+\varepsilon}$ *atoms* in D'. Observe that $G(X, Y)$ is biharmonic for $Y \in D'$ and that $G(X, \cdot)$ and $D_j G(X, \cdot)$ are supported on the unit size surface ball $\partial D' \cap \Gamma'$ on the boundary of

D. To obtain a size estimate on the tangential derivative of $D_jG(X,\cdot)$ on $\partial D\cap\Gamma$ observe that

$$\begin{aligned}\int_{\partial D'\cap\Gamma'}|\nabla_T D_jG(X,P)|\,d\sigma(P) &\le \left\{\int_{\partial D'}|\nabla_T D_j\Gamma(X,P)|^{2+\varepsilon}\,d\sigma(P)\right\}^{1/(2+\varepsilon)}\\ &\quad+\left\{\int_{\partial D'\cap\Gamma'}|\nabla_T D_j\gamma(X,P)|^{2+\varepsilon}\,d\sigma(p)\right\}^{1/(2+\varepsilon)}\\ &\le C+C\left\{\int_{\partial D}|N(\nabla_Q\nabla_Q\gamma)(X,\cdot)(Q)|^{2+\varepsilon}\,d\sigma(P)\right\}^{1/(2+\varepsilon)}\\ &\le C \qquad \text{by (2.4).}\end{aligned}$$

The tangential derivatives automatically have the cancellation since $D_jG(X,\cdot)$ is compactly supported and together with the size estimates, an application of theorem 9.6 [P–V1] yields (2.7).

THEOREM (2.8) (The (M.P.) for biharmonic functions). *Let D be a bounded starlike Lipschitz domain in $\mathbb{R}^3$ and let u be the L^2 solution of the Dirichlet problem in D with $|\nabla u|\in L^\infty(\partial D)$. Then there is a constant (depending only on the Lipschitz character) of D, and independent of* diam (D) *such that*

$$|\nabla u(X)|+|u(X)-u(X^*)|/d(X)\le C\|\nabla u\|_{L^\infty(\partial D)},\tag{2.9}$$

where $d(X)=\operatorname{dist}(X,\partial D)$ and X^* is the radial projection of X onto ∂D.

Proof. We will assume that X is near ∂D, i.e., that $d(X)\ll\operatorname{diam}(D)$ so that the ratio $d(X)/\operatorname{diam}(D)$ is smaller than some fixed constant which depends only on the Lipschitz constant of D. Once (2.9) is proved for all points X in this band near the boundary we shall use the dilation invariance both of the estimate and of the class of Lipschitz domains to rescale so that $\operatorname{diam}(D)=1$. In this situation, the Miranda–Agmon maximum principle for *smooth* domains will yield (2.9) for all interior X. In this way, it is assured that all constants are independent of diam (D).

There is a number α sufficiently small so that when $d(X)<\alpha\operatorname{diam}(D)$, the domain D' defined in the proof of Lemma 2.5 is also starlike. Recall the representation (2.1) of biharmonic functions. Two cases will be considered. In the first case, assume $u|_{\partial D}\equiv 0$. Then $u(x)=\int_{\partial D}(\partial u/\partial N)\,\Delta G(X,\cdot)(Q)\,d\sigma$ and, by interior estimates, it suffices to show that $|u(X)|\le Cd(X)$. This is equivalent to the estimate

$$\int_{\partial D}|\Delta_Q G(X,Q)|\,d\sigma(Q)\le Cd(X).\tag{2.10}$$

The estimate (2.10) rescales. That is, it suffices to prove (2.10) for every starlike Lipschitz domain under the assumption that $d(X) = 1$. For if $d(X) = r$, let $D' = \{X' : rX' \in D\}$, a dilation of D by the factor r. Let $\tilde{X} \in D'$ denote the point X/r. Then $\operatorname{dist}(\bar{X}, \partial D') = (1/r) \operatorname{dist}(X, \partial D) = 1$. By Lemma (2.5),

$$\int_{Q' \in \partial D'} |\Delta_{Q'} G'(\bar{X}, Q')| \, d\sigma(Q') \le C \tag{$*$}$$

where $G'(\cdot, \cdot)$ denotes the Green's function for Δ^2 in D'. But $G'(\bar{X}, Q')$ is just $(1/r)G(r\bar{X}, rQ') \equiv (1/r)G(X, Q)$, $Q \in \partial D$, where $G(\cdot, \cdot)$ is the Green's function for the original domain D. A change of variables in $(*)$ yields (2.10). Then (2.10) together with the fact that $|\partial u/\partial N| \in L^\infty(\partial D)$ prove (2.9).

In the second case, assume that $\partial u/\partial N_Q \equiv 0$ so that the representation (2.1) has the form

$$u(X) = \int_{\partial D} u(Q) \frac{\partial}{\partial N_Q} \Delta_Q G(X, Q) \, d\sigma(Q). \tag{2.11}$$

We will show that for any X in D,

$$\frac{|u(X) - u(X^*)|}{d(X)} \le C \tag{2.12}$$

where X^* is the projection of X onto ∂D. Fix now $X_0 \in D$ and, arguing as before, we assume that $d(X_0) = 1$. With X_0 fixed, assume also that $u(X_0^*) = 0$. Define Γ' and Δ' as in Lemmas 2.5, with $\Delta' = \Gamma' \cap \partial D$. The strategy is to use the representation (2.11), convert normal derivatives to tangential derivatives and integrate by parts. In order to replace the normal derivative by tangentials, we need the Riesz transforms.

Suppose h is a harmonic function (with nontangential limits in $L^2(\partial D)$) in a Lipschitz domain Ω, starlike with respect to the origin. Suppose further that $h(0) = 0$. Define the harmonic function $H(x)$ by

$$H(x) = \int_{t=0}^{1} h(tX) \frac{dt}{t}, \qquad X \in \Omega. \tag{2.13}$$

Then $h(X) = X \cdot \nabla H(X)$ and the vector ∇H restricted to the boundary of Ω is the vector of Riesz transforms of the function h. An easy calculation using the harmonicity of H shows that

$$\begin{aligned}\frac{\partial h(Q)}{\partial N} &= \frac{\partial}{\partial N}(X\cdot\nabla H)|_{X=Q}\\ &= \sum_{j,k} N^j X_k D_j D_k H|_{X=Q} + \frac{\partial H}{\partial N}\\ &= \sum_{j,k}(N^j D_k - N^k D_j)(X_k D_j H)|_{X=Q} + (1-n)\frac{\partial H}{\partial N}.\end{aligned}$$

Observe that, for any function F and fixed j and k $(N^j D_k - N^k D_j)F$ is a tangential derivative of F. If $\vec{e}_k$ denotes the unit vector in the x_k-direction, an application of the divergence theorem yields, for any function u on ∂D:

$$\begin{aligned}\int_{\partial D} u(Q)(N^j D_k v - N^k D_j v)\, d\sigma(Q) &= \int_D \operatorname{div}[uD_k v\vec{e}_j - uD_j v\vec{e}_k]\, dX\\ &= \int_D \operatorname{div}[vD_j u\vec{e}_k - vD_k u\vec{e}_j]\, DX\\ &= \int_{\partial D} v(Q)[N^k D_j u - N^j D_k u](Q)\, d\sigma(Q).\end{aligned}$$

Hence, if h is harmonic in D and H is defined by (2.13), we have

$$\begin{aligned}\int_{\partial D} u(Q)\frac{\partial h}{\partial N}(Q)\, d\sigma(Q) &= (1-n)\int_{\partial D} u(Q)\frac{\partial H}{\partial N}(Q)\, d\sigma(Q)\\ &\quad + \sum_{j,k}\int_{\partial D} u(Q)\frac{\partial}{\partial T_{j,k}}(X_k D_j H)(Q)\, d\sigma(Q)\\ &= (1-n)\int_{\partial D} u(Q)\frac{\partial H}{\partial N}(Q)\, d\sigma(Q)\\ &\quad + (-1)\sum_{j,k}\int \frac{\partial}{\partial T_{j,k}} u(Q)(Q_k D_j H)(Q)\, d\sigma(Q) \qquad (2.14)\end{aligned}$$

where $\partial/\partial T_{j,k}$ is the operator $(N^j D_k - N^k D_j)$. Furthermore, by area integral estimates for harmonic functions ([D]) together with a lemma of Stein ([S], p. 213) it can be shown that $\|N(X_i D_k H)\|_{L^p(\partial D)} \le C\|N(h)\|_{L^p(\partial D)}$ for $0 < p < \infty$. The estimate above, with $N(D_k H)$ in place of $N(X_i D_k H)$ appears in [V2] and the presence of the term X_i eliminates the dependence of the constants on the diameter of the domain.

