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Polynomial invariants of representations of quivers

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Let k be an algebraically closed field of arbitrary characteristic. Recall that if an affine algebraic group G over k acts on an affine variety Z then we get an induced action of G on the coordinate algebra $k[Z]$, given by $(x \cdot f)(z) = f(x^{-1}z)$, for $x \in G$, $f \in k[Z]$ and $z \in Z$. We consider here the space $R(Q, \alpha)$ of all k-representations of a quiver Q with given dimension vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. There is a natural action of the product of general linear groups $\mathrm{GL}(\alpha) = \mathrm{GL}(\alpha_1, k) \times \mathrm{GL}(\alpha_2, k) \times \cdots \times \mathrm{GL}(\alpha_n, k)$ on $R(Q, \alpha)$ and the purpose of this note is to describe generators for the algebra of polynomial invariants $k[R(Q, \alpha)]^{\mathrm{GL}(\alpha)_\theta}$ of the coordinate algebra $k[R(Q, \alpha)]$, where $\mathrm{GL}(\alpha)_\theta$ is the centralizer in $\mathrm{GL}(\alpha)$ of an element θ. In particular we show that $k[R(Q, \alpha)]^{\mathrm{GL}(\alpha)}$ is generated by the coefficients of the characteristic polynomials of products over oriented cycles. In characteristic zero this is a result of Le Bruyn and Procesi, [4], Theorem 1. I am very grateful to Dr. W. W. Crawley-Boevey for bringing this problem to my attention.

By a quiver we mean a quadruple $Q = (V, A, h, t)$, consisting of the vertex set $V = \{1, 2, \ldots, n\}$, a finite set A of arrows and maps $h : A \to V$, $t : A \to V$ which assign to an arrow $a \in A$ its head, $h(a)$, and tail, $t(a)$.

Let $E_1, E_2, \ldots, E_n$ be finite dimensional k-vector spaces and let $E = E_1 \oplus E_2 \oplus \cdots \oplus E_n$. Let $\alpha_i = \dim_k E_i$, $1 \le i \le n$ and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. We write $\mathrm{GL}(\alpha) = \mathrm{GL}(E_1) \times \mathrm{GL}(E_2) \times \cdots \times \mathrm{GL}(E_n)$ and identify $\mathrm{GL}(\alpha)$ with a subgroup of $\mathrm{GL}(E)$. Thus $\mathrm{GL}(\alpha)$ is the centralizer in $\mathrm{GL}(E)$ of a linear endomorphism σ of E which acts as multiplication by c_i on E_i, $1 \le i \le n$, for distinct scalars $c_1, c_2, \ldots, c_n \in k$. Then $R(Q, \alpha) = \Pi_{a \in A} \mathrm{Hom}_k(E_{t(a)}, E_{t(a)})$ is the space of all k-representations of Q on the spaces $E_1, E_2, \ldots, E_n$. Now $\mathrm{GL}(\alpha)$ acts rationally on $R(Q, \alpha)$ by $g \cdot (y_a)_{a \in A} = (g_{h(a)} y_a g_{t(a)}^{-1})_{a \in A}$, for $g = (g_1, g_2, \ldots, g_n) \in \mathrm{GL}(\alpha)$ and $(y_a)_{a \in A} \in R(Q, \alpha)$. For an endomorphism z of a k-vector space E of finite dimension d and non-negative integer $s \le d$ we let $\chi_s(z)$ denote $(-1)^s$ times the coefficient of t^{d-s} in the characteristic polynomial $\det(tI - z)$ of z (where I is the identity map on E). In the case in which Q has only one vertex the following becomes the description of generators of matrix invariants given in [3], §2, Theorem 1.

PROPOSITION. *The algebra of invariants* $k[R(Q,\alpha)]^{\mathrm{GL}(\alpha)}$ *is generated by the functions* $(y_a)_{a\in A}\mapsto\chi_s(y_{a_r}\cdots y_{a_2}y_{a_1})$, *where* $(a_1,a_2,\ldots,a_r)$ *is an oriented cycle and* $s\geq 0$.

