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Coverings of 1-convex manifolds with 1-dimensional exceptional set

MIHNEA COLTOIU

To the memory of C. Banica

§0. Introduction

By a classical result of K. Stein [15] it is known that every covering \tilde{X} of a Stein manifold X is itself Stein. The aim of this paper is to consider the case when X is a 1-convex manifold with 1-dimensional exceptional set and to study the convexity properties of \tilde{X} .

In [4] Grauert and Docquier have introduced several notions of convexity. For example a complex manifold Y is said to be p_3 -convex if it can be exhausted by a sequence $\{Y_\nu\}_{\nu \in \mathbb{N}}$ of relatively compact strongly pseudoconvex domains. Our main result is Theorem 2 which says that every covering \tilde{X} of a 1-convex manifold X with 1-dimensional exceptional set is p_3 -convex.

We recall also [4] that a complex manifold Y is said to be p_1 -convex if there exists a smooth plurisubharmonic exhaustion function $\varphi : Y \rightarrow \mathbb{R}$. Obviously every holomorphically convex manifold is p_1 -convex. In §3 we exhibit an example of a strongly pseudoconvex surface X whose universal covering \tilde{X} fails to be p_1 -convex, in particular \tilde{X} is not holomorphically convex. So, if we study the convexity properties of the coverings of 1-convex manifolds with 1-dimensional exceptional set, a natural condition is the p_3 -convexity.

For embeddable 1-convex manifolds X (e.g. strongly pseudoconvex surfaces) Napier [11] has shown that their coverings \tilde{X} have good meromorphic convexity properties: if $\{a_\nu\}_{\nu \in \mathbb{N}}$ is a discrete sequence of points in \tilde{X} then there exists a meromorphic function f on \tilde{X} which is holomorphic near $\{a_\nu\}_{\nu \in \mathbb{N}}$ and unbounded on $\{a_\nu\}_{\nu \in \mathbb{N}}$.

§1. Preliminaries

We assume all complex manifolds Hausdorff and countable at infinity. In [2] the following result is proved:

THEOREM 1. *Let X be a 1-convex manifold and $S \subset X$ its exceptional set. Then there is a strongly plurisubharmonic exhaustion function $\varphi : X \rightarrow [-\infty, \infty)$ such that $S = \{\varphi = -\infty\}$. Moreover φ can be chosen such that $\exp \varphi$ is smooth.*

Using the above result and a method of LeBarz [8] (see also ([13], p. 494)) we prove:

PROPOSITION 1. *Let X be a 1-convex manifold with exceptional set S and $p : \tilde{X} \rightarrow X$ any covering. Then there is a strongly plurisubharmonic function $\tilde{\varphi} : \tilde{X} \rightarrow [-\infty, \infty)$ such that $p^{-1}(S) = \{\tilde{\varphi} = -\infty\}$, $\exp \tilde{\varphi}$ is smooth and, for any open neighbourhood U of S , the restriction $\tilde{\varphi}|_{\tilde{X} \setminus p^{-1}(U)}$ is an exhaustion function on $\tilde{X} \setminus p^{-1}(U)$.*

Proof. We may assume that X and \tilde{X} are connected. Let $\{U_i\}_{i \in \mathbb{N}}$ be a locally finite open covering of X such that $U_i \subset \subset X$ and each U_i is biholomorphic to a ball (so U_i is evenly covered). We get a decomposition $p^{-1}(U_i) = \bigcup_k W_{i,k}$ into disjoint open sets with $W_{i,k}$ biholomorphic to U_i via the projection map p . Let $\{\varphi_i\}_{i \in \mathbb{N}}$ be a partition of unity corresponding to $\{U_i\}_{i \in \mathbb{N}}$ and define $f : \tilde{X} \rightarrow \mathbb{R}$ as follows: fix some W_{i_0, k_0} and define $\lambda_{i,k}$ as the length of the shortest chain $W_{i_0, k_0}, W_{i_1, k_1}, \dots, W_{i_s, k_s}$ such that $W_{i_v, k_v} \cap W_{i_{v+1}, k_{v+1}} \neq \emptyset$ and $(i_s, k_s) = (i, k)$. If we set $f = \sum_{i,k} (\varphi_i \circ p) \lambda_{i,k}$ then one has ([13], p. 494):

- (a) f is a smooth exhaustion function on \tilde{X} .
- (b) The levi form $L(f)|_{p^{-1}(U_i)}$ is bounded from below.