We shall use these observations to convert normal derivatives of $\Delta G(X, \cdot)(Q)$ (or $\Delta_Q \gamma$) into tangential derivatives as follows. Let φ, ψ be defined in $\mathbb{R}^n$ so that for $B = B(X_0^*, M)$ for some M with $B \cap \partial D = \Delta'$, then $\varphi \equiv 1$ on B, $\varphi \equiv 0$ on ${}^c(2B)$ and $\psi \equiv 1$ on ${}^c(2B)$, $\psi \equiv 0$ on B and $\varphi + \psi \equiv 1$. Then (2.11) may be written

$$u(x) = \int_{\partial D} u \cdot (\varphi + \psi) \frac{\partial}{\partial N_Q} \Delta_Q G(X, Q)\, d\sigma.$$

Consider one term in the sum:

$$\int_{\partial D} \varphi u \frac{\partial}{\partial N} \Delta_Q G(X, Q)\, d\sigma(Q) = \int_{\partial D} u\varphi \frac{\partial}{\partial N} \Delta_Q \Gamma\, d\sigma - \int_{\partial D} u\varphi \frac{\partial}{\partial N} \Delta_Q \gamma\, d\sigma.$$

Because $u(X_0^*) = 0$, our explicit knowledge of $(\partial/\partial N)\, \Delta_Q \Gamma$ shows that $|\int_{\partial D} u\varphi(\partial/\partial N)\, \Delta_Q \Gamma\, d\sigma| \le C\|\nabla u\|$. Now $\Delta_Q \gamma(X, \cdot)$ is harmonic so let H_γ denote the harmonic function associated to $\Delta_Q \gamma$ as in (2.13). Then

$$\begin{aligned} &-\int_{\partial D} u\varphi \frac{\partial}{\partial N} \Delta_Q \gamma(X, Q)\, d\sigma(Q) \\ &= \sum_{j,k} \int_{\partial D} \frac{\partial}{\partial T_{j,k}} (u\varphi)(Q)(Q_k D_j H_\gamma)(Q)\, d\sigma(Q) - (1-n) \int_{\partial D} \varphi u(Q) \frac{\partial H_\gamma}{\partial N}(Q)\, d\sigma(Q). \end{aligned}$$

For each j and k,

$$\begin{aligned} \left| \int_{\partial D} \frac{\partial}{\partial T_{j,k}} (u\varphi)(Q)(Q_k D_j H_\gamma)(Q)\, d\sigma(Q) \right| \le &\int_{\partial D \cap 2\Delta'} \left| \frac{\partial u}{\partial T_{j,k}}(Q) \right| |Q_k D_j H_\gamma(Q)|\, d\sigma(Q) \\ &+ \int_{\partial D \cap 2\Delta'} |u(Q)| \left| \frac{\partial \varphi}{\partial T_{j,k}} \right| |Q_k D_j H_\gamma)|\, d\sigma(Q). \end{aligned}$$

Since $|u(Q)| = |u(Q) - u(X_0^*)| \le \|\nabla u\|_{L^\infty(\partial D)} |Q - X_0^*|$ and $\|N(X_k D_j H_\gamma)\|_{L^{2+\varepsilon}} \le C\|\Delta_Q \gamma\|_{L^{2+\varepsilon}} \le C$, both terms above are dominated by a constant multiple of $\|\nabla u\|_{L^\infty(\partial D)}$. A similar estimate holds for $\int_{\partial D \cap \Delta'} |u|\, |\partial H/\partial N|\, d\sigma$ and it remains to bound the integral taken over ${}^c\Delta'$. We have,

$$\int_{\partial D} (u\psi)(Q) \frac{\partial}{\partial N} \Delta_Q G(X, Q)\, d\sigma(Q) = \int_{\partial D'} u\psi \frac{\partial}{\partial N} \Delta_Q G\, d\sigma$$

by definition of ψ, and in D', $\Delta_Q G(X, \cdot)$ is harmonic. Thus we apply (2.14) with $h = \Delta_Q G$ in D' to obtain

$$\int_{\partial D'} u\psi \frac{\partial}{\partial N} \Delta_Q G \, d\sigma(Q) = \sum_{j,k} \int_{\partial D'} \frac{\partial}{\partial T_{j,k}} (u\psi)[Q_k D_j H_G(Q)] \, d\sigma + (1-n) \int_{\partial D'} u\psi \frac{\partial}{\partial N} (H_G)(Q) \, d\sigma \tag{2.15}$$

where H_G is defined by (2.13) for $h = \Delta_Q G$. Fix a j and k and consider

$$\int_{\partial D'} \frac{\partial}{\partial T_{j,k}} (u\psi)(Q) Q_k (D_j H)(Q) \, d\sigma(Q) = \int_{\partial D' \cap {}^c(\Delta')} \frac{\partial u}{\partial T_{j,k}} \psi Q_k D_j H(Q) \, d\sigma(Q) + \int_{\partial D' \cap (2\Delta' \setminus \Delta')} u \frac{\partial \psi}{\partial T_{j,k}} Q_k D_j H(Q) \, d\sigma(Q). \tag{2.16}$$

Recall that (2.7) implies that

$$\| N(X_k D_j H) \|_{L^1(\partial D)} \le C.$$

Hence,

$$\left| \int_{\partial D' \cap {}^c\Delta'} \frac{\partial u}{\partial T_{j,k}} \psi Q_k D_j H(Q) \, d\sigma(Q) \right| \le C \| \nabla u \|_{L^\infty(\partial D)}.$$

A similar estimate holds for the second term in the sum (2.16). The lower order term in (2.15) is handled the same way using $\| N(X_k D_j H) \|_{L^{2+\varepsilon}(\partial D' \cap 2\Delta' \setminus \Delta')} \le C$. This last estimate follows from the $L^{2+\varepsilon}$ improvement on D', see p. 400.

§3. The Dirichlet problem for *BMO* data and Hölder continuity of the gradient of solutions.

A function f on ∂D is said to be in $BMO(d\sigma)$ if there is a constant C such that

$$\sup_{\Delta \subseteq \partial D} \left(\frac{1}{\sigma(\Delta)} \int_\Delta |f - f_\Delta| \, d\sigma \right] \le C \tag{3.1}$$

where $f_\Delta = (1/\sigma(\Delta)) \int_\Delta f\, d\sigma$ is the average of f on the surface ball Δ. The definition (3.1) leads to an improvement of itself, by the John–Nirenberg Theorem, namely

$$\sup \left(\frac{1}{\sigma(\Delta)} \int_\Delta |f - f_\Delta|^p \, d\sigma \right]^{1/p} \leq C_p$$

for any $p > 0$. In particular a BMO function belongs to $L^2(\partial D)$. We shall put a norm, $|\ |_*$, on $BMO(\partial D)$ by defining $\|f\|_* = \inf\{C : (3.1) \text{ holds}\} + |\int_{\partial D} f\, d\sigma|$. We shall prove in this section two natural extensions of the maximum principle for starlike domains. In the first place, the Dirichlet problem for Δ^2 will be shown to be solvable when the data is $BMO(d\sigma)$ and the appropriate Carleson measure estimate on solutions is obtained. Secondly, solutions with data vanishing on a surface ball on ∂D will have a Hölder continuous gradient at points in the domain near the surface ball. We begin with the theory for BMO data. As usual, $d(X)$ denotes dist $(X, \partial D)$ and $|\nabla\nabla u|^2 = \Sigma_{j,k} |D_j D_k u|^2$. For a biharmonic function u define the square function of its gradient by

$$S^2(\nabla u)(Q) = \int_{\Gamma(Q)} d(X)^{2-n} |\nabla\nabla u(X)|^2 \, dX.$$

We shall need the following theorem, but only in the special case $p = 2$.

