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Introduction

In [G1], we combined the slicing obstructions of Levine [L] with those of Casson and Gordon [CG1], [CG2], [Go] in a nontrivial way. Essentially we related the metabolizer for the Seifert form of a slice knot which Levine guaranteed to the characters on the homology of the 2-fold branched cover (with vanishing Casson–Gordon invariants) which Casson and Gordon guaranteed. In this paper we will simultaneously consider all the q-fold branch covers for q a prime power. We will define Γ^+ which strengthens the group Γ' of [G1]. In [GL2] Livingston and Gilmer applied these methods to concordances of two component links with linking number zero. They defined an algebraic group Ψ which enhanced Γ' to study concordances of links with two components. In this paper we will define an analogous strengthening Ψ^+ of Ψ. We also calculate enough Casson–Gordon invariants to decide when a genus one knot maps to zero in Γ^+. Finally we give a new example of a link which is a fusion of a boundary link but not concordant to a boundary link. Cochran and Orr [CO] were the first to discover such links. Livingston was the first to observe that our work on Casson–Gordon invariants could be applied to this problem [Li]. We would like to thank Chuck Livingston for many valuable conversations. We work in the smooth category.

§1. Seifert surfaces, branched covers and isometric structures

Let K be a knot in a homology 3-sphere S with Seifert surface F with intersection pairing $\langle\ ,\ \rangle$. Let $i_+ : H_1(F) \to H_1(S - F)$ denote the map which pushes a class off in the positive normal direction and let i denote the map given by pushing off the other way. Let θ denote the Seifert pairing, i.e. $\theta(x, y)$ is $lk(i_+ x, y)$. Let V denote the Seifert matrix for Θ with respect to some basis $\{a_1, \ldots, a_{2g}\}$ for $H_1(F)$. With respect to this basis the intersection form is then given by the matrix $V - V^t$. Let $\{\alpha_1, \ldots, \alpha_{2g}\}$ denote the basis for $H_1(S - F)$ such that $lk(a_i, \alpha_j) = \delta^i_j$ as in [R] p. 209. Then i_+ with respect to these bases is given by V^t. Define

$c : H_1(S-F) \to H_1(F)$ by $\langle cx, y\rangle = lk(x, y)$, so c is given by the matrix $(V^t - V)^{-1}$ with respect to the above bases. We can describe c geometrically as follows. First represent x by a curve γ. Let γ bound an embedded surface G in $S-K$ transverse to F. Orient the curve $G \cap F$ so that the positive normal to G followed by the positive normal of F, followed by the tangent to $G \cap F$ is a right handed system. Then $G \cap F$ represents $c(x)$. This is because linking with γ is given by intersection with G in S, and for a curve on F intersection with G is the same as intersection in F with $G \cap F$.

Following Kervaire [Ke], and then Stoltzfus [St], we define the associated isometric structure $s : H_1(F) \to H_1(F)$ by the equation $\theta(x, y) = \langle sx, y\rangle$. We see that s is $c \circ i_+$ and so is given by the matrix $G = (V^t - V)^{-1}V^t$. Note $s - Id$ is given by $G - I = (V^t - V)^{-1}V$ and so $s - Id$ is actually $c \circ i_-$. The map s can be described geometrically and is related to Cochran's derivatives [C]. If $x \in H_1(F)$ is represented by a curve γ on F we can push γ off of F in the positive normal direction and obtain γ_+. Let γ_+ bound a surface G in $S-K$ transverse to F, then $G \cap F$ represents $s(x)$. $s - Id$ can be described similarly; one simple pushes off in the negative direction. See [GL3] where this connection between s and Cochran's invariants is explored.

Let M^q denote the q-fold branched cyclic cover of S along K. Let X denote S slit along F i.e. the complement of the interior of B, a bicollar neighborhood of F. M^q (denoted M for short from now on) can be constructed from q copies of X say: $X_1, \dots, X_q$. Let F_+ and F_- denote the copies of F in X lying just above and below F in S. Let $F^i_\pm$ denote the copy of $F_\pm$ lying in X_i. Then M is formed by identifying F^1_- with F^2_+, F^2_- with $F^3_+, \dots$, and F^q_- with F^1_+. Thus $H_1(M)$ is generated by the homology of the pieces. In fact if we let α^j_i denote α_i in $H_1(X_j)$ then $\{\alpha^j_i\}$ generate $H_1(M)$ subject to relations coming from the q copies of F. So the following two matrices are presentation matrices for $H_1(M)$. We follow the convention that the rows of a presentation matrix give the relations among the generators:

$$V^* = \begin{bmatrix} V^t & -V & 0 & 0 & 0 \\ 0 & V^t & -V & 0 & 0 \\ 0 & 0 & V^t & -V & 0 \\ 0 & 0 & 0 & V^t & -V \\ -V & 0 & 0 & 0 & V^t \end{bmatrix}$$

or

$$G^\wedge = \begin{bmatrix} G & I-G & 0 & 0 & 0 \\ 0 & G & I-G & 0 & 0 \\ 0 & 0 & G & I-G & 0 \\ 0 & 0 & 0 & G & I-G \\ I-G & 0 & 0 & 0 & G \end{bmatrix}.$$

We have only indicated the matrices for $q = 5$. $G^\wedge$ is obtained from V^* by multiplying on left by a block matrix with $(V' - V)^{-1}$ down the diagonal. It follows from the following lemma due to Seifert [S] p. 576–579 that $\{\alpha_i^! \mid 1 \le i \le 2g\}$ generates $H_1(M)$ and $(G)^q - (G - I)^q$ is a presentation matrix for $H_1(M)$ with this set of generators. Formally let y be $I - G$. Note Seifert finds a similar but different presentation matrix for a different set of generators. Compare [R] p. 214. [F] p. 155, and [B–Z] p. 115, [K2] p. 241.

LEMMA 1. *There is a sequence of reversible row and column operations over $Z[y]$ leading from the $q \times q$ matrix*

$$\begin{bmatrix} 1-y & y & 0 & \cdot & 0 \\ 0 & 1-y & y & \cdot & 0 \\ 0 & 0 & 1-y & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ y & 0 & 0 & \cdot & 1-y \end{bmatrix} \quad \textit{to} \quad \begin{bmatrix} 1 & * & * & \cdot & * \\ 0 & 1 & * & \cdot & * \\ 0 & 0 & 1 & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & y^q - (y-1)^q \end{bmatrix}.$$

Moreover the column operations together perform a cyclic permutation of the columns putting the first column last, the second first, and the third second etc.