Let $\varphi : X \rightarrow [-\infty, \infty)$ be a strongly plurisubharmonic exhaustion function having the properties stated in Theorem 1. By the conditions (a) and (b) we easily see that there exists a smooth convex strictly increasing function $\theta : \mathbb{R} \rightarrow \mathbb{R}$, $\theta(t) = At$ ($A > 0$) near $-\infty$, $\lim_{t \rightarrow \infty} \theta(t) = \infty$ such that $\tilde{\varphi} = \theta \circ \varphi \circ p + f$ is strongly plurisubharmonic on \tilde{X} . From the definition of $\tilde{\varphi}$ it follows that it has all the required properties. Thus the proof of Proposition 1 is complete.

Another important ingredient for the proof of Theorem 2 is the following result due to Siu ([12], Corollary 1):

PROPOSITION 2. *Let A be a closed complex submanifold of a complex manifold Y . If A is Stein, then there exists a biholomorphic map from a neighbourhood W of A in Y onto an open neighbourhood of the zero cross section of the normal bundle of A in Y such that its restriction to A agrees with the canonical map from A onto the zero cross section. As a consequence, there is a holomorphic retract from W onto A .*

In fact we shall need this result in the case when A is a connected non-compact complex curve, which, by Behnke–Stein theorem, is Stein. The curve A will be obtained by removing finitely many points from the exceptional curve S of X .

§2. Proof of the main result

We begin by recalling the following definition [4]: a complex manifold Y is said to be p_3 -convex if there exists an increasing sequence $\{Y_\nu\}_{\nu \in \mathbb{N}}$ of relatively compact strongly pseudoconvex domains such that $Y = \bigcup_{\nu \in \mathbb{N}} Y_\nu$.

The aim of this paragraph is to prove the following:

THEOREM 2. *Let X be a 1-convex manifold with 1-dimensional exceptional set S and $p : \tilde{X} \rightarrow X$ any covering. Then \tilde{X} is p_3 -convex.*

The following lemma shows that it suffices to prove the above theorem for a suitable small neighbourhood of the exceptional set S .

LEMMA 1. *Let X be a 1-convex manifold, S its exceptional set and $p : \tilde{X} \rightarrow X$ any covering. Assume that there exists an open neighbourhood U of S in X such that $\tilde{U} = p^{-1}(U)$ is p_3 -convex. Then \tilde{X} is p_3 -convex.*

Proof. Let $K \subset \tilde{X}$ be any compact subset of \tilde{X} . We prove the existence of a strongly pseudoconvex neighbourhood $D \subset \subset \tilde{X}$ of K . Let $\tilde{\varphi} : \tilde{X} \rightarrow [-\infty, \infty)$ be a strongly plurisubharmonic function on \tilde{X} having the properties stated in Proposition 1 and let $\alpha > 0$ be such that $K \subset \{\tilde{\varphi} < \alpha\}$. Choose also strongly pseudoconvex neighbourhoods $U_1 \subset \subset U_2$ of S such that $\tilde{U}_1 = p^{-1}(U_1)$ and $\tilde{U}_2 = p^{-1}(U_2)$ are p_3 -convex. Since $\tilde{\varphi}|_{\tilde{X} \setminus \tilde{U}_1}$ is an exhaustion function there is a compact set L containing K with $\{\tilde{\varphi} < \alpha\} \cap \mathbb{C}L \subset \tilde{U}_1$ and because \tilde{U}_2 is p_3 -convex there is a strongly pseudoconvex domain $M \subset \subset \tilde{X}$, $M \subset \tilde{U}_2$ such that $L \cap \tilde{U}_2 \subset M$. If we define D by

$$D = \begin{cases} \{\tilde{\varphi} < \alpha\} & \text{in } \mathbb{C}\tilde{U}_1 \\ \{\tilde{\varphi} < \alpha\} \cap M & \text{in } \tilde{U}_2 \end{cases}$$

then obviously D is a strongly pseudoconvex neighbourhood of K , so the proof of Lemma 1 is complete.