THEOREM 3.2 ([P–V2]). *For u biharmonic in a Lipschitz domain $D \subseteq \mathbb{R}^n$ with $|\nabla u(P)| = 0$ for some $P \in D$ and $0 < p < \infty$.*

$$\|S(\nabla u)\|_{L^p(d\sigma)} \approx \|N(\nabla u)\|_{L^p(d\sigma)}$$

with a comparability constant depending only on dimension and the Lipschitz character of D.

Given a surface ball $\Delta(Q, r) \subset \partial D$, define the Carleson region associated to $\Delta(Q, r)$ by $T(\Delta) = B(Q, r) \cap D$. For a Lipschitz domain $D \subseteq \mathbb{R}^n$, if h is the solution to the Dirichlet problem for the Laplacian in D with boundary values $f \in BMO\ (d\sigma)$ then the Carleson estimate

$$\sup_{\Delta \subseteq \partial D} \left(\iint_{T(\Delta)} d(X) |\nabla h|^2 \frac{dX}{\sigma(\Delta)} \right)^{1/2} \leq C \|f\|_*$$

is valid (see [F–K–N]). The analog of this result for biharmonic functions is the following theorem for starlike Lipschitz domain $D \subseteq \mathbb{R}^3$.

THEOREM 3.3. *Let* $\vec{f} = \langle f_0, f_1, f_2, f_3 \rangle \in WA_2$ *and let* u *be the solution to the* L^2 *biharmonic Dirichlet problem in* $D \subseteq \mathbb{R}^3$ *which satisfies*

$$u|_{\partial D} = f_0$$

$$\frac{\partial u}{\partial N} = \sum_j f_j N^j,$$

nontangentially. If $f_j \in BMO(\partial D)$ *for* $j = 1, 2, 3$ *then there exists a constant* C *depending only on the Lipschitz character of* D *and the norms* $\|f_j\|_*$ *such that*

$$\sup_{\Delta \subseteq \partial D} \left\{ \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} d(X)\, |\nabla\nabla u(X)|^2\, dX \right\} \leq C, \tag{3.4}$$

where $T(\Delta)$ *is the Carleson region associated to* Δ.

It may seem at first more natural to ask for Carleson measure results on $|\nabla\nabla u|$ merely under the assumption that $\partial u/\partial N \in BMO$. In fact, the hypotheses of Theorem 3.3 state that all derivatives of u (restricted to ∂D) belong to $BMO(\partial D)$, a condition which is certainly implied by (3.4) but does not necessarily follow from the assumption that $\partial u/\partial N \in BMO(\partial D)$.

Proof of Theorem 3.3. Fix a surface ball $\Delta \subseteq \partial D$ with center Q_0 and radius r_0. From the compatibility conditions, we have $|\nabla_T f_0| \leq c\, \Sigma_j |f_j|$. If $4\Delta \equiv \Delta(Q_0, 4r_0)$ and $(f_j)_{4\Delta} \equiv (1/\sigma(4\Delta)) \int_{4\Delta} f_j(Q)\, d\sigma(Q)$, let $c_j = (f_j)_{4\Delta}$ and define $v(X) = \Sigma_j c_j X_j$. Then $\nabla\nabla v \equiv 0$ and $(\partial/\partial N)(u - v) = \Sigma_j (f_j - c_j) N_j$ and hence, by subtracting v from u, we may assume that $c_j = 0$, $j = 1, 2, 3$. We may also assume that $\int_{4\Delta} f_0\, d\sigma = 0$ by subtracting a constant from u.

Now let $\psi \in C_0^\infty(\mathbb{R}^3)$ be such that $\psi \equiv 1$ on 2Δ and $\psi \equiv 0$ on ${}^c4\Delta$, and $|\nabla\psi| \sim 1/r_0$ on $4\Delta \backslash 2\Delta$. Define

$$f_0^{(1)} = f_0 \psi, \quad \text{and}$$

$$f_0^{(2)} = f_0 (1 - \psi).$$

Then there are functions $g_i, h_i, i = 1, 2, 3$ such that both $\langle f_0^{(1)}, g_1, g_2, g_3 \rangle$ and $\langle f_0^{(2)}, h_1, h_2, h_3 \rangle$ are Whitney arrays. For simplicity, we calculate this in local coordinates. Thus if the boundary of D is locally defined by the graph of the Lipschitz function φ, we see that

$$\frac{\partial}{\partial x_j} f_0^{(1)}(x, \varphi(x)) = \left(f_j + f_3 \frac{\partial \varphi}{\partial x_j} \right) \psi + f_0(x, \varphi(x)) \left(\frac{\partial \psi}{\partial x_j} + \frac{\partial \psi}{\partial x_3} \frac{\partial \varphi}{\partial x_j} \right)$$

for $j = 1, 2$ and so

$$g_j = f_j\psi + \left(f_0 \frac{\partial\psi}{\partial x_j}\bigg|_{\partial D}\right)$$

and

$$g_3 = f_3\psi + \left(f_0 \frac{\partial\psi}{\partial x_3}\bigg|_{\partial D}\right).$$

We shall set $u^{(1)}$ to be the biharmonic solution to the Dirichlet problem with data $\langle f_0^{(1)}, \Sigma_j g_j N_j\rangle$ and similarly define $u^{(2)}$ so that $u = u^{(1)} + u^{(2)}$. By Theorem 3.2, the quantity (3.4) for $u^{(1)}$ may be estimated using L^2 norms:

$$\begin{aligned}\frac{1}{\sigma(\Delta)}\iint_{T(\Delta)} d(X)\,|\nabla\nabla u^{(1)}|^2\,dX &\le \frac{1}{\sigma(\Delta)}\int S^2(\nabla u^{(1)})(Q)\,d\sigma(Q)\\ &\le \frac{1}{\sigma(\Delta)}\int N^2(\nabla u^{(1)})\,d\sigma \le \frac{1}{\sigma(\Delta)}\int |\nabla u^{(1)}|^2\,d\sigma.\end{aligned}$$

We return to local coordinates to calculate the L^2 norms of the derivatives if $u^{(1)}$ on the boundary

$$\left(\int_{\partial D} |f_j\psi|^2\,d\sigma\right)^{1/2} \le \left(\int_{4\Delta} |f_j - (f_j)_{4\Delta}|^2\,d\sigma\right)^{1/2} \le \|f_j\|_* \sigma(\Delta), \text{ by assumption.}$$

And, by Poincaré's inequality,

$$\begin{aligned}\left(\int\left|f_0\frac{\partial\psi}{\partial x_j}\right|^2 d\sigma\right)^{1/2} &\le \frac{1}{r_0}\left(\int_{4\Delta/2\Delta}\left|f_0 - ⨍_{4\Delta} f_0\right|^2 d\sigma\right)^{1/2}\\ &\le \left(\int_{4\Delta\setminus 2\Delta} |\nabla_T f_0|^2\,d\sigma\right)^{1/2}\left(\int_{4\Delta}\sum_j |f_j|^2\,d\sigma\right)^{1/2} \le \sum_j \|f_j\|_* \sigma(\Delta).\end{aligned}$$

It remains to estimate (3.4) when $u^{(2)}$ is substituted for u. The solution $u^{(2)}$ has the representation

$$u^{(2)}(X) = \int_{\partial D}\sum_j h_j N^j(Q)\Delta_Q G(X, Q)\,d\sigma(Q) + \int_{\partial D} f_0^{(2)}(Q)\frac{\partial}{\partial N_Q}\Delta_Q G(X, Q)\,d\sigma(Q),$$

where $G(X, Y)$ is the biharmonic Green's function for D. We consider first the term