Proof. This is Seifert's proof. We thank Steve Weintraub for translating the passage. For $n \le q - 1$, consider the square matrix P_n of order $q - n + 1$:

$$\begin{bmatrix} -(y-1)^n & y^n & 0 & \cdot & 0 \\ 0 & 1-y & y & \cdot & 0 \\ 0 & 0 & 1-y & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ y & 0 & 0 & \cdot & 1-y \end{bmatrix}.$$

Let w_n denote $(1 - y^n)/(1 - y)$. Starting with P_n with $n \le q - 2$, add w_n times row two to row one, then add $y - 1$ times the new row one to row two, finally interchange column one and column two. The result is:

$$\begin{bmatrix} 1 & -(y-1)^n & (w_n)y & \cdot & 0 \\ 0 & -(y-1)^{n+1} & y^{n+1} & \cdot & 0 \\ 0 & 0 & 1-y & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & y & 0 & \cdot & 1-y \end{bmatrix}.$$

Note that if we delete the first row and column we obtain P_{n+1}. Thus starting with P_1, the matrix we are interested in, we may do these operations and obtain a matrix with P_2 nestled within. Then we may operate on this new matrix so we preform the prescribed operations on the P_2, and continue in this way. We obtain an almost triangular matrix with the 2×2 matrix P_{q-1} in the lower right corner. If we now operate on the whole matrix to do the indicated operations on P_{q-1}, we obtain the desired upper triangular matrix. $\square$

We are really interested in $H^1(M, Q/Z)$, the characters on $H_1(M)$. Define ε_q to be the endomorphyism of $H_1(F)$ given by $s^q - (s-1)^q$ and $A^q \subseteq H_1(F, Q/Z)$ to be the kernel of $\varepsilon_q \otimes id_{Q/Z}$. Given $\Sigma\, r_i a_i \in A^q$ where $r_i \in Q/Z$, we can define say $\chi \in H^1(M, Q/Z)$ by $\chi(\alpha_i) = r_i$. This is consistent with the relations given by the rows of $G_q = (G)^q - (G-I)^q$ precisely because $[s^q - (s-1)^q](\Sigma\, r_i a_i) = 0$. So $H^1(M, Q/Z)$ is isomorphic to A^q. This isomorphism only depends on the choice of the lift of X which we label X_1. As the group of covering translations cyclically permutes the X_i and preserves the Casson–Gordon invariants $\tau(K, \chi)$, we may identify $H^1(M, Q/Z)$ with A^q and view $\tau(K, \ )$ which we denote simply τ as a function on A^q.

Note if we have any matrix A presenting a torsion abelian group B, we can identify its dual group $\operatorname{Hom}(B, Q/Z)$ with the set of column vectors with Q/Z coefficients which become integral when multiplied by A. We will call such a vector a dual vector to the matrix.

A s-invariant direct summand H of $H_1(F)$ such that $H = H^\perp$ (with respect to the intersection pairing $\langle\ ,\ \rangle$) is called a metabolizer for the isometric structure. *From now on q will always denote a power of a prime* (say q'). Then G_q is congruent mod q' to $(-1)^q$ and so A^q is finite with order congruent to $\pm 1 \bmod q'$.

LEMMA 2. *If H is a metabolizer for the isometric structure, then the order of $A^q \cap (H \otimes Q/Z)$ is the square root of the order of A^q.*

Proof. As in [G1] p. 309, we may extend a basis $\{x_1, \ldots, x_G\}$ for H to a basis, $\{x_1, \ldots, x_g, x_1^\#, \ldots, x_g^\#\}$ for $H_1(F)$ such that with respect to this basis, the Seifert matrix

$$V = \begin{bmatrix} 0 & C+I \\ C^t & E \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} C^t + I & E^t \\ 0 & -C \end{bmatrix}.$$

So ε_q is given by the matrix

$$G_q = \begin{bmatrix} (C_q)^t & * \\ 0 & C_q \end{bmatrix} \qquad \text{where } C_q = C^q - (C-1)^q.$$

Thus $A^q \cap (H \otimes Q/Z)$ is given by the dual vectors to $(C_q)^t$ and so has order the determinant of $(C_q)^t$. Similarly the order of A^q is the determinant of G_q. $\square$

§2. Slice knots

We say K is homology slice if there is some homology 4-ball D with $\partial D = S$ and a smoothly embedded 2-disk Δ in D with $\partial\Delta = K$. Let $A^q{}_p$ denote the p-primary component of A^q.

THEOREM 1. *If F is a Seifert surface for a homology slice knot K then there is a metabolizer H for the isometric structure on $H_1(F)$ such that $\tau(A^q{}_p \cap (H \otimes Q/Z))$ vanishes for all prime powers q and primes p.*

Proof. We begin by using transversality to find a 3-manifold in D with boundary $F \cup \Delta$. Let $H \subset H_1(F)$ be the kernel of the map induced by inclusion to $H_1(R)$/torsion. H is a direct summand. Lefshetz duality shows twice the rank of H is the rank of $H_1(F)$. Curves on F representing elements of H rationally bound surfaces in R. These surfaces made transverse provide null bordisms of the intersections of the curves. Thus $H \subseteq H^\perp$. As H is a direct summand, it follows that $H = H^\perp$. One may also use these surfaces to see $\theta(H \times H) = 0$. From this we may conclude algebraically that H is s-invariant (see [Ke] p. 95). It is perhaps more enlightening to note that s extends to a map s' on $H_1(R)$. s' is defined by pushing a curve in R off R in the positive normal direction, picking a surface in $D - \Delta$ which bounds this curve, and taking the intersection of this surface with R.

It remains to prove that the characters in $A^q \cap H \otimes Q/Z$ extend to characters on W^q, the q-fold branched cyclic cover of D along Δ. As then by theorem 2 of [C–G1], τ vanishes on $A^q{}_p \cap H \otimes Q/Z$. We will give a description of $H^1(W^q, Q/Z)$ parallel to that given above for $H^1(M^q, Q/Z)$. First it would be best to present the above description slightly differently. We have the following isomorphisms (with Q/Z coefficients understood): $H^1(X) \approx H^2(S, X) \approx H^2(B, \partial B) = H_1(B) \approx H_1(F)$ given by, the coboundary, excision, Lefshetz duality, and a homotopy equivalence. This isomorphism (in the reverse direction and with integral coefficients) sends the class of a curve γ on F to the map on $H_1(S - F)$ given by linking with γ. Thus with respect to the dual basis to $\{\alpha_1, \ldots, \alpha_{2g}\}$ and the basis $\{a_1, \ldots, a_{2g}\}$ the above isomorphism is given by the identity matrix. A character in $H^1(M, Q/Z)$ arises as a collection of characters in each $H^1(X_i, Q/Z)$ which are compatible in that they agree on the lifts of F. In fact we have an exact sequence

$$0 \to H^1(M, Q/Z) \to \bigoplus_{1 \le i \le q} H^1(X_i, Q/Z) \to \bigoplus_{1 \le i \le q} H^1(F_i, Q/Z).$$