LEMMA 2. *Let W be a Stein manifold and $A \subset W$ a closed complex submanifold. Then there exists a smooth plurisubharmonic function $\varphi : W \rightarrow [0, \infty)$ such that $A = \{\varphi = 0\}$, φ is strongly plurisubharmonic on $W \setminus A$ and*

(α) *if $U \subset W$ is any open subset, $\psi \in C^\infty(U)$ is plurisubharmonic and its restriction $\psi|_{A \cap U}$ is strongly plurisubharmonic, then for any $\varepsilon > 0$ the function $\psi + \varepsilon\varphi$ is strongly plurisubharmonic on U .*

Proof. Let \mathcal{I} be the ideal sheaf of A and let $g_1, \dots, g_k \in \Gamma(W, \mathcal{I})$ be a set of generators of \mathcal{I} on W . If we set $h = \sum_{i=1}^k |g_i|^2$ then h is plurisubharmonic, $h \geq 0$

and $A = \{h = 0\}$. Also if $z \in A$ then the Levi form $L(h)(t) > 0$ for any vector $t \in T_z W \setminus T_z A$. If $g_{k+1}, \dots, g_m \in \Gamma(W, \mathcal{J})$ give an immersion at any point $z \in W \setminus A$ then $\varphi = \sum_{i=1}^m |g_i|^2$ has all the required properties.

Let us recall now some elementary results of algebraic topology which we shall need in the proof of Theorem 2.

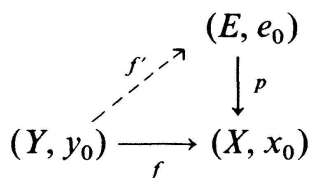
If V is a complex manifold and $A \subset V$ is a closed analytic subset (or, more generally, a semi-analytic subset), by the triangulation theorem [9] and the results in [14], the following conditions are equivalent:

- (a) the inclusion $A \hookrightarrow V$ is a weak homotopic equivalence,
- (b) the inclusion $A \hookrightarrow V$ is a homotopic equivalence,
- (c) A is a deformation retract of V ,
- (d) A is a strong deformation retract of V .

In order to lift some holomorphic retracts from the base space to the covering space we shall need also the following topological results (see for instance [6]):

PROPOSITION 2 (Covering homotopy theorem ([6], p. 18)). *Let $(E, e_0), (X, x_0)$ be topological spaces with base points and $p : (E, e_0) \rightarrow (X, x_0)$ a covering map. Let (Y, y_0) be arbitrary and $f : (Y, y_0) \rightarrow (X, x_0)$ a map which has a lifting $f' : (Y, y_0) \rightarrow (E, e_0)$. Then every homotopy $F : Y \times I \rightarrow X$ with $F(y, 0) = f(y)$ for all $y \in Y$ can be lifted to a homotopy $F' : Y \times I \rightarrow E$ with $F'(y, 0) = f'(y)$ for all $y \in Y$ (here I denotes the interval $[0, 1]$).*

PROPOSITION 3 (Lifting criterion ([6], p. 22)). *Assume that the topological spaces E, X, Y are connected and locally pathwise connected. Consider the diagram*



where p is a covering map and f is arbitrary. Then there exists a lifting f' of f ($p \circ f' = f$) iff $f_* \pi_1(Y, y_0) \subset p_* \pi_1(E, e_0)$.

One application of these two results is described in the following:

REMARK 1. Let V be a complex manifold, $A \subset V$ a closed complex submanifold, $p : \tilde{V} \rightarrow V$ a covering map and $\tilde{A} = p^{-1}(A)$. If A is a deformation retract of V then \tilde{A} is also a deformation retract of \tilde{V} .

This follows easily from the Covering homotopy theorem and the equivalence of the conditions (a), (b), (c), (d). Moreover every connected component \tilde{A}_i of \tilde{A} is contained in precisely one connected component \tilde{V}_i of \tilde{V} and every connected component of \tilde{V} contains precisely one connected component of \tilde{A} . For every i the set \tilde{A}_i is a deformation retract of \tilde{V}_i .

REMARK 2. Let $V, \tilde{V}, A, \tilde{A}$ be as above and assume that A is a deformation retract of V . Let $r : V \rightarrow A$ be any continuous retract (not necessarily a deformation retract). Then there is a continuous retract $\tilde{r} : \tilde{V} \rightarrow \tilde{A}$ such that the diagram

$$\begin{array}{ccc}
 \tilde{V} & \xrightarrow{\tilde{r}} & \tilde{A} \\
 p \downarrow & & \downarrow p \\
 V & \xrightarrow{r} & A
 \end{array} \tag{*}$$

is commutative. In particular, if r is a holomorphic retract then so is \tilde{r} .