$h(X) = \int_{\partial D} \Sigma_j h_j N^j(Q) \Delta_Q G(X, Q)\, d\sigma(Q)$, i.e., $\Delta^2 h = 0$, $h \equiv 0$ on ∂D and $\partial h/\partial N = \Sigma_j h_j N^j$, where $\langle f_0^{(2)}, h_1, h_2, h_3 \rangle$ is the Whitney array associated to $u^{(2)}$. To estimate $(1/\sigma(\Delta)) \iint_{T(\Delta)} d(X) |\nabla\nabla h(X)|^2\, dX$, we begin by obtaining a pointwise estimate on $h(X)$. Fix $X \in T(\Delta)$. Set $\Delta_k \equiv \Delta(Q_0, 2^k r_0)$ and $R_k = \Delta_k - \Delta_{k-1}$, so that

$$|h(X)| = \left| \sum_{k>0} \int_{R_k} \sum_{j=1}^{3} h_j N^j(Q) \Delta_Q G(X, Q)\, d\sigma(Q) \right|$$
$$\leq \sum_k \int_{R_k} \sum_j |h_j(Q)|\, |\Delta_Q G(X, Q)|\, d\sigma(Q).$$

Fix a h and k. We have

$$|h_j| \leq |f_j| + |f_0|\, |\nabla \psi|.$$

The basic estimate on atoms used to prove Theorem 9.6 (see Lemma 5.7 of [P–V1]) when specialized to $G(X, \cdot)$ is the following, for $d(X) = 1$ and $r_0 = 1$

$$\int_{|Q| \approx 2^k} |\Delta_Q G(X, Q)|^2\, d\sigma(Q) \leq C(2^k)^{-2-\varepsilon} \tag{3.5}$$

for some ε depending only on the Lipschitz character of D. Rescaling this estimate yields

$$\int_{R_k} |\Delta_Q G(X, Q)|^2\, d\sigma(Q) \leq C(2^k)^{-2-\varepsilon} \cdot \left(\frac{d(X)}{r_0}\right)^{2+\varepsilon}. \tag{3.6}$$

Recall that $c_j = (f_j)_{4\Delta} \equiv 0$ and a standard BMO estimate shows that $|(f_j)_{\Delta_k} - c_j| \leq Ck \|f_j\|_*$. Thus

$$\int_{R_k} |f_j(Q)|\, |\Delta_Q G(X, Q)|\, d\sigma(Q)$$
$$\leq \int_{R_k} |\Delta_Q G(X, Q)| \{ |f_j - (f_j)_{\Delta_k}| + |(f_j)_{\Delta_k} - c_j| \}\, d\sigma(Q).$$

The term $\int_{R_k} |\Delta_Q G(X, Q)|\, |(f_j)_{\Delta_k} - c_j|\, d\sigma(Q)$ has the bound

$$Ck \|f\|_* \cdot 2^k r_0 \cdot \left\{ \int_{R_k} |\Delta_Q G(X, Q)|^2\, d\sigma(Q) \right\}^{1/2} \leq Ck \|f_j\|_* 2^{-k\varepsilon} \left(\frac{d(X)}{r_0}\right)^{\varepsilon} \cdot d(X),$$

by (3.6).

The other term has a better estimate:

$$\int_{R_k} |f_j(Q) - (f_j)_{\Delta_k}|\,|\Delta_Q G(X, Q)|\, d\sigma(Q)$$
$$\leq \left(\int_{R_k} |f_j(Q) - (f_j)_{\Delta_k}|^2\, d\sigma\right)^{1/2} \cdot (2^k)^{-1-\varepsilon} \cdot \left(\frac{d(X)}{r_0}\right)^{1+\varepsilon}$$
$$\leq C 2^k r_0 \|f_j\|_* \cdot (2^k)^{-1-\varepsilon} (d(X)/r_0)^{1+\varepsilon}$$
$$\leq C \|f_j\|_* 2^{-k\varepsilon} \left(\frac{d(X)}{r_0}\right)^{\varepsilon} \cdot d(X).$$

Finally, to bound $|h|$, we need an estimate for $\int_{R_k} |f_0|\,|\nabla\psi|\,|\Delta_Q G(X, Q)|\, d\sigma(Q)$. Again, we use Poincaré's inequality, and the fact that $|\nabla_T f_0| \leq \Sigma_j |f_j|$, so that only the term $k = 2$ does not vanish in the sum.

Altogether, interior estimates will give a bound on $\iint_{I^+} d(X)\,|\nabla\nabla h(X)|^2\, dX$ where I is a dyadic subcube of D and $I^+ = \{(\bar{x}, y) : \bar{x} \in I \text{ and } \ell(I)/2 < \operatorname{dist}(y, \partial D) < \ell(I)\}$. We have

$$\iint_{I^+} d(X)|\nabla\nabla h(X)|^2\, dX \leq \ell(I) \iint_{I^+} |\nabla\nabla h(X)|^2\, dX$$
$$\leq \frac{\ell(I)}{\ell(I)^4} \iint_{\tilde{I}^+} |h(X)|^2\, dX \tag{3.7}$$

(here $\tilde{I} = \frac{3}{2}I$).

For $X \in I^+$,

$$|h(X)| \leq \sum_k 2^{-k\varepsilon} k \|f_j\|_* \cdot \left(\frac{d(X)}{r_0}\right)^{\varepsilon} \cdot d(X) \leq C \|f_j\|_* \frac{\ell(I)^{1+\varepsilon}}{r_0}$$

and so (3.7) is bounded by

$$C\|f_j\|_* \ell(I)^{-3} \iint_{I^+} \left(\frac{\ell(I)^{1+\varepsilon}}{r_0^{\varepsilon}}\right)^2 dX \leq C \|f_j\|_* \frac{\ell(I)^{2+\varepsilon}}{r_0^{\varepsilon}}.$$

Then,

$$\sum_{I \subset \Delta} \frac{\ell(I)^{2+\varepsilon}}{r_0^{\varepsilon}} \leq C\sigma(\Delta)$$

as desired.

To complete the Carleson measure estimate, we have only to show that

$$\frac{1}{\sigma(\Delta)}\iint_{T(\Delta)} d(X)|\nabla\nabla F(X)|^2\,dX \le C$$

where $F(X) = \int_{\partial D} f_0^{(2)}(Q)(\partial/\partial N_Q)\,\Delta_Q G(X,Q)\,d\sigma(Q)$. Because $f_0^{(2)}$ is supported on ${}^c\Delta$ we may form a new domain D' relative to D and convert normal derivatives to tangential derivatives as in the proof of Theorem 2.8. Our basic estimate (3.5) is valid for $N(\Delta_Q G(X,Q))$ replacing $\Delta_Q G(X,Q)$ and consequently all estimates are the same, so the details are omitted.

We turn now to results on Hölder continuity of the gradient of solutions when the data is sufficiently smooth. Recall that $g \in \Lambda_\alpha$ if

$$\sup_{Q_1,Q_2\in\partial D}\frac{|g(Q_1)-g(Q_2)|}{|Q_1-Q_2|^\alpha} \le C.$$

THEOREM 3.8. *Let u be the solution to the L^2 biharmonic Dirichlet problem in the starlike domain D with data*

$$u|_{\partial D} = f_0$$

$$\frac{\partial u}{\partial N} = \sum_{j=1}^{3} f_j N^j$$

where $\langle f_0, f_1, f_2, f_3\rangle \in WA_2$, and N^j is the j^{th} component of the normal vector. Assume that $f_j \in \Lambda_\alpha$ where $\alpha < \varepsilon$, for ε defined by (3.5). Then $\nabla u \in \Lambda_\alpha(D)$; in fact

$$\|\nabla u\|_{\Lambda_\alpha(D)} + \sup_{X\in D} d(X)^{1-\alpha}|\nabla\nabla u(X)| \le C\sum_{j=1}^{3}\|f_j\|_{\Lambda_\alpha(\partial D)},$$

where C depends only on the Lipschitz constant of D.