The second map is given by V^* with respect to bases built from the dual bases of $\{\alpha_j\}$ and $\{a_j\}$. We can replace each $H^1(X_i, Q/Z)$ by a copy of $H_1(F, Q/Z)$ via the isomorphism given above. We can also replace each $H^1(F_i, Q/Z)$ by a copy of

$H_1(F, Q/Z)$ via Poincare duality. The Poincare duality isomorphism d from $H^1(F)$ to $H_1(F)$ (defined by capping with the fundamental class) is characterized by $\langle d(\omega), \alpha\rangle = \omega(\alpha)$. Thus with respect to the basis $\{\alpha_j\}$ and its dual basis d is given by $(V' - V)^{-1}$. Thus the map α below is given by $G^\wedge$ with respect to the basis $\{a_j\}$, i.e. "pieces" of the map are given by s and $I - s$. This gives us the short exact sequence

$$0 \to H^1(M, Q/Z) \to \bigoplus_{1 \le i \le q} H_1(F, Q/Z) \xrightarrow{\alpha} \bigoplus_{1 \le i \le q} H_1(F, Q/Z).$$

Let Y denote the complement of a bicollar neighborhood of R in D. We have the following commutative diagram with Q/Z coefficients understood

$$\begin{tikzcd}
H^1(Y) \arrow[r, phantom, "\approx"] \arrow[d] & H^2(D, Y) \arrow[r, phantom, "\approx"] \arrow[d] & H^2(R \times I, R \times S^0 \cup \Delta \times I) \arrow[r, phantom, "\approx"] \arrow[d] & H_2(R \times I, F \times I) \arrow[r, phantom, "\approx"] \arrow[d, "\partial"] & H_2(R, F) \arrow[d, "\partial"] \\
H^1(X) \arrow[r, phantom, "\approx"] & H^2(S, X) \arrow[r, phantom, "\approx"] & H^2(F \times I, F \times S^0 \cup K \times I) \arrow[r, phantom, "\approx"] & H_1(F \times I) \arrow[r, phantom, "\approx"] & H_1(F)
\end{tikzcd}$$

Let Y_i denote the lifts of Y in the W. In the same way we have an exact sequence

$$0 \to H^1(W, Q/Z) \to \bigoplus_{1 \le i \le q} H_1(Y_i, Q/Z) \to \bigoplus_{1 \le i \le q} H^1(R, Q/Z).$$

Using the isomorphisms above we can replace each $H_1(Y_i, Q/Z)$ with $H_2(R, F, Q/Z)$ and each $H^2(R, Q/Z)$ via Lefshetz duality also with $H_2(R, F, Q/Z)$. This new exact sequence fits into a commutative diagram with the one above

$$\begin{tikzcd}
0 \arrow[r] & H^1(W, Q/Z) \arrow[r] \arrow[d, "\iota^*"] & \bigoplus_{1 \le i \le q} H_2(R, F, Q/Z) \arrow[r] \arrow[d, "\oplus\partial"] & \bigoplus_{1 \le i \le q} H_2(R, F, Q/Z) \arrow[d, "\oplus\partial"] \\
0 \arrow[r] & H^1(M, Q/Z) \arrow[r] & \bigoplus_{1 \le i \le q} H_1(F, Q/Z) \arrow[r, "\alpha"] \arrow[d, "\oplus\mu"] & \bigoplus_{1 \le i \le q} H_1(F, Q/Z) \\
& & \bigoplus_{1 \le i \le q} H_1(R, Q/Z) &
\end{tikzcd}$$

The first vertical map is given by restriction. Now $H \otimes Q/Z$ is the kernel of μ_*, so $\bigoplus_{1 \le i \le q} H \otimes Q/Z$ is the kernel of $\bigoplus \mu_*$. Therefore the image of ι^* is contained in

$$\left(\bigoplus_{1 \le i \le q} H \otimes Q/Z\right) \cap \text{kernel}\,(\alpha).$$

We must identify the subgroup of $H^1(M, Q/Z)$ given by

$$\left(\bigoplus_{1 \leq i \leq q} H \otimes Q/Z\right) \cap \text{kernel}\,(\alpha)$$

with that given by $A^q \cap H \otimes Q/Z$. Lemma 1 shows that one can find integral determinant ± 1 $2gq \times 2gq$ matrices R and C which can be written as block matrices whose blocks are polynomials in G (moreover C is a block permutation matrix whose last column in block form consists of zero except for the first entry which is I), such that $R(G^\wedge)C = G^+$ has the form

$$\begin{bmatrix} I & * & * & * & * \\ 0 & I & * & * & * \\ 0 & 0 & I & * & * \\ 0 & 0 & 0 & I & * \\ 0 & 0 & 0 & 0 & (-1)^{q-1}G_q \end{bmatrix}$$

where the stars mean some unspecified polynomials in G. The power of minus one, could be absorbed into the last row of R. In the future we may need to give R explicitly which will be easier if we leave it alone. Again we have only given the $q = 5$ case. Any element of A^q can be written in terms of the basis $\{a_i\}$ as a dual vector to G_q. This vector can be extended uniquely to a dual vector to G^+. Multiplication by C gives an isomorphism of the group of dual vectors to G^+ to the dual vectors of $G^\wedge$. In this way we obtain an isomorphism from the group of dual vectors to G_q to the group of dual vectors to $G^\wedge$. Because H is s-invariant, this map induces an isomorphism from the subgroup of dual vectors to G_q in $H \otimes Q/Z$ to the subgroup of dual vectors to $G^\wedge$ in $\bigoplus_{1 \leq i \leq q} H \otimes Q/Z$. Thus we have identified the subgroups mentioned at the beginning of this paragraph.

By Lemma 2, $A^q \cap H \otimes Q/Z$ has order the square root of the order of A^q. By Lemma 3 of [C–G1], the image of $H_1(M)$ in $H_1(W)$ has order the square root of the order of $H_1(M)$ which is the order of A^q. It follows that the image of ι^* has the same order as $A^q \cap H \otimes Q/Z$. As the image of ι^* has been identified with a subgroup of $A^q \cap H \otimes Q/Z$, we see that the characters in $A^q \cap H \otimes Q/Z$ extend to characters on W^q and the proof of the theorem is complete. □

§3. Linking forms and Levine's Group

In this section, we prepare the way for the proof of the cancellation lemma for Γ^+ in the next section. We discuss the linking form on M^q, the proof of the cancellation lemma for Levine's Group G_- and discuss the Witt group of torsion linking forms.