This can easily be seen in the following way:

First we may assume that all the spaces $V, \tilde{V}, A, \tilde{A}$ are connected. From the Lifting criterion, one can conclude that an \tilde{r} making the above diagram commutative exists, noting that the induced map $\pi_1(\tilde{A}) \rightarrow \pi_1(\tilde{V})$ is an isomorphism (by Remark 1). Indeed, let $i : A \hookrightarrow V, \tilde{i} : \tilde{A} \hookrightarrow \tilde{V}$ be the inclusions maps and consider the commutative diagram

$$\begin{array}{ccc}
 \tilde{V} & \xleftarrow{\tilde{i}} & \tilde{A} \\
 p \downarrow & & \downarrow p \\
 V & \xleftarrow{i} & A
 \end{array} \tag{**}$$

If $\alpha \in \pi_1(\tilde{V})$ then there is a unique $\beta \in \pi_1(\tilde{A})$ with $\tilde{i}_*(\beta) = \alpha$. It follows that $r_*(p_*(\alpha)) = r_*(p_*(\tilde{i}_*(\beta))) = r_*(i_*(p_*(\beta))) = p_*(\beta)$ where the last equality holds because r is a retract. Hence the conditions of the Lifting criterion are satisfied and the map \tilde{r} making (*) commutative exists. Restricting the diagram (*) to A , the uniqueness theorem for liftings ([6], p. 17) shows that the restriction $\tilde{r}|_A = \text{id}$ so \tilde{r} is a retract. Clearly \tilde{r} is holomorphic if r has this property.

LEMMA 3. Let X be a 1-convex manifold with 1-dimensional exceptional set S and $p : \tilde{X} \rightarrow X$ any covering. Then there is an open neighbourhood U of S such that $\tilde{U} = p^{-1}(U)$ is p_3 -convex.

Proof. We may assume that X, \tilde{X}, S are connected and that the covering $p : \tilde{X} \rightarrow X$ has infinite fibers. Let $M = \{s_1, \dots, s_k\} \subset S$ be a finite set such that $A = S \setminus M$ is non-singular and Stein. Since A is a closed Stein submanifold of $X \setminus M$

it follows from Proposition 2 that there exists an open neighbourhood W of A in $X \setminus M$ and a biholomorphic map from W onto a neighbourhood of the zero cross section of the normal bundle $N_{A|X \setminus M}$ such that its restriction to A agrees with the canonical map from A onto the zero cross section. So we have an induced holomorphic retract $r : W \rightarrow A$ and we also may assume that W is Stein [12].

We set $\tilde{S} = p^{-1}(S)$, $\tilde{A} = p^{-1}(A)$ and let V be an open neighbourhood of A in $X \setminus M$, $\bar{V} \subset W$ (adherence with respect to $X \setminus M$), such that A is a deformation retract of V . If we denote $\tilde{V} = p^{-1}(V)$ and we consider r as a map $V \rightarrow A$ then r can be lifted (by Remark 2) to a holomorphic retract $\tilde{r} : \tilde{V} \rightarrow \tilde{A}$ such that the diagram

$$\begin{array}{ccc}
 \tilde{V} & \xrightarrow{\tilde{r}} & \tilde{A} \\
 \downarrow p & & \downarrow p \\
 V & \xrightarrow{r} & A
 \end{array}
 \tag{*}$$

is commutative. We may also assume that W and V are small enough such that if $K \subset\subset \tilde{A}$ then $\tilde{r}^{-1}(K) \subset\subset \tilde{X}$ (condition (C)). We choose balls $T_1, \dots, T_k \subset\subset X$ centered at s_1, \dots, s_k with $\bar{T}_i \cap \bar{T}_j = \emptyset$ if $i \neq j$ such that some neighbourhoods of $\bar{T}_1, \dots, \bar{T}_k$ are evenly covered and $T_1 \cap S, \dots, T_k \cap S$ are connected. Let also $L_1 \subset\subset T_1, \dots, L_k \subset\subset T_k$ be sufficiently small concentric balls such that:

- (1) $x \in S \setminus T_i$ implies $r^{-1}(x) \cap \bar{L}_i = \emptyset$, $i = 1, \dots, k$,
- (2) $r^{-1}(L_i \cap A) \subset T_i$, $i = 1, \dots, k$.