Proof. The theorem is a consequence of the basic estimate (3.5) and the L^∞ bound on the gradient. We mimic the simple argument of [DK2]. It suffices to show, after rescaling and invoking the dilation invariance of the constants in the estimate, that

$$|\nabla\nabla u(X)| \le C\sum_{j=1}^{3}\|f_j\|_{\Lambda_\alpha}$$

when $d(X) \equiv 1$. We have, as always, the representation

$$u(X) = \int_{\partial D} f_0(Q) \frac{\partial}{\partial N_Q} \Delta_Q G(X, Q)\, d\sigma(Q) + \int_{\partial D} \sum_j f_j(Q) N^j(Q) \Delta_Q G(X, Q)\, d\sigma(Q).$$

After subtracting a constant (namely $f_0(Q_0)$ where Q_0 is the radial projection of X onto ∂D) and substracting the linear function $\Sigma_{j=1}^3 f_j(Q_0)X_j$ we may assume that $f_0(Q_0) \equiv 0$ and that $(\partial u/\partial N)(Q_0) \equiv 0$. We shall treat each term in the sum above separately. The argument for the term $\int_{\partial D} f_0(Q)(\partial/\partial N_Q)\Delta_Q G(X, Q)_c$ is similar to that of Theorem 2.8 – normal derivatives are converted to tangential derivatives and Riesz potentials are introduced. All estimates are the same as those for the second term so we shall give these estimates only. We have

$$\left| \int_{\partial D} \sum_j f_j(Q) N^j(Q) \Delta_Q G(X, Q)\, d\sigma(Q) \right| \le \sum_j \|f_j\|_{\Lambda_\alpha} \int_{\partial D} |Q - Q_0|^\alpha \, |\Delta_Q G(X, Q)|\, dQ$$

and the basic estimate (3.5) immediately shows, when $\alpha < \varepsilon$, that

$$\int_{\partial D} |Q - Q_0|^\alpha \, |\Delta_Q G(X, Q)|\, d\sigma(Q) \le C.$$

Interior estimates give the desired bound on $|\nabla \nabla u(X)|$, since $d(X) = 1$.

REMARK 3.9. We now observe (following a suggestion of C. Kenig) that an analog of the above theorem with *continuous* data follows from the Hölder continuous case and yields a "classical" solution, i.e., a solution with ∇u continuous up to the boundary. To see this, approximate the continuous data by C^α functions and use the L^∞ bound to estimate the difference between the solutions to the approximating data and the continuous data.

§4. The maximum principle for C^1 domains

The essential reason that the maximum principle for the gradient of biharmonic functions is valid on C^1 domains in all dimensions is that the Dirichlet problem for Δ^2 is solvable in every p, $1 < p < \infty$ ([V2]). This fact allows us to obtain an atom's estimate in every dimension by using solvability of the Dirichlet problem for p near 1. Indeed, by the remarks following the proof of Lemma 2 of [V2], we know that the Dirichlet problem (1.1) is solvable for all $1 < p < \infty$ when D is a Lipschitz domain whose Lipschitz constant is smaller than some $\varepsilon_0 > 0$. We shall state and prove the following results for Lipschitz graphs with small constant in order to have available the rescaling and dilation invariance techniques which simplify the arguments.

An L^∞ bound on $|\nabla u|$ follows, as in §2, from the atom's estimate. For arbitrary Lipschitz domains, this estimate was only valid in $\mathbb{R}^3$ but we shall prove it for Lipschitz domains with *small* constant in $\mathbb{R}^n$, all $n \geq 2$. From the L^∞ estimate and the basic estimate on the Green's function, one can obtain Hölder estimates on gradients of solutions as well as Carleson estimates for *BMO* data. All of this will follow from the appropriate analog of Lemma 5.7 of [P–V1].

Let D be the domain above the graph of a compactly supported Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}$ with Lipschitz constant smaller than $\varepsilon_0 > 0$. We will need the following Cacciopoli type inequality, which is in fact valid for any Lipschitz constant.

LEMMA 4.1 (Lemma 5.6 of [P–V1]). *Let $\Omega_1 \subset \Omega_2$ be bounded Lipschitz domains with $\Omega_2 \subset D$. Let $\Delta^2 u = 0$ in D with $N(\nabla\nabla u) \in L^2(\partial D)$. Let $1/p + 1/p' = 1$ and $0 < d = \operatorname{dist}(\Omega_1, D\backslash\Omega_2)$. Then there is a constant that depends only on $1 < p < \infty$ and the Lipschitz constant so that*

$$\begin{aligned}\int_{\Omega_1} |\nabla\nabla u(X)|^2\, dX \leq C(&\|\nabla u\|_{L^{p'}(\partial D \cap \partial\Omega_2)} \|N(\nabla\nabla u)\|_{L^p(\partial D)} \\ &+ d^{-1}\|u\|_{L^{p'}(\partial D \cap \partial\Omega_2)} \|N(\nabla\nabla u)\|_{L^p(\partial D)} \\ &+ d^{-1}\|\nabla u\|_{L^2(\Omega_2)} \|\nabla\nabla u\|_{L^2(\Omega_2)} \\ &+ d^{-2}\|u\|_{L^2(\Omega_2)} \|\nabla\nabla u\|_{L^2(\Omega_2)}.\end{aligned}$$

Proof. The proof uses the equation together with several integrations by parts, and is given in [P–V1].

The following lemma gives the atomic estimate for which a Hardy space regularity result follows. This was Lemma 5.7 of [P–V1] which was valid on an arbitrary Lipschitz domain only in $\mathbb{R}^3$ (and $\mathbb{R}^2$). Recall that a $(1, q)$ atom a is a function supported in a surface ball $\Delta(Q, r) = \{P \in \partial D : |P - Q| < r\}$ such that $\|a\|_{L^q(\partial D)} \leq \sigma(\Delta)^{1/q - 1}$ and $\int a\, d\sigma = 0$.

THEOREM 4.2. *Let a be a $(1, q)$ atom, $q \geq 2 - \varepsilon$ on ∂D and let u be a solution to the following regularity problem*

(i) $\Delta^2 u = 0$

(ii) $\displaystyle\lim_{X \to Q, X \in \Gamma(Q)} D_n u(X) = 0$ a.e.

(iii) $\displaystyle\lim_{\substack{X \to Q \\ X \in \Gamma(Q)}} \sum_{j=1}^{n-1} \langle T_j(Q), \nabla D_j u(X)\rangle = a(Q)$ a.e.

Then,

$$\int_{\partial D} N(\nabla\nabla u)\,d\sigma \leq C, \tag{4.3}$$

where C depends on ε but not on the support of the function φ for $D = \{(x, y): y > \varphi(x)\}$.

(Cf. Theorem 4.6 of [P–V1] for this formulation of the regularity problem above a graph.)

The following lemma is a special case of a more general theorem due to J. L. Lions. Since the argument is brief and shows the dependence of the constants, we shall give it here. This estimate will be used in the proof of Theorem 4.2.

LEMMA 4.4. *Let u be a C^2 function in a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$. Then there exists a constant C such that for any $\varepsilon \leq 1$,*

$$\left(\int_\Omega |\nabla u(x)|^2 \frac{dx}{|\Omega|}\right)^{1/2} \leq C(\varepsilon) \int_\Omega |\nabla u(x)| \frac{dx}{|\Omega|} + \varepsilon R \left(\int_\Omega |\nabla\nabla u(x)|^2 \frac{dx}{|\Omega|}\right)^{1/2}, \tag{4.5}$$

where $R = \operatorname{diam}\Omega$, and $C(\varepsilon) \sim 1/\varepsilon^M$ for some $M = M(n)$.

Proof. Without loss of generality, assume that $\operatorname{diam}\Omega = 1$. Let $v = D_i u$ for some $1 \leq i \leq n$. We may assume that $\int_\Omega v\,dx = 0$. We then claim that for some $\theta = \theta(n)$,

$$\left(\int_\Omega v^2\right)^{1/2} \leq \left(\int_\Omega |\nabla v|^2\right)^{(1-\theta)/2} \cdot \left(\int_\Omega |v|\right)^{\theta}. \tag{4.6}$$

The estimate (4.6) gives (4.5) with constants ε and $1/\varepsilon^M$, where $M = (1-\theta)/\theta$.