Proof. First suppose that Q is the quiver which has m arrows between each pair of vertices, i.e. there is a positive integer m such that for each $(p,q)\in V\times V$ there are precisely m arrows $a\in A$ with $t(a)=p$ and $h(a)=q$. We write A as a disjoint union $A=A_1\cup A_2\cup\cdots\cup A_m$ in such a way that for each $(p,q)\in V\times V$ and $1\leq r\leq m$ there is exactly one element $a\in A_r$ with $t(a)=p$ and $h(a)=q$. We regard $\mathrm{End}_k(E)$ as a $\mathrm{GL}(\alpha)$-module via conjugation and $\mathrm{End}_k(E)^m$ as the direct sum $\mathrm{End}_k(E)\oplus\mathrm{End}_k(E)\oplus\cdots\oplus\mathrm{End}_k(E)$. Then we have an isomorphism of $\mathrm{GL}(\alpha)$-modules (and varieties) $\phi: R(Q,\alpha)\to\mathrm{End}_k(E)^m$ given by $\phi((y_a)_{a\in A})=(\bar{y}_1,\bar{y}_2,\ldots,\bar{y}_m)$, for $(y_a)_{a\in A}\in R(Q,\alpha)$, where $\bar{y}_r=\Sigma_{a\in A_r}y_a$, for $1\leq r\leq m$. Now the comorphism $\phi^*: k[\mathrm{End}_k(E)^m]\to k[R(Q,\alpha)]$ induces an isomorphism $k[\mathrm{End}_k(E)^m]^{\mathrm{GL}(\alpha)}\to k[R(Q,\alpha)]^{\mathrm{GL}(\alpha)}$ on invariants. By [3], §2 Theorem 2, we have that $k[\mathrm{End}_k(E)^m]^{\mathrm{GL}(\alpha)}$ is generated by the functions $(\bar{y}_1,\bar{y}_2,\ldots,\bar{y}_m)\mapsto\chi_u(\sigma^{q_1}\bar{y}_{i_1}\sigma^{q_2}\bar{y}_{i_2}\cdots\sigma^{q_r}\bar{y}_{i_r})$, for $r\geq 1$, $q_1,q_2,\ldots,q_r\geq 0$, $u\geq 0$ and $(i_1,i_2,\ldots,i_r)$ an r-tuple with entries in $\{1,2,\ldots,m\}$. Therefore $k[R(Q,\alpha)]^{\mathrm{GL}(\alpha)}$ is generated by functions of the form

$$(y_a)_{a\in A}\mapsto\chi_u\left(\sigma^{q_1}\left(\sum_{a\in A_{i_1}}y_a\right)\sigma^{q_2}\left(\sum_{a\in A_{i_2}}y_a\right)\cdots\sigma^{q_r}\left(\sum_{a\in A_{i_r}}y_a\right)\right)$$

with r, $(i_1,i_2,\ldots,i_r)$, and $q_1,q_2,\ldots,q_r$ as above. However, $\sigma y_a=c_{h(a)}y_a$ for $a\in A$, so the above function is

$$(y_a)_{a\in A}\mapsto\chi_u\left(\left(\sum_{a\in A_{i_1}}c_{h(a)}^{q_1}y_a\right)\left(\sum_{a\in A_{i_2}}c_{h(a)}^{q_2}y_a\right)\cdots\left(\sum_{a\in A_{i_r}}c_{h(a)}^{q_r}y_a\right)\right).$$

However, as is well known, a signed coefficient $\chi_u(z_1+z_2)$, of the characteristic polynomial of a sum of endomorphisms z_1,z_2, is a linear combination of products of the coefficients of the characteristic coefficients in monomials in z_1 and z_2. (Also, this follows from the main result of [3], since the function $(z_1,z_2)\mapsto\chi_u(z_1+z_2)$ is a polynomial invariant for the action of the general linear group by simultaneous conjugation on pairs on endomorphisms.) Hence the above function is a linear combination of products of functions of the form