Clearly these conditions may easily be satisfied if W is chosen from the beginning small enough.

We now consider an exhaustion $\{D_\lambda\}_{\lambda \in \mathbb{N}}$ of \tilde{S} by relatively compact domains with smooth boundary, $D_\lambda \subset\subset D_{\lambda+1}$, such that $\partial D_\lambda \cap \text{Sing}(\tilde{S}) = \emptyset$ and ∂D_λ does not intersect $p^{-1}(S \cap \bar{T})$, where $T = T_1 \cup \dots \cup T_k$. Since \tilde{S} is 1-dimensional each D_λ is strongly pseudoconvex. We set $U_1 = V \cup L_1 \cup \dots \cup L_k$ and $\tilde{U}_1 = p^{-1}(U_1)$. We first describe an exhaustion of \tilde{U}_1 by relatively compact domains in \tilde{X} whose boundaries (relative to \tilde{U}_1) are pseudoconvex (not necessarily strongly pseudoconvex). Then, by a simple perturbation argument and Lemma 2, we may achieve that their boundaries (relative to \tilde{U}_1) become strongly pseudoconvex. Finally, replacing U_1 by a sufficiently small strongly pseudoconvex neighbourhood U of the exceptional set S , we get the desired exhaustion of \tilde{U} by domains whose boundaries (relative to \tilde{X}) are strongly pseudoconvex.

In order to obtain the exhaustion of \tilde{U}_1 we define the open sets $\tilde{D}_\lambda \subset \tilde{U}_1$, $\lambda \in \mathbb{N}$, as follows:

Let $\{b_1, \dots, b_m\}$, where $m = m(\lambda)$, be the subset of $p^{-1}(M)$ consisting of those points in $p^{-1}(M)$ contained in D_λ . Consider the decomposition into connected components $p^{-1}(L_i) = \bigcup_{j \in \mathbb{N}} L_i^j$, $i = 1, \dots, k$ and denote by B_1, \dots, B_m those connected components L_i^j containing b_1, \dots, b_m . We set

$\tilde{D}_\lambda = \tilde{r}^{-1}(D_\lambda \setminus \{b_1, \dots, b_m\}) \cup B_1 \cup \dots \cup B_m$. Clearly $\{\tilde{D}_\lambda\}_{\lambda \in \mathbb{N}}$ is an increasing sequence of open subsets of \tilde{U}_1 and $\bigcup_{\lambda \in \mathbb{N}} \tilde{D}_\lambda = \tilde{U}_1$. To see that $\tilde{D}_\lambda \subset \subset \tilde{X}$ it suffices to verify that $\tilde{r}^{-1}(D_\lambda \setminus \{b_1, \dots, b_m\}) \subset \subset \tilde{X}$ and by the condition (C) it is enough to show that $\tilde{r}^{-1}(B_t \cap \tilde{S} \setminus b_t) \subset \subset \tilde{X}$, $t = 1, \dots, m$. But this follows immediately from the commutativity of the diagram (*) and the condition (2). Now we study the pseudoconvexity of the boundary of \tilde{D}_λ (relative to \tilde{U}_1). First we remark that by the condition (1) and the commutativity of the diagram (*) it follows that $\bar{B}_t \cap \tilde{V} \subset \tilde{r}^{-1}(D_\lambda \setminus \{b_1, \dots, b_m\})$ and if E is another component L_i^j , different from B_1, \dots, B_m , then $\bar{E} \cap \tilde{r}^{-1}(D_\lambda \setminus \{b_1, \dots, b_m\}) = \emptyset$. Hence the boundary of \tilde{D}_λ (relative to \tilde{U}_1) is precisely the boundary $\tilde{r}^{-1}(D_\lambda \setminus \{b_1, \dots, b_m\})$ (relative to \tilde{V}). To describe this boundary we choose an open neighbourhood M_λ of ∂D_λ in \tilde{S} , $\bar{M}_\lambda \cap p^{-1}(S \cap \bar{T}) = \emptyset$, and a smooth strongly subharmonic function φ_λ defining D_λ in M_λ . Then $\varphi_\lambda \circ \tilde{r}$ is a plurisubharmonic defining function for \tilde{D}_λ in \tilde{U}_1 so each \tilde{D}_λ has a pseudoconvex boundary (relative to \tilde{U}_1). To obtain the desired exhaustion by strongly pseudoconvex domains we need Lemma 2. By this lemma there is a smooth plurisubharmonic function $\varphi : W \rightarrow [0, \infty)$ such that: $A = \{\varphi = 0\}$, φ is strongly plurisubharmonic on $W \setminus A$ and φ has the property (α). If we set $\tilde{\varphi} = \varphi \circ p|_{\tilde{V}}$ then for any $\varepsilon > 0$ the function $\varphi_\lambda \circ \tilde{r} + \varepsilon \tilde{\varphi}$ is strongly plurisubharmonic on $\tilde{r}^{-1}(M_\lambda)$ and its restriction to M_λ is φ_λ . The functions $\varphi_\lambda \circ \tilde{r} + \varepsilon_\lambda \tilde{\varphi}$ with $\varepsilon_\lambda > 0$ sufficiently small define in an obvious way the exhaustion of \tilde{U}_1 by domains \tilde{D}'_λ having strongly pseudoconvex boundaries relative to \tilde{U}_1 (after replacing V if necessary by a smaller open subset V' with $A \subset V'$ and $\bar{V}' \subset V$, where the adherence is taken with respect to $X \setminus M$). If we choose now a strongly pseudoconvex neighbourhood U of S , $U \subset \subset U_1$, it follows then immediately, from our previous remarks, that $\tilde{U} = p^{-1}(U)$ is p_3 -convex. The proof of Lemma 3 is complete.