We begin with some generalities. Suppose N is a rational homology 3-sphere and thus has a nonsingular linking form $\ell : H_1(N)x \cdot H_1(N) \to Q/Z$. Define $c : H_1(N) \to \mathrm{Hom}\,(H_1(N), Q/Z) = H^1(N, Q/Z)$ by $c(x)(y) = \ell(x, y)$. Then we may define a linking form β on $H^1(N, Q/Z)$ (called the dual linking form) by $\beta(c(x), c(y)) = -\ell(x, y)$. We insert the minus to be consistent with earlier papers [G1], [G2], and [GL1]. Recall if N is the boundary of a 4-manifold W with $H_1(W)$ zero, and Z is a matrix for the intersection form on $H_2(W)$ with respect to a basis $\{z_i\}$, then Z is a presentation matrix for $H_1(N)$ with respect to generators given by the boundaries of the Lefshetz duals to the z_i. Moreover ℓ on these generators is given by the entries in $-Z^{-1}$. Thus if $H^1(N, Q/Z)$ is identified with the dual vectors to Z and if u and v are dual vectors to Z we have $\beta(u, v) = u'Zv$. We need to see how the dual linking form β on M^q can be described on A^q in terms of s and $\langle\ ,\ \rangle$.

PROPOSITION 1. *For each q there is a polynomial with integral coefficients f_q, such that $\beta(x, y) = \langle f_q(s)x, y\rangle$ (f_q is not unique).*

Proof. Recall an element of A^q is a dual vector v to $G^q - (G - I)^q$. This can be extended to a dual vector v^+ to G^+ whose block entries are all given by polynomials in s applied to V. Then $C(v^+)$ is a dual vector to $G^\wedge$ and thus V^* again with all block entries polynomials in s applied to v. Now we can relate the presentation of $H_1(M^q)$ given by V^* to the presentation given by the $(q-1) \times (q-1)$ block matrix with $2g \times 2g$ blocks

$$V^{\#} = \begin{bmatrix} V+V' & -V & 0 & 0 \\ -V' & V+V' & -V & 0 \\ 0 & -V' & V+V' & -V \\ 0 & 0 & -V' & V+V' \end{bmatrix}$$

for the intersection pairing on the q-fold branched cover of D along a copy of F pushed slightly into D (again we only show the $q = 5$ case) (see [V], [K1] corollary 5.7, [K2] p. 283, [AK]) This is a presentation matrix with respect to the generating set

$$\left\{ \sum_{k=1}^{j} \alpha_i^k \;\middle|\; 1 \le i \le 2g,\ 1 \le j \le q-1 \right\}$$

ordered lexigraphically (i comes first). Let D be the $(q-1) \times q$ block matrix with $2g \times 2g$ blocks with I down the diagonal and I everywhere beneath the diagonal and zeros elsewhere. Then $v^{\#} = DCv^{+}$ is the corresponding dual vector to $V^{\#}$. Its block entries are all polynomials in s applied to v. As in [G1], we see that for v, w in A_q given as dual vectors to G_q, we have $\beta^q(v, w) = v^{\#\prime}V^{\#}w^{\#}$. As the nonzero block entries of $V^{\#}$ are just $V + V'$, V and $-V$, and since $a'Vb = \langle sa, b\rangle$, and $a'V'b = b'Va = \langle sb, a\rangle = -\langle a, sb\rangle = \langle (s-1)a, b\rangle$ (here we write a and b sometimes as elements of $V \otimes Q/Z$, and sometimes in terms of our basis as column vectors with Q/Z coefficients), it is clear that $\beta^q(v, w)$ can be written as $\langle f_q(s)v, w\rangle$ where f_q is as above. $\square$

Remark. The above proof gives a method to find f_q. We may take f_2 to be $(2s-1)$. We do not need to know a formula for f_q. There are other simpler ways to give formulas for ℓ. However this formulation is useful to us.

We now wish to recall Levine's group G_- and its formulation in terms of isometric structures. We consider triples $(U, \langle\ ,\ \rangle, s)$ where U is a finitely generated free Z-module, $\langle\ ,\ \rangle$ is a nonsingular skew symmetric bilinear form and s is an endomorphism of U such that $\langle sx, y\rangle + \langle x, sy\rangle = \langle x, y\rangle$ for all x, y in U. Such triples are called isometric structures. We say a triple is metabolic if there is a s-invariant submodule H of such that $H = H^{\perp}$. Such an H is called a metabolizer. Note H is a direct summand as $H = H^{\perp}$. The following cancellation lemma is due to Levine [L]. The proof that Kervaire [Ke] p. 87 gave can be adapted to prove the new cancellation lemmas we need. We motivate Kervaire's proof with a geometric proof which, as far as we know, is original with us. It only applies to the skew symmetric case. Kervaire's proof will be applied below to linking forms on finite groups as well. That the same argument works in these two cases will be the key to the proofs of the cancellation lemmas for the more complicated new Witt type groups of the next section. This is why we repeat Kervaire's proof here. The direct sum of two triples is defined in the expected way.

LEMMA 3. *If* $(U_1, \langle\ ,\ \rangle_1, s_1)$ *is metabolic, and* $(U_1, \langle\ ,\ \rangle_1, s_1) \oplus (U_2, \langle\ ,\ \rangle_2, s_2)$ *is metabolic then* $(U_2, \langle\ ,\ \rangle_2, s_2)$ *is metabolic.*

Geometric Proof. $\langle\ ,\ \rangle_1$ and, $\langle\ ,\ \rangle_2$ are isomorphic to the intersection forms on two surfaces F_1 and F_2. Let H_1 be a metabolizer for the first triple. We may represent a basis for H_1 by a collection of disjoint simple closed curves on F_1. Thus we may attach 2-handles to $F_1 \times I$ along this collection of curves on one boundary component and then a 3-handle to form a 3-manifold X_1 with boundary F_1. The kernel of the map induced by the inclusion of F_1 into X_1 on the first homology is H_1. Let H be a metabolizer to the direct sum. We can build a 3-manifold X with boundary the disjoint union of F_1 and F_2 and with H the kernel of the map on first homology. First we add a one handle to $(F_1 \cup F_2) \times I$ so that the connected sum is a boundary component. Then we represent a basis for H by simple closed curves on $F_1 \# F_2$ and add 2-handles and a 3-handle as above.