To prove (4.6), let $\alpha_n = 4/(n+2)$ and choose p_n so that $p_n'\alpha_n = 1$, where $p_n' = p_n/(p_n - 1)$. Then

$$\int_\Omega |v|^2 = \int_\Omega |v|^{2-\alpha_n}|v|^{\alpha_n} \leq \left(\int_\Omega |v|^{(2-\alpha_n)p_n}\right)^{1/p_n} \cdot \left(\int_\Omega |v|^{\alpha_n p_n'}\right)^{1/p_n'}.$$

Then $(2-\alpha_n)p_n = 2n/(n-2)$ and by Sobolev's Theorem, since $\int_\Omega v = 0$,

$$\left(\int_\Omega |v|^{2n/(n-2)}\right) \leq \left(\int_\Omega |\nabla v|^2\right)^{n/(n-2)}.$$

Thus,

$$\left(\int_\Omega v^2\right)^{1/2} \le \left(\int_\Omega |\nabla v|^2\right)^{(n/(n-2))\cdot(1/(2p_n))} \cdot \left(\int_\Omega |v|\right)^{1/(2p_n')}$$

and if $\theta = (2p_n')^{-1} = (p_n - 1)/2p_n$, then $1 - \theta = (p_n + 1)/2p_n$ and $p_n + 1 = 2n/(n-2)$.

Proof of Theorem 4.2. By translation and dilation invariance, we may fix $\varphi(0) = 0$ and assume that the atom a is supported in the unit size surface ball centered at the origin with $\|a\|_{L^p(d\sigma)} \le C$, for $1 \le p \le q$.

Define, for $x \in \mathbb{R}^{n-1}$ $(n > 3)$, as in [P–V1],

$$\tilde{u}(x) = \int_{\mathbb{R}^{n-1},} |x - y|^{3-n} a(y, \varphi(y))\, dy,$$

so that

$$\Delta_x \tilde{u}(x) = a(x, \varphi(x)) \quad \text{a.e.}$$

The support and cancellation properties of $a(x, \varphi(x))$ yield the following estimates:

$$\begin{aligned} |\tilde{u}(x)| &\le C|x|^{2-n} \\ |\nabla \tilde{u}(x)| &\le C|x|^{1-n} \\ |\nabla\nabla \tilde{u}(x)| &\le C|x|^{-n}. \end{aligned} \tag{4.7}$$

for all $|x| \ge 2$. By Weyl's Lemma and the Liouville Theorem, $\tilde{u}(x)$ differs from $u(x, \varphi(x))$ by a linear function of x; by subtracting this linear function we may assume $\tilde{u}(x) = u(x, \varphi(x))$ a.e.

The estimate (4.3) follows from the L^2 regularity theory via Schwarz's inequality when the domain of integration is restricted to $\{(x, \varphi(x) : |x| \le 2\}$. For $Q = (x, \varphi(x))$, define

$$\Gamma_1(Q) = \Gamma(Q) \cap \{X : \text{dist}\,(X, \partial D) > |x|\}$$

$$\Gamma_2(Q) = \Gamma(Q) \cap \{X : \text{dist}\,(X, \partial D) \le |x|\}$$

and

$$N_1(\nabla\nabla u)(Q) = \sup_{X \in \Gamma_1(Q)} |\nabla\nabla u(x)|, \qquad N_2(\nabla\nabla u)(Q) = \sup_{X \in \Gamma_2(Q)} |\nabla\nabla u(x)|.$$

Then $N(\nabla\nabla u) \leq N_1(\nabla\nabla u) + N_2(\nabla\nabla u)$ and each term will be handled separately. We first observe that for $|x|$ large, $|\nabla_x u| \leq C|x|^{1-n}$ and since $D_n u|_{\partial D} \equiv 0$, u has Dirichlet data in $L^p(d\sigma)$ for any $p > 1$, In [S–S], the Dirichlet problem for the biharmonic equation above a graph with small Lipschitz constant was solved for L^p data, $1 < p < \infty$. This fact could also be proved from the results of [V2] for bounded C^1 domains. In any case we have, for this u, $\|N(\nabla u)\|_{L^p(d\sigma)} \leq C_p$, where C_p depends only on p and the dimension.

For $n \geq 3$ fixed, we fix a $p < (n-1)/(n-2)$ so that $\|N(\nabla u)\|_{L^p(d\sigma)} \leq C_0$, where C_0 depends only on n.

Thus for $X \in D$,

$$|\nabla u(X)| \leq Cd(X)^{-(n-1)/p}$$

and by interior estimates,

$$|\nabla\nabla u(X)| \leq C'd(X)^{1-n-\varepsilon}$$

if ε is defined by $p = (n-1)/(n-2+\varepsilon)$. Consequently the estimate (4.3) holds if $N(\nabla\nabla u)$ is replaced by $N_1(\nabla\nabla u)$.

We now proceed as in [P–V1] to estimate $\|N_2(\nabla\nabla u)\|_{L^1(d\sigma)}$, applying the Cacciopoli estimate of Lemma 4.1.

Let $\Lambda_R = \{(x, \varphi(x)) : R < |x| < 2r\}$, and for $1 \leq \tau \leq 4$ define

$$\Omega_\tau = \{X \in D : X = (x, \varphi(x) + s), \tau^{-1}R < |x| < 2\tau R, 0 < s < 4R\}$$

so that $\partial\Omega_1 \cap \partial D = \Lambda_R$. Then

$$\int_{\Lambda_R} N_2(\nabla\nabla u)(Q)\, d\sigma(Q) \leq R^{(n-1)/2} \cdot \left(\int_{\Lambda_R} N_2^2(\nabla\nabla u)(Q)\, d\sigma(Q)\right)^{1/2}. \tag{4.8}$$

By the L^2 regularity theory for bounded starlike Lipschitz domains,

$$\int_{\partial\Omega_1 \cap \partial D} (N_2(\nabla\nabla u))^2(Q)\, d\sigma(Q)$$
$$\leq C\left(\int_{\partial\Omega_\tau \setminus \partial D} |\nabla\nabla u(P)|^2\, d\sigma_\tau(P) + \sum_{j=1}^{n-1} \int_{\partial D \cap \partial\Omega_\tau} \left|\frac{\partial}{\partial T_j} \nabla u\right|^2 d\sigma(Q)\right)$$

for all $1 < \tau < 2$ and C is independent of τ and R. By (4.7), the last term is of order R^{-n-1}. Averaging in τ yields

$$\int_{\partial\Omega_1 \cap \partial D} N_2(\nabla\nabla u)^2 \, d\sigma \leq C\left(R^{-1} \int_{\Omega_2} |\nabla\nabla u(X)|^2 \, dX + R^{-n-1}\right). \tag{4.9}$$

The Cacciopoli estimate of Lemma 4.1 will be used to bound the solid integral of $|\nabla\nabla u|$ over Ω_2. We first observe that for any $1 < q < \infty$, the regularity problem with data in (L^q, L^q_1) is solvable on Lipschitz graphs with Lipschitz norm $\leq \varepsilon_0$, by the duality between this problem and the Dirichlet problem. (This duality was shown in [P–V1] where it was used in dimension 3 for the Dirichlet problem above any Lipschitz graph.) Thus, for a fixed q, namely $1/q = 1 - (n - 2 + \varepsilon)/(n-1)$, and the assumption that a is a unit $(1, q)$ atom, we have $\|N(\nabla\nabla u\|_{L^q(\partial D)} \leq C$. For this fixed q, C depends only on the dimension (and ε_0). In addition, we have the following estimates from (4.7):

$$\begin{aligned} &\|u\|_{L^2(\Omega_2)} \leq C\{R\|\nabla u\|_{L^2(\Omega_2)} + R^{(-n+4)/2}\} \\ &\|\nabla u\|_{L^p(\partial\Omega_2 \cap \partial D)} \leq CR^{(-n-1)/p} \\ &\|u\|_{L^p(\partial\Omega_2 \cap \partial D)} \leq R^{-1} \cdot R^{-(n-1)/p}. \end{aligned} \tag{4.10}$$