Proof of Theorem 2. Theorem 2 is a direct consequence of Lemma 1 and Lemma 3.

If X is a 1-convex manifold of dimension 2 we shall call it a 1-convex surface (or strongly pseudoconvex surface). From Theorem 2 we get:

COROLLARY 1. *Let X be a 1-convex surface. Then every covering \tilde{X} of X is p_3 -convex.*

Let now $D \subset \mathbb{C}^n$ be the half-open unit polydisc, i.e.

$$D = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1| \leq 1, |z_\nu| < 1, \nu = 2, \dots, n\}$$

and set

$$\delta D = \{(z_1, \dots, z_n) \in D \mid |z_1| = 1\}.$$

According to [4] a complex manifold Y is called p_6 -convex if there is no biholomorphic map φ from an open neighbourhood of D onto an open subset of Y satisfying the following two conditions:

- (a) $\varphi(\delta D) \subset\subset Y$,
- (b) $\varphi(D) \not\subset\subset Y$.

In [4] it is proved that every p_3 -convex manifold is p_6 -convex. From Theorem 2 it follows:

COROLLARY 2. *Let X be a 1-convex manifold with 1-dimensional exceptional set. Then every covering \tilde{X} of X is p_6 -convex.*

We also obtain easily from Theorem 2:

COROLLARY 3. *Let X be a 1-convex manifold with 1-dimensional exceptional set and \tilde{X} any covering of X . The \tilde{X} is holomorphically convex iff \tilde{X} is a proper modification of a Stein space at a discrete set.*

§3. Some counterexamples

In this paragraph we show that in general the covering spaces (even the universal covering spaces) of 1-convex surfaces are not holomorphically convex.

We recall [4] that a complex manifold Y is said to be p_1 -convex if there exists a smooth plurisubharmonic exhaustion function $\varphi : Y \rightarrow \mathbb{R}$. Clearly any holomorphically convex manifold is p_1 -convex. We shall exhibit an example of a 1-convex surface X whose universal covering \tilde{X} fails to be p_1 -convex, in particular X is not holomorphically convex. To construct X we shall need the following two results:

GRAUERT'S CRITERION [5]. *Let X be a 2-dimensional complex manifold and $S \subset X$ a 1-dimensional connected compact analytic subset with irreducible components S_1, \dots, S_m . Then S is exceptional iff the intersection matrix $(S_i S_j)$ is negative definite.*

LEMMA 4. *Let X be a complex manifold and $S \subset X$ an exceptional set. Then there exists a strongly pseudoconvex neighbourhood V of S such that S is a deformation retract of V .*