$$(y_a)_{a\in A}\mapsto\chi_s(y_{a_r}\cdots y_{a_2}y_{a_1})$$

for $a_1,a_2,\ldots,a_r\in A$, $s\geq 0$. Now, for $a,b\in A$, we have $y_a\in\mathrm{Hom}_k(E_{t(a)},E_{h(a)})$ and $y_b\in\mathrm{Hom}_k(E_{t(b)},E_{h(b)})$ so that y_ay_b is zero unless $h(a)=t(b)$. Thus $y_{a_r}\cdots y_{a_2}y_{a_1}$ is zero unless $h(a_1) = t(a_2)$, $h(a_2) = t(a_3), \ldots, h(a_r) = t(a_1)$. Moreover, $y_{a_r} \cdots y_{a_2} y_{a_1}$ belongs to $\mathrm{Hom}_k(E_{t(a_1)}, E_{h(a_r)}) \leq \mathrm{End}_k(E)$ and, for an element z of $\mathrm{Hom}_k(E_i, E_j)$, we have $\chi_s(z) = 0$ for all $s > 0$ unless $i = j$. Hence $k[R(Q, \alpha)]^{\mathrm{GL}(\alpha)}$ is generated by the functions $(y_a)_{a \in A} \mapsto \chi_s(y_{a_r} \cdots y_{a_2} y_{a_1})$, where $(a_1, a_2, \ldots, a_r)$ is an oriented cycle and $s \geq 0$.

To conclude we make use of the elementary remark that if $L = M \oplus N$ is a direct sum decomposition of a finite dimensional rational H-module L, where H is an affine algebraic group over k, then the restriction map $k[L] \to k[M]$ is a split surjection of H-modules and hence the induced map $k[L]^H \to k[M]^H$ is surjective. So now let Q be arbitrary. Let m be a positive integer such that for each $(p, q) \in V \times V$ the number of arrows $a \in A$ satisfying $t(a) = p$ and $h(a) = q$ is at most m. Let $\hat{Q}$ be a (complete) quiver on the same vertex set V with set of arrows $\hat{A}$ containing A such that for each $(p, q) \in V \times V$ there are exactly m arrows $a \in \hat{A}$ with $t(a) = p$ and $h(a) = q$. Let Q' be the complement of Q in $\hat{Q}$, i.e. the quiver on vertex set V with arrows $A' = \hat{A} \backslash A$. We identify $R(Q, \alpha)$ with the subspace of $R(\hat{Q}, \alpha)$ consisting of the elements $(y_a)_{a \in \hat{A}}$ such that $y_a = 0$ for $a \notin A$. We similarly identify $R(Q', \alpha)$ with a subspace of $R(\hat{Q}, \alpha)$. Then $R(\hat{Q}, \alpha) = R(Q, \alpha) \oplus R(Q', \alpha)$ is a decomposition of $\mathrm{GL}(\alpha)$-modules. Hence the map $k[R(\hat{Q}, \alpha)]^{\mathrm{GL}(\alpha)} \to k[R(Q, \alpha)]^{\mathrm{GL}(\alpha)}$ is surjective. By the case already considered $k[R(\hat{Q}, \alpha)]^{\mathrm{GL}(\alpha)}$ is generated by the functions $(y_a)_{a \in \hat{A}} \mapsto \chi_s(y_{a_r} \cdots y_{a_2} y_{a_1})$, where $(a_1, a_2, \ldots, a_r)$ is an oriented cycle in $\hat{A}$, and $s \geq 0$. By restricting these functions to $R(Q, \alpha)$ we get that $k[R(Q, \alpha)]^{\mathrm{GL}(\alpha)}$ is generated by the functions $(y_a)_{a \in A} \mapsto \chi_s(y_{a_r} \cdots y_{a_2} y_{a_1})$, where $(a_1, a_2, \ldots, a_r)$ is an oriented cycle in A, and $s \geq 0$.

We now generalize this result to give generators for the invariants of $k[R(Q, \alpha)]$, for the action of a centralizer in $\mathrm{GL}(\alpha)$. In the case $[V] = 1$ this is [3], §2, Theorem 2, and in general follows from the Proposition above in the same way that [3], §2, Theorem 2 follows from [3], §2, Theorem 1.

Let $\theta_i \in \mathrm{End}_k(E_i)$ and let $\mathrm{GL}(E_i)_{\theta_i}$ be the centralizer of θ_i in $\mathrm{GL}(E_i)$, $1 \leq i \leq n$. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathrm{End}_k(E_1) \oplus \mathrm{End}_k(E_2) \oplus \cdots \oplus \mathrm{End}_k(E_n)$ and let $\mathrm{GL}(\alpha)_\theta = \mathrm{GL}(E_1)_{\theta_1} \times \mathrm{GL}(E_2)_{\theta_2} \times \cdots \times \mathrm{GL}(E_n)_{\theta_n}$.