Now let X_2 be the union of X_1 and X along F_1. The boundary of X_2 is F_2. Let H_2 be the smallest direct summand including the kernel of the map on first homology induced by the inclusion of F_2 in X_2. A well known argument shows that $H = H^{\perp}$. Moreover the kernel is easily seen to be $\rho((H_1 \oplus U_2) \cap H)$ (here ρ denotes projection on U_2) or $(H_1 + H) \cap U_2$ (where we think of all these groups as subgroups of $U_1 \oplus U_2$). One way to see this is to represent a null homology of a curve on F_2 in X by a surface which is transverse to $F_1 \# F_2$. Thus H_2 is s-invariant. □

Kervaire's Proof. Let $\hat{H}$ denote $(H_1 \oplus U_2) \cap H$. One easily sees that $(H_1 + H) \cap U_2$ i.e. $\rho(\hat{H})$ is s-invariant and included in its own perp. It remains to show it has the right rank. We write U_1 as $H_1 \oplus C$. Let $\pi : U_1 \oplus U_2 \to C$ denote the projection, and $\bar{H}$ be $\pi(H)$. We have a short exact sequence: $0 \to \hat{H} \to H \to \bar{H} \to 0$. There is also a short exact sequence $0 \to \hat{H} \cap U_1 = H \cap H_1 \to \hat{H} \to \rho(\hat{H}) \to 0$. We have that $H \cap H_1$ and $\bar{H}$ are orthogonal with respect to $\langle\,,\rangle_1$. As if $v \in \bar{H}$, then there exist $w \in H_1$, and $x \in U_2$ such that $(w \oplus v) \oplus x \in H$. So if $u \in H \cap H_1$, $\langle u, v\rangle_1 = \langle u, v + w\rangle_1 = \langle u, w + v + x\rangle = 0$. Here $\langle\,,\rangle$ denotes the form on $U_1 \oplus U_2$. As H_1 is also orthogonal to $H \cap H_1$, we have $H_1 \oplus \bar{H}$ is orthogonal to $H \cap H_1$. Thus we have:

$$\operatorname{rank}(H_1 \oplus \bar{H}) + \operatorname{rank}(H \cap H_1) \leq \operatorname{rank}(U_1),$$

$$\operatorname{rank}(H) = \operatorname{rank}(\hat{H}) + \operatorname{rank}(\bar{H}),$$

$$\operatorname{rank}(\hat{H}) = \operatorname{rank}(H \cap H_1) + \operatorname{rank}(\rho(\hat{H})),$$

$$2\operatorname{rank}(H_1) = \operatorname{rank}(U_1),$$

$$2\operatorname{rank}(H) = \operatorname{rank}(U_1) + \operatorname{rank}(U_2).$$

It follows that $2\operatorname{rank}(\rho(\hat{H})) \geq \operatorname{rank}(H_2)$. As $\rho(\hat{H})$ is self annihilating and $\langle\,,\rangle_2$ is nonsingular, we must have equality. Let H_2 be the direct summand generated by $\rho(\hat{H})$. It is our desired metabolizer for U_2. □

To complete the definition of G_-, define $-(U, \langle\,,\rangle, s) = (U, -\langle\,,\rangle, s)$, and $(U_1, \langle\,,\rangle, s)$ equivalent to $(U_2, \langle\,,\rangle, s)$ if $(U_1, \langle\,,\rangle, s) \oplus -(U_2, \langle\,,\rangle, s)$ is metabolic. The cancellation lemma shows this relation to transitive. G_- is the group of equivalence classes of such tuples. The sum is induced by direct sum.

One can define the Witt group $W(Q/Z)$ of torsion linking forms in much the same way. See [AHV] for a slightly different approach. We consider pairs $(\mathcal{U}, \mathcal{S})$

where $\mathscr{U}$ is a finite abelian group and $\mathscr{I}$ is a nonsingular symmetric bilinear form $\mathscr{I}: \mathscr{U} \times \mathscr{U} \to Q/Z$. Such a pair is called a symmetric torsion form. $(\mathscr{U}, \mathscr{I})$ is said to be metabolic if there exist a subgroup $\mathscr{H} \subseteq \mathscr{U}$ with $\mathscr{H} = \mathscr{H}^{\perp}$. It follows that $|\mathscr{H}|^2 = |\mathscr{U}|$. Moreover if $|\mathscr{H}|^2 = |\mathscr{U}|$, and $\mathscr{H} \subseteq \mathscr{H}^{\perp}$, then $\mathscr{H} = \mathscr{H}^{\perp}$. Thus the order of a subgroup plays a totally analogous role to that played by rank above. One may prove the cancellation lemma by the same argument except one should not take the summand generated by $\rho(\hat{\mathscr{H}})$ in the last step. The desired metabolizer is $\rho(\hat{\mathscr{H}})$. One goes on to define $W(Q/Z)$ in an analogous way.

LEMMA 4. *If* $(U, \langle\ ,\ \rangle, s)$ *is an isometric structure and* $A^q \subseteq U \otimes Q/Z$ *is the kernel of* $(s^q - (s-1)^q) \otimes id_{Q/Z}$, *then* $(A^q, \langle f_q(s)\ ,\ \rangle)$ *is a symmetric torsion form. If* H *is a metabolizer for* $(U, \langle\ ,\ \rangle, s)$, *then* $A^q \cap H \otimes Q/Z$ *is a metabolizer for* $(A^q, \langle f_q(s)\ ,\ \rangle)$.

Proof. Every isometric structure arises as the isometric structure of some knot. The q-fold branched cover of S^3 along a knot is a rational homology sphere, and a rational homology sphere's dual linking form is a symmetric torsion form. So by Proposition 1, $(A^q, \langle f_q(s)\ ,\ \rangle)$ is a symmetric torsion form. Because H is s-invariant and H is self annihilating under $\langle\ ,\ \rangle$, $A^q \cap H \otimes Q/Z$ is self annihilating under $\langle f_q(s)\ ,\ \rangle$. By Lemma 2, $A^q \cap H \otimes Q/Z$ has order the square root of the order of A^q. So it is a metabolizer. □

Remark. It follows that there is a homomorphism from G_- to $W(Q/Z)$ sending the class of $(U, \langle\ ,\ \rangle, s)$ to the class of $(A^q, \langle f_q(s)\ ,\ \rangle)$. Let $\mathscr{HC}$ denote the group of concordance classes of knots in homology 3-spheres were we allow a concordance between knots to live in an integral homology cobordism of homology 3-spheres. Let $\mathscr{RH}$ denote the group of rational homology cobordism classes of rational homology 3-spheres. There is a well defined homomorphism from $\mathscr{HC}$ to $\mathscr{RH}$ given by taking the q-fold branched cover. These maps fit into a commutative diagram with Levine's map from $\mathscr{HC}$ to G_-, and the map from $\mathscr{RH}$ to $W(Q/Z)$ obtained by taking the dual linking form.

$$\begin{tikzcd} \mathscr{RH} \arrow[r] & W(Q/Z) \\ \mathscr{HC} \arrow[r] \arrow[u] & G_- \arrow[u] \end{tikzcd}$$

§4. Some more Witt type groups

We begin by defining Γ^+ and a homomorphism from $\mathscr{HC}$ to Γ^+. The class of a knot maps to zero if and only if it satisfies the conclusion of Theorem 1. We consider tuples $(U, \langle\ ,\ \rangle, s, \tau^q{}_p)$ where $(U, \langle\ ,\ \rangle, s)$ is an isometric structure and for each prime power q and prime p, $\tau^q{}_p$ are functions. The domain of $\tau^q{}_p$ is $A^q{}_p \subseteq U \otimes Q/Z$ which is defined to be the p-primary part of the kernel of $(s^q-(s-1)^q) \otimes id_{Q/Z}$. We also specify that $\tau^q{}_p(0)=0$ and that $\tau^q{}_p(-x)=\tau^q{}_p(x)$ for all x. The range of $\tau^q{}_p$ is W_p. For p odd W_p is the direct limit of the Hermitian Witt groups of the field of rational functions over $Q(\zeta_{p^r})$ tensored with $Z_{(2)}$. If p is two, we must tensor with Q instead. See [GL2] section 5. Note we would like to go back and retroactively redefine τ slightly in the groups Γ' and, Ψ in this way i.e. defined on p-primary parts with range W_p, even and vanishing on zero.