Proof. Let $\varphi \geq 0$ be a real-analytic plurisubharmonic function in a neighbourhood of S such that $S = \{\varphi = 0\}$ and φ is strongly plurisubharmonic outside S . Such a function exists because S is exceptional. Since φ is real-analytic it follows from the ‘‘Curve selection lemma’’ [7] that for $\varepsilon_0 > 0$ small enough φ has no critical points in $\{\varphi < \varepsilon_0\} \setminus S$. So, for any $0 < \varepsilon < \varepsilon_0$, $\{\varphi \leq \varepsilon\}$ is a deformation retract of $V = \{\varphi < \varepsilon_0\}$. Choose an open neighbourhood V_1 of S , $V_1 \subset \subset \{\varphi < \varepsilon_0\}$ such that S is a deformation retract of V_1 and let $0 < \varepsilon_1 < \varepsilon_0$ be such that $\{\varphi \leq \varepsilon_1\} \subset V_1$. Because S is a deformation retract of V_1 and $\{\varphi \leq \varepsilon_1\}$ is a deformation retract of V it follows that the inclusion $S \hookrightarrow V$ is a weak homotopic equivalence. But (S, V) is a polyhedral pair so we deduce [14] that S is a deformation retract of V . The proof of Lemma 4 is complete.

We now begin constructing our example. First we make some remarks:

(i) Let X_1, X_2 be complex manifolds (Hausdorff), $U_1 \subset X_1$ and $U_2 \subset X_2$ open subsets and $\varphi : U_1 \rightarrow U_2$ a biholomorphic map. Let X be obtained by glueing X_1 and X_2 via the map φ . In general X is not Hausdorff but one can easily verify that the necessary and sufficient condition on X to be Hausdorff is the following: for every $x_1 \in \partial U_1$ (boundary relative to X_1) and every $x_2 \in \partial U_2$ (boundary relative to X_2) there exist open neighbourhoods $V_1 \subset X_1$ of x_1 and $V_2 \subset X_2$ of x_2 such that $\varphi(U_1 \cap V_1) \cap (U_2 \cap V_2) = \emptyset$.

(ii) We construct a complex manifold X of dimension 2 (complex surface) containing an exceptional curve S such that S has two irreducible components $S = S_1 \cup S_2$, $S_1 \cong \mathbb{P}^1$, $S_2 \cong \mathbb{P}^1$, the intersection $S_1 \cap S_2$ consists of two points and at these points S_1 and S_2 meet transversally. Let $S_1 = \mathbb{P}^1$ be the zero cross section of $\mathcal{O}(-3) = Y_1$ and $S_2 = \mathbb{P}^1$ the zero cross section of $\mathcal{O}(-3) = Y_2$. We glue suitable neighbourhoods of S_1 in Y_1 and of S_2 in Y_2 such that S_1 and S_2 meet transversally at two points. The precise construction is as follows: Let B^2 be the unit ball in \mathbb{C}^2 and $f : B^2 \rightarrow B^2$ the automorphism given by $f(z_1, z_2) = (z_2, z_1)$. Choose $p_1 \neq q_1$, $p_1, q_1 \in S_1$ and $p_2 \neq q_2$, $p_2, q_2 \in S_2$. Let $E_1 \subset \subset Y_1, F_1 \subset \subset Y_1, \bar{E}_1 \cap \bar{F}_1 = \emptyset$ be open neighbourhoods of p_1 and of q_1 respectively, such that there exist biholomorphic maps $\tau_1 : E_1 \rightarrow B^2, \psi_1 : F_1 \rightarrow B^2$. We also assume that, via the maps τ_1, ψ_1 the set S_1 corresponds to $z_1 = 0$ and the points p_1, q_1 to the origin $O \in \mathbb{C}^2$. Similarly we consider open neighbourhoods $E_2 \subset \subset Y_2, F_2 \subset \subset Y_2, \bar{E}_2 \cap \bar{F}_2 = \emptyset$ of p_2 and of q_2 respectively, and biholomorphic maps $\tau_2 : E_2 \rightarrow B^2, \psi_2 : F_2 \rightarrow B^2$ such that via these maps S_2 corresponds to $z_1 = 0$ and the points p_2, q_2 to the origin $O \in \mathbb{C}^2$. We have induced isomorphisms φ_1, φ_2 where $\varphi_1 : E_1 \rightarrow E_2$ is defined by $\varphi_1 = \tau_2^{-1} \circ f \circ \tau_1$ and $\varphi_2 : F_1 \rightarrow F_2$ by $\varphi_2 = \psi_2^{-1} \circ f \circ \psi_1$; so we get an isomorphism $\varphi = (\varphi_1, \varphi_2) : E_1 \cup F_1 \rightarrow E_2 \cup F_2$. Let V be an open neighbourhood of the zero cross section in $\mathcal{O}(-3)$ and set $X_1 = V \cup E_1 \cup F_1, X_2 = V \cup E_2 \cup F_2, U_1 = E_1 \cup F_1, U_2 = E_2 \cup F_2$. From the previous remark it follows that for sufficiently small V we can glue X_1 and