We recall, from [2], the notion of a good pair of varieties. Let G be a reductive affine algebraic group over k. By a *good filtration* of a rational G-module M we mean an ascending filtration $0 = M_0 \leq M_1 \leq M_2 \leq \cdots$ such that, for each $i > 0$, the section M_i / M_{i+1} is either zero or isomorphic to a module induced from a one dimensional module for a Borel subgroup of G. We call Z a good G-variety if the coordinate algebra $k[Z]$ admits a good G-module filtration. By a *good pair* of G varieties we mean a pair (Z, T), where Z is an affine G-variety, T is a closed G-stable subvariety of Z and the G-modules $k[Z]$ and I_T admit good G-module filtrations, where $I_T \leq k[Z]$ is the ideal of T. Recall from [2], §1.3, that if (Z, T) is a good pair of G-varieties then Z and T are good G-varieties. The main point about good pairs, as far as invariant theory is concerned, is that the restriction map on fixed points $k[Z]^G \to k[T]^G$ is surjective (by [2], Proposition 1.4a).

THEOREM. *The algebra of invariants* $k[R(Q,\alpha)]^{\mathrm{GL}(\alpha)_\theta}$ *is generated by the functions* $(y_a)_{a\in A} \mapsto \chi_s(\theta_{h(a_r)}^{q_r} y_{a_r} \cdots \theta_{h(a_2)}^{q_2} y_{a_2} \theta_{h(a_1)}^{q_1} y_{a_1})$, *for* $(a_1, a_2, \ldots, a_r)$ *an oriented cycle*, $q_1, q_2, \ldots, q_r \geq 0$ *and* $s \geq 0$.

Proof. Let Q^+ be the quiver (V^+, A^+, t^+, h^+) obtained from Q by adding n extra loops, one at each vertex. Thus we have $V^+ = V$, $A^+ = A \cup L$, the disjoint union of A and $L = \{l_1, l_2, \ldots, l_n\}$, $t^+|_A = t$, $h^+|_A = h$, and $t^+(l_i) = h^+(l_i) = i$, for $1 \leq i \leq n$. We let $\alpha^+ = \alpha$ and take

$$R(Q^+, \alpha^+) = \prod_{a\in A^+} \mathrm{Hom}_k(E_{t(a)}, E_{h(a)}) = R(Q,\alpha) \times S$$

where $S = \Pi_{l\in L} \mathrm{Hom}_k(E_{t(l)}, E_{h(l)}) = \bigoplus_{i=1}^n \mathrm{End}_k(E_i)$.

For any affine $\mathrm{GL}(\alpha)$-variety X we have

$$k[X]^{\mathrm{GL}(\alpha)_\theta} \cong (k[X] \otimes \mathrm{Ind}_{\mathrm{GL}(\alpha)_\theta}^{\mathrm{GL}(\alpha)} k)^{\mathrm{GL}(\alpha)}$$

by Frobenius reciprocity and the tensor identity for induction (see also [5], (1.4)). But we have the natural isomorphism $\mathrm{Ind}_{\mathrm{GL}(\alpha)_\theta}^{\mathrm{GL}(\alpha)} k \cong k[\overline{C(\theta)}]$, by [2], Theorem 2.2a (iii), where $\overline{C(\theta)}$ is the Zariski closure of the $\mathrm{GL}(\alpha)$ conjugacy class $C(\theta)$ of θ. Thus we get $k[X]^{\mathrm{GL}(\alpha)_\theta} \cong k[\mathrm{GL}(\alpha) \times \overline{C(\theta)}]^{\mathrm{GL}(\alpha)}$. Explicitly, we have the isomorphism $\xi : k[X \times \overline{C(\theta)}]^{\mathrm{GL}(\alpha)} \to k[X]^{\mathrm{GL}(\alpha)}$, $\xi(f)(x) = f(x, \theta)$, for $f \in k[X \times \overline{C(\theta)}]^{\mathrm{GL}(\alpha)}$, $x \in X$. Assume now that X is a good G-variety. Then $(S, \overline{C(\theta)})$ is a good pair of $\mathrm{GL}(\alpha)$-varieties by [2], Theorem 2.2a (ii), so that $(X \times S, X \times \overline{C(\theta)})$ is a good pair of $\mathrm{GL}(\alpha)$-varieties, by [2], Proposition 1.3c (i). Hence the restriction $k[X \times S]^{\mathrm{GL}[\alpha]} \to k[X \times \overline{C(\theta)}]$ is surjective, by [2], Lemma 2.3a (or Proposition 1.4a). Thus, for a good $\mathrm{GL}(\alpha)$-variety X, we have the surjective map $\eta : k[X \times S]^{\mathrm{GL}(\alpha)} \to k[X]^{\mathrm{GL}(\alpha)_\theta}$, $\eta(f)(x) = f(x, \theta)$, for $f \in k[X \times \mathrm{End}_k(E)^n]^{\mathrm{GL}(\alpha)}$, $x \in X$.