A metabolizer for the isometric structure $(U, \langle\ ,\ \rangle, s)$ such that $\tau^q{}_p(A^q{}_p \cap H \otimes Q/Z)$ vanishes for all prime powers q and primes p is said to be a metabolizer for $(U, \langle\ ,\ \rangle, s, \tau^q{}_p)$. We say $(U, \langle\ ,\ \rangle, s, \tau^q{}_p)$ is metabolic if it has a metabolizer.

LEMMA 5. *If* $(U_1, \langle\ ,\ \rangle_1, s_1, \tau^q{}_{p_1})$ is metabolic, and $(U_1, \langle\ ,\ \rangle_1, s_1, \tau_1{}^q{}_p) \oplus (U_2, \langle\ ,\ \rangle_2, s_2, \tau_2{}^q{}_p)$ is metabolic then $(U_2, \langle\ ,\ \rangle_2, s_2, \tau_2{}^q{}_p)$ is metabolic.

Proof. We begin exactly as in the algebraic proof of Lemma 3. We wish to show that H_2, the smallest direct summand including $p(\hat{H})$ is a metabolizer. Given a subgroup B of $U_1 \oplus U_2$, let $\tilde{B}$ denote $(A_1{}^q{}_p \oplus A_2{}^q{}_p) \cap B \otimes Q/Z$, suppressing the dependence on p and q. We only need to check that $\tau_1{}^q{}_p \oplus \tau_2{}^q{}_p$ vanishes on $\widetilde{H_2}$. By Lemma 4, $\widetilde{H_1}$ is a metabolizer for the torsion form on $A_1{}^q{}_p$ and $\tilde{H}$ is metabolizer for the torsion form on $A_1{}^q{}_p \oplus A_2{}^q{}_p$. Now $\tau_1{}^q{}_p \oplus \tau_2{}^q{}_p$ vanishes on $\rho((\widetilde{H_1} \oplus \widetilde{U_2}) \cap \tilde{H})$ using additivity of τ on direct sums and the vanishing of τ on $\widetilde{H_1}$ and $\tilde{H}$. But $\rho((\widetilde{H_1} \oplus \widetilde{U_2}) \cap \tilde{H})$ equals $(\widetilde{H_1} + \tilde{H}) \cap \widetilde{U_2}$ which also has the correct order to be a metabolizer for the induced linking form on $A_1{}^q{}_p \oplus A_2{}^q{}_p$ (by the proof of the cancellation lemma for $W(Q/Z)$). Finally it is easy to see that $(\widetilde{H_1} + \tilde{H}) \cap \widetilde{U_2} \subset \widetilde{H_2}$. As these two groups have the same order they must be the same and we are done. □

As in [G1], one defines $-(U, \langle\ ,\ \rangle, s, \tau^q{}_p)$ to be $(U, -\langle\ ,\ \rangle, s, -\tau^q{}_p)$ and two tuples to be equivalent if the direct sum of one with minus the other is metabolic. The equivalence classes form a group Γ^+ with the operation induced by direct sum. There is a homomorphism γ^+ from $\mathscr{HC}$ to Γ^+. The class of a knot K in S is sent to the tuple $(H_1(F), \langle\ ,\ \rangle, s, \tau^q{}_p)$, where $\langle\ ,\ \rangle$ is the intersection pairing on $H_1(F)$, s is the isometric structure, and $\tau^q{}_p$ is given by $\tau^q{}_p(a)=\tau(a)-\tau(0)$ for $a \in A^q{}_p$. γ^+ is well defined by Theorem 1. Here we must note that the Casson–Gordon invariants are even, and are additive under connected sums. The hardest is additivity [Lith] p. 335. Or one can adapt the proof of additivity in [G1] for 2-fold covers.

We may also define a group Ψ^+ as equivalence classes of tuples $(U, \langle\ ,\ \rangle, s, \tau^q{}_p, \lambda)$ where λ is an element of U. H is a metabolizer if it meets the conditions given in the definition Γ^+ and λ is an element of H. Everything goes through as in the definition of Φ in [GL3], and of Ψ in [GL2]. Note in [GL2] we could have specified in the definition of metabolic that λ was an element of H rather than λ was orthogonal to H, as $H = H^\perp$. There is an injection from Γ^+ to Ψ^+ obtained by taking λ to be zero. Let B^+ denote the image. There is a left inverse to this injection given by a map which forgets λ. Thus Ψ^+ is isomorphic to $\Gamma^+ \oplus \Psi^+/B^+$. The following proposition is sometimes a convenient way to show an element is not in B^+ or just non-zero in Ψ^+.

PROPOSITION 2. *Suppose* $\mathscr{X} = (U, \langle\ ,\ \rangle, s, \tau^q{}_p, \lambda)$ *represents an element of* B^+, *and let L be the smallest s-invariant summand of U which includes* λ. *Then* $\tau^q{}_p$ *vanishes on* $(L \otimes Q/Z) \cap A^q{}_p$.

Proof. Since $\mathscr{X} \in B^+$, there is a $\mathscr{X}'$ with λ zero such that $\mathscr{X} \oplus \mathscr{X}'$ is metabolic. So the s-invariant metabolizer H for $\mathscr{X} \oplus \mathscr{X}'$ must include $\lambda \oplus 0$. It follows that H includes $s^i(\lambda) \oplus 0$ for all i. As the $s^i(\lambda)$ generate $L \otimes Q$ as a vector space, $H \otimes Q/Z$ includes $(L \oplus 0) \otimes Q/Z$. The result follows as τ is additive, vanishes on zero and vanishes on $A^q{}_p \cap H \otimes Q/Z$. □

Remark. The reason we included $\tau^q{}_p(0) = 0$ in the definition of Γ^+ and Ψ^+ is so that this proposition would hold.