These easy estimates follow from the definition of nontangential maximal function and calculus. For example, if $X^* = (x, \varphi(x))$ when $X = (x, \varphi(x) + s)$,

$$\begin{aligned} \|u\|^2_{L^2(\Omega_2)} &= \int_{\Omega_2} |u(x)|^2 \, dX \lesssim \int_{\Omega_2} |u(X) - u(X^*)|^2 \, Dx + \int_{\Omega_2} |u(X^*)|^2 \, DX \\ &\lesssim \int_{\Omega_2} \left(\int_0^s \frac{\partial u}{\partial y}(x, \varphi(x) + y) \, dy\right)^2 dX + \left(\frac{1}{R^{n-2}}\right)^2 \cdot R^n \\ &\lesssim \int_{Q \in \Lambda_R} \int_{\rho=0}^{R} \left(\int_{y=0}^{\rho} |\nabla u(Q) + (\hat{o}, y)| \, dy\right)^2 d\rho \, d\sigma(Q) + R^{-n+4} \\ &\lesssim R^2 \int_{\Omega 2} |\nabla u(X)|^2 \, dX + R^{-n+4}. \end{aligned}$$

From (4.10), Lemma 4.1 with $d = R$ becomes

$$\begin{aligned} \int_{\Omega_2} |\nabla\nabla u(X)|^2 \leq C\Big\{&R^{-(n-1)/p} + R^{-1}\|\nabla u\|_{L^2(\Omega_3)} \|\nabla\nabla u\|_{L^2(\Omega_3)} \\ &+ R^{-2}\|u\|_{L^2(\Omega_3)} \|\nabla\nabla u\|_{L^2(\Omega_3)}\Big\}. \end{aligned} \tag{4.11}$$

By Lemma 4.4, and α to be determined later,

$$\begin{aligned}\|\nabla u\|_{L^2(\Omega_3)} &\leq C_\alpha R^{n/2} R^{-(n-1)/p} \|N(\nabla u)\|_{L^p(\partial D)} + \alpha R \|\nabla\nabla u\|_{L^2(\Omega_3)} \\ &\leq C_\alpha R^{n/2-(n-1)/p} + \alpha R \|\nabla\nabla u\|_{L^2(\Omega_3)}.\end{aligned}$$

Because $\|u\|_{L^2(\Omega_3)} \leq C\{R^{-1}\|\nabla u\|_{L^2(\Omega_3)} + R^{(-n+4)/2}\}$, inequality (4.11) becomes

$$\begin{aligned}\int_{\Omega_2} |\nabla\nabla u|^2 \leq C\{&R^{-(n-1)/p} + C_\alpha R^{(n-2)/2} R^{-(n-1)/p} \|\nabla\nabla u\|_{L^2(\Omega_3)} \\ &+ \alpha \|\nabla\nabla u\|_{L^2(\Omega_3)} + R^{-n/2} \|\nabla\nabla u\|_{L^2(\Omega_3)}\}. \qquad (4.12)\end{aligned}$$

Now,

$$\|\nabla\nabla u\|_{L^2(\Omega_3)} \leq C\left(R \int_{\partial\Omega_3 \cap \partial D} (N_2(\nabla\nabla u))^2 \, d\sigma\right)^{1/2}.$$

With this estimate, (4.12) becomes

$$\begin{aligned}\int_{\Omega_2} |\nabla\nabla u(x)|^2 \, dX \leq C\Bigg\{&R^{-(n-1)/p} + C_\alpha R^{(n-2)/2} R^{-(n-1)/p} R^{1/2} \\ &\cdot \left(\int_{\partial\Omega_3 \cap \partial D} N_2(\nabla\nabla u)^2 \, d\sigma\right)^{1/2} + \alpha R \int_{\partial\Omega_3 \cap \partial D} N_2(\nabla\nabla u)^2 \, d\sigma \\ &+ R^{-(n+1)/2} \left(\int_{\partial\Omega_3 \cap \partial D} N_2(\nabla\nabla u)^2 \, d\sigma\right)^{1/2}\Bigg\}.\end{aligned}$$

Observe now that $C_\alpha \sim 1/\alpha^M$ for the M of Lemma 4.4 and thus we choose $\alpha = R^{-\varepsilon/2M}$ and recall that $p = (n-1)/(n-2+\varepsilon)$. Hence,

$$\begin{aligned}\int_{\Omega_2} |\nabla\nabla u(x)|^2 \, dX \leq C\Bigg\{&R^{-(n-2+\varepsilon)} + R^{(-n+3-\varepsilon)/2} \left(\int_{\partial\Omega_3 \cap \partial D} N_2(\nabla\nabla u)^2 \, d\sigma\right)^{1/2} \\ &+ R^{1-\varepsilon/2M} \int_{\partial\Omega_3 \cap \partial D} N_2(\nabla\nabla u)^2 \, d\sigma\Bigg\}.\end{aligned}$$

By (4.8) and (4.9), the above estimate yields

$$\int_{\Delta_R} N_2(\nabla\nabla u)\,d\sigma \le R^{(n-1)/2}\left(\int_{\partial\Omega_1\cap\partial D} N_2(\nabla\nabla u)^2\,d\sigma\right)^{1/2}$$

$$\le C\left\{R^{-\varepsilon/2} + R^{(n-1)/4}R^{-\varepsilon/4}\left(\int_{\partial\Omega_3} N_2(\nabla\nabla u)^2\,d\sigma\right)^{1/4} + R^{(n-1)/2}R^{-\varepsilon/4}\left(\int_{\partial\Omega_3} N_2(\nabla\nabla u)^2\,d\sigma\right)^{1/2}\right\}. \tag{4.13}$$

Let

$$\alpha_j = \int_{\{2^j < |Q| \le 2^{j+1}\}} N_2(\nabla\nabla u)\,d\sigma$$

and

$$\beta_j = \left(\int_{\{2^j < |Q| \le 2^{j+1}\}} N_2(\nabla\nabla u)^2\,d\sigma\right)^{1/2} \qquad \text{for } j \ge N,$$

where N will be determined later. In this notation, and with $R = 2^j$,

$$\int_{\Delta_R} N_2(\nabla\nabla u)\,d\sigma = \alpha_j$$

and

$$\left(\int_{\partial\Omega_3(R)} N_2(\nabla\nabla u)^2\,d\sigma\right)^{1/2} \le \beta_{j-1} + \beta_j + \beta_{j+1}.$$

Therefore (4.17) becomes

$$\sum_{j\ge N} \alpha_j \le \sum_{j\ge N} (2^j)^{(n-1)/2}\beta_j$$

$$\le C\sum_{j\ge N} 2^{-\varepsilon_j/2} + C\sum_{j\ge N}(2^j)^{(n-1)/4}2^{-\varepsilon_j/2}(\beta_{j-1}+\beta_j+\beta_{j+1})^{1/2} + C\sum_{j\ge N}(2^j)^{(n-1)/2}2^{-\varepsilon_j/4}(\beta_{j-1}+\beta_j+\beta_{j+1}). \tag{4.14}$$

This inequality shows that $\Sigma_{j \geq N} \alpha_j \leq C'$, and since $\Sigma_{j \leq N} \alpha_j \leq C_0$, this completes the proof. To see this, observe that each β_j is bounded from above by an absolute constant and that N may be chosen sufficiently large so that the last term in (4.18) is less than $\frac{1}{2}(\Sigma_{j \geq N} (2^j)^{(n-1)/2} \beta_j)$. The second term in (4.18) is estimated using Cauchy–Schwarz.

At this point, the passage from solvability in $H^1_{1,\mathrm{at}}$ above a graph to solvability on bounded C^1 is achieved in the same way it was achieved in dimension 3 for Lipschitz domains in [P–V1]. Here the main fact we are using about C^1 domains is that a covering by cylinders of the domain may be chosen with the cylinders small enough so that the Lipschitz constant of each local graph is as small as we wish. The localization arguments needed to conclude the weak maximum principle from the $H^1_{1,\mathrm{at}}$ regularity result (and its implications thereby for the Green's function) are discussed in the following section.

§5. The maximum principle on arbitrary Lipschitz domains

In this section we give the localization arguments which prove, from the corresponding results on starlike domains, that (M.P.) is valid for Lipschitz domains in $\mathbb{R}^3$. These same arguments can be used to show that (M.P.) is valid for any C^1 domain in $\mathbb{R}^n$, so we confine ourselves to proving the following.