X_2 via the map φ and we get a complex manifold X containing a compact analytic curve S with two irreducible components $S = S_1 \cup S_2$, $S_1 \cong \mathbb{P}^1$, $S_2 \cong \mathbb{P}^1$, which meet transversally at two points. The intersection matrix is $\begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}$ which is negative definite. By Grauert's criterion S is exceptional and our construction is finished.

(iii) Let X be a 1-convex surface such that its exceptional curve S has two irreducible components $S = S_1 \cup S_2$, $S_1 \cong \mathbb{P}^1$, $S_2 \cong \mathbb{P}^1$ and the intersection $S_1 \cap S_2$ consists of two points where the intersection is transversal. Replacing X by a smaller strongly pseudoconvex neighbourhood of S we may assume (by Lemma 4) that S is a deformation retract of X . Let $p : \tilde{X} \rightarrow X$ be the universal covering of X . We assert that \tilde{X} is not p_1 -convex (hence it is not holomorphically convex). To see this it is enough to verify that $\tilde{S} = p^{-1}(S)$ is not p_1 -convex. Since S is a deformation retract of X , it follows (from Remark 1 which clearly extends to singular A) that \tilde{S} is a deformation retract of \tilde{X} , hence \tilde{S} is the universal covering of S . But \tilde{S} has a very simple description: its decomposition into irreducible components can be written $\tilde{S} = \bigcup_{i \in \mathbb{Z}} \tilde{S}_i$, $\tilde{S}_i \cong \mathbb{P}^1$, \tilde{S}_i meets \tilde{S}_{i-1} in one point, \tilde{S}_i meets \tilde{S}_{i+1} in one point and \tilde{S}_i does not meet any other component (so \tilde{S} is an infinite necklace [10]; topologically \tilde{S} is an infinite union of spheres each having one point in common with the next). Obviously \tilde{S} is not p_1 -convex because, by the maximum principle, any plurisubharmonic function on \tilde{S} must be constant.

So we have shown:

THEOREM 3. *There exists a 1-convex surface X such that its universal covering \tilde{X} is not holomorphically convex.*

REMARK 3. In our example the exceptional curve is not irreducible, so it is natural to ask if one can produce an example having the properties in Theorem 3 and with irreducible exceptional curve. Such an example can be obtained as follows: We glue suitable neighbourhoods of the zero cross sections in $\mathcal{O}(-1)$ and $\mathcal{O}(-5)$ exactly as before and we get a complex surface Z containing a compact analytic curve C with two irreducible components $C = C_1 \cup C_2$, $C_1 \cong \mathbb{P}^1$, $C_2 \cong \mathbb{P}^1$ such that the intersection $C_1 \cap C_2$ consists of two points where the intersection is transversal. The intersection matrix is $\begin{pmatrix} -1 & 2 \\ 2 & -5 \end{pmatrix}$ which is negative definite, hence by Grauert's criterion, C is exceptional. Because the selfintersection $C_1^2 = -1$ it follows that C_1 is a curve of the first kind, so it can be contracted to a point in the complex (non-singular) surface X via the contraction map $h : Z \rightarrow X$. Then $S = h(C)$ is an irreducible exceptional curve (rational with one singular point). If we replace X by a smaller strongly pseudoconvex neighbourhood of S such that S is a deformation retract of this neighbourhood we get the desired example. The proof is exactly as in (iii) and so it is omitted.

REMARK 4. In ([11], Theorem 6.2) Napier has proved the following result: Let X be a 1-convex surface with exceptional set S and $p : \tilde{X} \rightarrow X$ any covering. Then \tilde{X} is holomorphically convex iff $\tilde{S} = p^{-1}(S)$ is holomorphically convex. This last result explains why the exceptional curves in our previous examples are singular (for smooth S it follows that \tilde{S} is holomorphically convex).

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