We may now assign to the concordance class of a two component link (K_1, K_2) with linking number zero an element $\psi^+(K_1, K_2)$ in Ψ^+. We begin by picking a Seifert surface F for K_1 in $S - K_2$ setting $\lambda \in H_1(F)$ equal to $c([K_2])$ where $[K_2]$ denotes the homology class represented by K_2 in $H_1(S - K_1)$ and c denotes the map of section 1. So λ is really the homology class on F represented by the Sato–Levine curve of the link [Sa] and $s^i(\lambda)$ are the homology classes represented by Cochran's derivatives. We define $\psi^+(K_1, K_2)$ to be the equivalence class of the tuple $(H_1(F), \langle\ ,\ \rangle, s, \lambda)$ where $\langle\ ,\ \rangle$ and s are as in the definition of γ^+. The proof of the Theorem 9 in [GL2] together with the above shows:

THEOREM 2. *The map* ψ^+ *is well-defined.*

COROLLARY (2.1). *If* (K_1, K_2) *is concordant to a boundary link,* $\Psi^+(K_1, K_2) \in B^+$.

Using Proposition 2, we have

COROLLARY (2.2). *Let L be the smallest s-invariant summand of $H_1(F)$ which includes λ. If (K_1, K_2) is concordant to a boundary link, then $\tau^q{}_p$ vanishes on $(L \otimes Q/Z) \cap A^q{}_p$.*

COROLLARY (2.3). *Suppose (K_1, K_2) is concordant to a boundary link, and $\gamma^+(K_1)$ is zero (for example if K_1 is slice), then $\psi^+(K_1, K_2)$ is zero.*

Proof. An element in B^+ which maps to zero under the forgetful map is zero. □

§5. Algebraically slice genus one knots

Recall a knot is called algebraically slice if it has a half dimensional summand of the homology of a Seifert surface which is self-annihilating with respect to the Seifert form. This summand is called a metabolizer for the Seifert form (or isometric structure). Let K be an algebraically slice knot with a genus one Seifert surface F. Then there are two (up to sign) primitive elements in $H_1(F)$ which are self annihilating with respect to the Seifert form, and thus potential generators for a metabolizer.

PROPOSITION 3. *The curves representing a non-trivial homology class on a punctured torus are all isotopic.*

Proof. We may deduce this from the same but well known fact for a closed torus. Consider the isotopy on the closed torus: every time the knot passes over the deleted point we are effectively taking a connected sum of the curve with a loop around the hole. Thus we only need to see that the result of this one move does not change the isotopy class on a punctured torus. This can be checked readily from a picture. □

The two curves on F generating metabolizers are precisely the curves on F that one would want to surger F along in the 4-ball to obtain a slice disk. This is Levine's method to show that algebraically slice knots in high dimensions are slice. In [G1], we showed that the Casson–Gordon invariants associated to the 2-fold branched cover which should vanish if the knot were slice are given by certain signatures of one of these curves when viewed as knots in S. Here we will do the same for all prime power covers. Actually we are working with a slightly refined Casson–Gordon invariant here (for odd primes we only tensor with Z localized at 2 instead of Q), and so the invariants are given by certain Witt invariants of these curves. As these Witt invariants are themselves obstructions to the knots in the band being algebraically slice, we can view the Casson–Gordon invariants as some sort of secondary obstruction.

Let $x \in H_1(F)$ generate a metabolizer, and extend $\{x\}$ to form a basis $\{x, y\}$ for $H_1(F)$. With respect to this basis (replacing y by $-y$ if necessary, the Seifert matrix V has the form

$$V = \begin{bmatrix} 0 & -(m+1) \\ -m & a \end{bmatrix}.$$

Moreover

$$V - V^t = (V^t - V)^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} m+1 & -a \\ 0 & -m \end{bmatrix},$$

and

$$G - I = \begin{bmatrix} m & -a \\ 0 & -(m+1) \end{bmatrix}.$$

Now the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector for both G and $G - I$ with eigenvalues $m + 1$ and m respectively. Similarly the vector $\begin{bmatrix} a \\ 2m+1 \end{bmatrix}$ is an eigenvector for both G and $G - I$ with eigenvalues $-m$ and $-(m+1)$ respectively. Thus if we let $\mu = (m+1)^q - m^q$, then we have that

$$G^q - (G - I)^q = P \begin{bmatrix} \mu & 0 \\ 0 & (-1)^{q+1}\mu \end{bmatrix} P^{-1}$$

or

$$\frac{\mu}{2m+1} \begin{bmatrix} 2m+1 & -(1+(-1)^q)a \\ 0 & (-1)^q(2m+1) \end{bmatrix}.$$

So we see that the order of $H_1(M^q)$ is μ^2 and A^q is the cyclic subgroup of order μ generated by $x \otimes (1/\mu)$.

Now we can calculate τ on ${A^q}_p$. We follow the procedure used in the proof of Theorem 5 [GL2]. We express our answer in terms of the Witt knot invariants $w_{i/d}(J)$ (defined in [GL2]) which live in the group of Witt classes of Hermitian forms over the cyclotomic field $Q(\zeta_d)$. The signature of this form (using the usual embedding in the complex numbers) is an ordinary signature of $J : \sigma_{i/d}(J)$. Let χ denote the character which corresponds to $x \otimes (1/d)$ and assigns $1/d$ to the lift to X_1 of a meridian of the first band of K. Here d is a prime power which divides μ. We need to see what value χ assigns to the different lifts of the meridian. For this we consider the derivation of the presentation matrix V^*, and note that $m+1$ and m are relatively prime to μ. The dual vector $\begin{bmatrix} 1/d \\ 0 \end{bmatrix}$ to $G^q-(G-1)^q$ is the beginning of a unique dual vector to V^* whose further 2×1 block entries are given by applying

$$V^{-1}V' = \begin{bmatrix} \dfrac{m+1}{m} & -a \\ 0 & \dfrac{m}{m+1} \end{bmatrix}$$

repeatedly to the previous block entry. Thus χ assigns to the different lifts of the meridian of the first band the value $(m^*(m+1))^i$ where m^*m is congruent to 1 modulo d.

Let J be the knot in S obtained when we represent x by a simple closed curve on F. Suppose we form a new knot K' by tieing $-J$ (reversing both the string and ambient orientation) in the first band of F. For a knot in S^3, tying the knot in the band is equivalent to replacing a tubular neighborhood of the meridian of the band by the exterior of the knot. In this way we may make sense of the operation if S is just a homology sphere. Moreover this is key to the method we used to calculate τ for algebraically doubly slice knots in [GL2]. Now J' the knot given by representing x' on F' is $J\#(-J)$ which is slice. So K' is slice by Levine's method with metabolizer generated by x'. Thus $\tau(x\otimes(j/d))$ is zero by Theorem 1. We calculate the effect of replacing the meridian of the first band of F by the exterior of $-J$ on τ as in [GL2], and we thus prove:

THEOREM 3. *Let K be a knot with a genus one Seifert surface F and Seifert matrix*

$$V = \begin{bmatrix} 0 & -(m+1) \\ -m & a \end{bmatrix}$$

with respect to a basis $\{x, y\}$ for $H_1(F)$. Let $\langle x\rangle$ denote the subgroup of $H_1(F)$ generated by x. Then $A^q\cap(\langle x\rangle\otimes Q/Z)$ is a cyclic group of order μ generated by $x\otimes(1/\mu)$ where μ is $(m+1)^q-m^q$. If d is a power of a prime p dividing μ, $\tau(x\otimes(c/d))$ is the image in W_p of $\Sigma_{i\in cT} w_{i/d}(J)$ under the map (injective for p odd) induced by the inclusion of $Q(\zeta_d)$ in its field of rational functions. Here J is the knot in S obtained by representing x by a curve on F, and viewing that curve in S. T is the multiplicative subgroup of $Z_d{}^*$ generated by $m^*(m+1)$, where m^* is the inverse of m modulo d, and c is any element in $Z_d{}^*$.