THEOREM 5.1. *Let D be a Lipschitz domain in $\mathbb{R}^3$ and u be the solution to the L^2 Dirichlet problem for Δ^2 in D satisfying $|\nabla u| \in L^\infty(\partial D)$. Then there is a constant C, which depends only on the Lipschitz character of D, such that*

$$\sup_{X \in D} |\nabla u(X)| \leq C \|\nabla u\|_{L^\infty(\partial D)}. \tag{5.2}$$

Proof of 5.1. It suffices to prove (5.2) when the supremum is taken over all $X \in D$ with dist $(X, \partial D) < \varepsilon$. (We may also assume that diam $(D) = 1$.) In this case a C^∞ surface C contained entirely in $D \cap \{X : \operatorname{dist}(X, \partial D) < \varepsilon\}$ may be constructed with $|\nabla u| \in L^\infty(C)$ and u biharmonic in the domain determined by C. By the Miranda–Agmon maximum principle for smooth domains we could conclude that $|\nabla u| \in L^\infty(D)$. So, for some $\varepsilon > 0$, it must be shown that

$$\sup_{\substack{X \in D \\ \operatorname{dist}(X, \partial D) < \varepsilon}} |\nabla u(X)| \leq C \|\nabla u\|_{L^\infty(\partial D)}. \tag{5.3}$$

We now observe that (5.3) follows from estimate (2.6) of Lemma 2.5 when $u \equiv 0$ on ∂D and so we further assume that our biharmonic function satisfies $\partial u / \partial N \equiv 0$ on ∂D, $u \equiv f$ on ∂D with $|\nabla_T f| \in L^\infty(\partial D)$.

Let $\{Z_i\}_{j=1}^{k}$ be finitely many double truncated right circular cylinders such that $\partial D \subseteq \bigcup_{i=1}^{k}(\frac{1}{2}Z_i \cap \partial D)$ and such that, for all i, $4Z_i \cap D$ is a starlike Lipschitz domain. Fix a cylinder Z_i. For $\frac{1}{2} \leq \alpha \leq 4$, let Δ_α denote $\alpha Z_i \cap \partial D$ and Ω_α denote $\alpha Z_i \cap D$. Let $r \equiv \text{rad}(\Delta)$, let B be a ball centered at the center of the surface disk Δ_1 of radius r and let ψ be a C^∞ function satisfying $\psi \equiv 1$ on B and $\psi \equiv 0$ on ${}^c(2B)$. We now define two biharmonic functions u_1 and u_2 in D by specifying $\partial u_1/\partial N = \partial u_2/\partial N = 0$ and $u_1|_{\partial D} = \psi(f - c_0)$ and $u_2|_{\partial D} = (1-\psi)(f - c_0)$ where $c_0 = \int_{\Delta_2} f(d\sigma/\sigma(\Delta_2))$. Then, in D, $u = u_1 + u_2 + c_0$ and so $\Delta u = \Delta u_1 + \Delta u_2$. We want to show that u_1 and u_2 have bounded gradients at points in Ω_2 within ε of the boundary of D.

First observe that u_1 has (L_1^2, L^2) Dirichlet data in Ω_2 since

$$\int_{\partial\Omega_2} |\nabla u_1|^2\, d\sigma \leq \int_{\Delta_2} N(\nabla u_1)^2\, d\sigma \leq \int_{\partial D} |\nabla u_1|^2\, d\sigma = \int_{\partial D} |\nabla_T u_1|^2\, d\sigma$$

and $\nabla_T u_1 = (\nabla_T \psi)(f - C_0) + \psi \nabla_T f$.

Now $|\nabla_T \psi| \leq C r^{-1}$ and

$$\frac{1}{r^2}\int_{2\Delta} |f - C_0|^2\, d\sigma \leq C \int_{2\Delta} |\nabla_T f|^2\, d\sigma \leq C \|\nabla_T f\|_\infty^2 \sigma(\Delta)$$

so that $|\nabla u_1| \in L^2(\partial\Omega_2)$.

We turn now to the L^∞ estimates on $|\nabla u_1|$. If $X \in \partial\Omega_2$ and $\text{dist}(X, \partial D) \geq C_1 r$ then $|\nabla u_1(X)| \leq C(\fint_{\Delta_2}(N(u_1)^2\, d\sigma)^{1/2} \leq \|\nabla_T f\|_\infty$ by the L^2 estimate above. To bound $|\nabla u_1(X)|$ when X is near ∂D requires the Hölder continuity of the gradient. That is, consider u_1 in the (starlike) domain Ω_4 where (L_1^2, L^2) Dirichlet data satisfies the same estimates obtained when we consider u_1 as a function in Ω_2. On $\Delta_4 \backslash \Delta_1$, both $u_1|_{\partial D}$ and $\partial u_1/\partial N$ vanish. Let $G_4(X, Y)$ be the Green's function for the operator Δ^2 in Ω_4 and write

$$\begin{aligned} u_1(X) &= \int_{\partial\Omega_4} \frac{\partial u_1}{\partial N}(Q)\, \Delta_Q G_4(X, Q)\, dQ + \int_{\partial\Omega_4} u_1(Q) \frac{\partial}{\partial N_Q} \Delta_Q G_4(X, Q)\, dQ \\ &= I + II. \end{aligned}$$

Fix $X \in \partial\Omega_2$ with $\text{dist}(X, \partial D) < c_1 r$. Then

$$I \leq \left\| \frac{\partial u_1}{\partial N} \right\|_{L^2(\partial\Omega_4)} \cdot \left(\int_{\partial\Omega_4 \backslash \Delta_4} |\Delta_Q G_4(X, Q|^2\, dQ \right)^{1/2},$$

since $\partial u_1/\partial N \equiv 0$ on Δ_4. By the basic estimate,

$$\left(\int_{\partial\Omega_4\setminus\Delta_4} |\Delta_Q G_4(X,Q|^2\,dQ\right)^{1/2} \le C\left(\frac{\delta(X)}{r}\right)^{1+\eta},$$

and $\|\partial u_1/\partial N\|_{L^2} \le \|\nabla_T f\|_\infty \sigma(\Delta)^{1/2} = r\|\nabla_T r\|_\infty$. Therefore,

$$I \le C\delta(X)\|\nabla_T f\|_\infty \left(\frac{\delta(X)}{r}\right)^{\eta}.$$

Because the domain Ω_4 is starlike, term II has the same estimates. (Observe that u_1 vanishes on $\Delta_4\setminus\Delta_2$ and the Hölder estimates will be applied here. So even though u_1 does not vanish Δ_1, we are in a position to apply the Hölder estimate away from Δ_1.) Hence,

$$|\nabla u_1(X)| \le C\left(\frac{\delta(X)}{r}\right)^{\eta}\|\nabla_T f\|_\infty \le C\|\nabla_T f\|_\infty$$

for dist $(X, \partial D) \le c_1 r$.

Consider now the function u_2 in the domain Ω_1. We have

$$\int_{\partial\Omega_{1/2}} |\nabla u_2(P)|^2\,dP \le \int_{\Delta} N(\nabla u_2)^2\,d\sigma(P) \le C\int_{\partial D} |\nabla u_2(P)|^2\,d\sigma(P)$$
$$\le \|\nabla_T f\|_\infty^2 \sigma(\partial D).$$

In addition, both u_2 and $\partial u_2/\partial N$ vanish on Δ_1. For any $X \in \Omega_1$, we may write

$$u_2(X) = \int_{\partial\Omega_1\setminus\Delta_1} \frac{\partial u_2}{\partial N}\Delta_Q G_1(X,Q)\,dQ + \int_{\partial\Omega_1\setminus\Delta_1} u_2(Q)\frac{\partial}{\partial N_Q}\Delta_Q G_1(X,Q)\,dQ.$$

Thus if ε is sufficiently small ($\varepsilon < \min\{\sigma(\partial D), r\}$) and $X \in \Omega_{1/2}$ with dist $(X, \partial D) < \varepsilon$, these L^2 estimates on the starlike domain Ω_1 together with the basic estimate will show that $|\nabla u_2(X)| \le C\|\nabla_T f\|_\infty$.

Thus $|\nabla u(X)| \le C\|\nabla_T f\|_\infty$ for $X \in \frac{1}{2}Z \cap D$ when dist $(X, \partial D) < \varepsilon$ and the argument may be repeated in each of the finitely many cylinders.

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