Now take $X = R(Q, \alpha)$. Then $X \times S = R(Q^+, \alpha^+)$ so every $\mathrm{GL}(\alpha)_\theta$ invariant of $k[R(Q,\alpha)]$ has the form $f|_{y_l = \theta_{h(l)}, l\in L}$, for some $f \in k[R(Q^+, \alpha^+)]^{\mathrm{GL}(\alpha)}$. But by the Proposition $k[R(Q^+, \alpha^+)]^{\mathrm{GL}(\alpha)}$ is generated by the functions $(y_b)_{b\in A^+} \mapsto \chi_s(y_{b_t} \cdots y_{b_2} y_{b_1})$, with $(b_1, b_2, \ldots, b_t)$ an oriented cycle. Specializing y_l to $\theta_{h(l)}$, for $l \in L$, gives generators of the form described in the theorem.

Remark. We now deduce that the Proposition is valid over $\mathbb{Z}$ and stable under base change. The arguments are entirely analogous to those of [3], §3,1 so we shall be brief about the details. Let $E_{i,\mathbb{Z}}$ be a free $\mathbb{Z}$-module of rank α_i, $1 \le i \le n$. Let $\mathrm{GL}(\alpha)$ be the product of general linear group schemes corresponding to $E_{1,\mathbb{Z}}, E_{2,\mathbb{Z}}, \ldots, E_{n,\mathbb{Z}}$. Let $R(Q,\alpha)_{\mathbb{Z}} = \Pi_{a \in A} \mathrm{Hom}_{\mathbb{Z}}(E_{t(a),\mathbb{Z}}, E_{h(a),\mathbb{Z}})$ and let $\mathbb{Z}[R(Q,\alpha)]$ be the symmetric algebra on the dual abelian group $\mathrm{Hom}_{\mathbb{Z}}(R(Q,\alpha)_{\mathbb{Z}}, \mathbb{Z})$. Let $J = \mathbb{Z}[R(Q,\alpha)]^{\mathrm{GL}(\alpha)}$ and let J' be the subring generated by the coefficients of the characteristic polynomials of products, taken over oriented cycles, of elements of $\mathrm{Hom}_{\mathbb{Z}}(E_{t(a),\mathbb{Z}}, E_{h(a),\mathbb{Z}})$, $a \in A$. Let $A = \{a_1, a_2, \ldots, a_m\}$ have cardinality m. Then there is a natural multigrading $\mathbb{Z}[R(Q,\alpha)] = \bigoplus_{\omega \in \mathbb{N}_0^m} \mathbb{Z}[R(Q,\alpha)]_\omega$, such that a non-zero element of the dual of $\mathrm{Hom}_{\mathbb{Z}}(E_{t(a_r),\mathbb{Z}}, E_{h(a_r),\mathbb{Z}})$ has degree $(0,\ldots,0,1,0,\ldots,0)$ (1 in the rth position), for $1 \le r \le m$. This induces multigradings on $J, J', k[R(Q,\alpha)]$ and $k[R(Q,\alpha)]^{\mathrm{GL}(\alpha)}$. For $\omega \in \mathbb{N}_0^m$, the component $k[R(Q,\alpha)]_\omega$ has a good $\mathrm{GL}(\alpha)$-filtration (e.g. by [1], Corollary 3.2.6 and the fact that $R(Q,\alpha)$ is a good $\mathrm{GL}(\alpha)$-variety). Furthermore, the formal character of $k[R(Q,\alpha)]_\omega$ is independent of the field k and determines $\dim_k k[R(Q,\alpha)]_\omega^{\mathrm{GL}(\alpha)}$, which is therefore also independent of the field k. By the Proposition, the natural map $k \otimes_{\mathbb{Z}} J'_\omega \to k[R(Q,\alpha)]_\omega^{\mathrm{GL}(\alpha)}$ is surjective for every algebraically closed field k. It follows that $J = J'$ and that the natural map $k \otimes_{\mathbb{Z}} J \to k[R(Q,\alpha)]^{\mathrm{GL}(\alpha)}$ is an isomorphism, for every algebraically closed field k.

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