Combining Theorems one and three, we have:

THEOREM 4. *Suppose K is a homology slice knot with a genus one Seifert surface F. Then there is a nontrivial class $x \in H_1(F)$ with $\theta(x, x)$ zero such that if J is a simple closed curve on F representing x viewed as a knot, then $\Sigma_{i \in cT} w_{i/d}(J)$ is zero. Here d is prime power dividing $\mu = (m+1)^q - m^q$, q is a prime power, $m = (\sqrt{\det(K)} - 1)/2$, where c and T are as above. Consequently $\Sigma_{i \in cT} \sigma_{i/d}(j)$ is zero.*

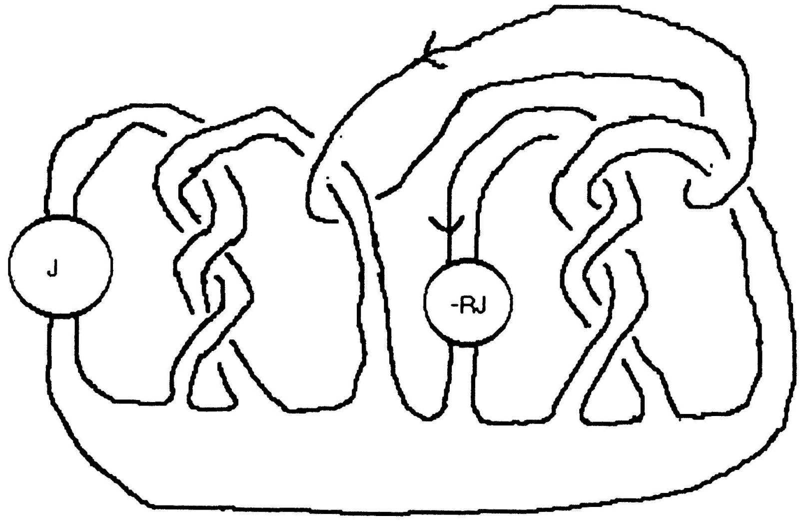
A genus one knot maps to zero in Γ^+ if and only if it satisfies the conclusion of this theorem. Daryl Cooper [Co] in his thesis proved a stronger theorem than the signature version of Theorem 4 using different methods. His theorems at present only apply to the genus one case. See [GL2] for a discussion. In joint work with Livingston, we have made use of the torsion Witt invariants to show certain knots are not slice even though the knots "in the bands" are torsion in Levine's group [GL4].

§6. Examples

Examples 1, 2 and 3 [GL2] of links not concordant to boundary links can all be deduced from Corollary (2.2). We will give a new example of a link which is a fusion of a boundary link but not concordant to a boundary link. Moreover while this will be detected using Corollary (2.1), it cannot be detected by Corollary (2.2).

We begin by discussing Example 3 of [GL2]. The Seifert matrix with respect the basis given by the bands from left to right, each oriented counterclockwise is the direct sum of two copies of the matrix labelled V in the previous section with $a = 0$. So the matrix for s is the direct sum of two copies of the matrix labelled G in the previous section with $m = 1$. We may calculate λ is $[1, 0, 0, 1]'$, and $s\lambda$ is $[2, 0, 0, -1]'$. Thus the smallest s-invariant summand including λ and $s\lambda$ is the summand denoted H_B in [GL2]. This is ruled out by the signatures of J_1 as in [GL2].

Our new example is modelled on Example 3. K_1 is the same except J_2 is tied in the third band instead of the fourth. Here we take $m = 1$, $J_1 = J$, $J_2 = -RJ$. We let $R(K)$ denote the reverse of K obtained by reversing the string orientation only. J should be any knot for which the Witt sum of Theorem 3 (for $m = 1$, and some q, d and c) does not vanish. K_2 is an unknot linking the second and fourth band once in a simple way. See Figure. The isometric structure is given by the same matrix as before. Now $\lambda = [1, 0, -1, 0]'$, and so $s(\lambda)$ is just 2λ. Thus L is just the subgroup generated by λ. Note λ is represented by a curve on F which when viewed as a knot in S^3 is the slice knot $J \# -J$. We see that τ must vanish on $(L \otimes Q/Z) \cap A^q{}_p$ using additivity of the Casson–Gordon Invariant and Theorem 3. Now we may look at the list of metabolizers for the Seifert form worked out in Example 3 [GL2]. Only



Figure

two contain λ: H_C generated by $[1, 0, 0, 0]'$ and $[0, 0, 1, 0]'$, and H_E with $p = -1$ and $q = 1$ generated by $[1, 0, -1, 0]'$ and $[0, 1, 0, 1]'$. We can rule out H_C as $[1, 0, 0, 0]'$ is represented by a curve of knot type J and we assume the Witt sum of Theorem 3 (for $m = 1$, and some q, d and c) for J does not vanish. For H_E we consider $[1, 1, -1, 1]' \otimes (1/3)$ in $A^2{}_3$. We can see τ of this character is nonzero using Theorem 3.5 of [G1] (warning in [G1] τ takes values in $W_p \otimes Q$) and additivity. We view F as the connected sum of two genus one Seifert surfaces. Note that $[1, 1]'$ can be represented on the first surface by a curve with the knot type of J, and $[-1, 1]'$ can be represented by a curve on the second surface with the knot type of $-J \# a$ trefoil. Thus H_E is ruled out as well, and the link cannot be concordant to a boundary link.

Addendum. These results also apply to locally flat topologically embedded slice disks and concordances. We only need to note that Quinn [Q] has pointed out that map transversality continues to hold in these dimensions as 4-manifolds are almost smoothable. Moreover the Casson–Gordon invariant may be calculated with nonsmooth null bordisms as $\Omega_*^{\mathrm{Top}}(\ ) \otimes Z_{(2)}$ and $\Omega_*^{\mathrm{Top}}(\ ) \otimes Q$ are generalized homology theories and similar arguments to those made in the smooth category work.

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